Open Access Library Journal
2016, Volume 3, e3106
ISSN Online: 2333-9721
ISSN Print: 2333-9705

Oscillator with Distributed Nonlinear Structure on a Segment of Lossy Transmission Line

Vasil G. Angelov

Department of Mathematics, University of Mining and Geology "St. I. Rilski", Sofia, Bulgaria
Email: angelov@mgu.bg

How to cite this paper: Angelov, V.G. (2016) Oscillator with Distributed Nonlinear Structure on a Segment of Lossy Transmission Line. *Open Access Library Journal*, **3**: e3106.
http://dx.doi.org/10.4236/oalib.1103106

Received: September 28, 2016
Accepted: November 3, 2016
Published: November 7, 2016

Abstract

We consider a model of self-oscillator with distributed amplifying structure realized on a segment of lossy transmission line. The distributed structure of tunnel diode type generates nonlinearity of polynomial type in the hyperbolic transmission line system. The transmission line is terminated by nonlinear reactive elements at both ends. This means that using Kirchhoff's law we obtain nonlinear boundary conditions. Then a mixed problem for lossy transmission line system is formulated. We give a new approach to present the mixed problem in a suitable operator form and using fixed point method we prove existence-uniqueness of a solution. To apply the theorem proved one has to check just several inequalities. We demonstrate conditions obtained on a numerical example.

Subject Areas

Multimedia/Signal Processing

Keywords

Oscillator Amplifier, Lossy Transmission Line, Nonlinear Distributed Structure, Fixed Point Method

1. Introduction

The present paper is devoted to investigation of self-oscillators with distributed amplifying structure of tunnel diode type realized on a segment of lossy transmission line. The transmission line is terminated by nonlinear reactive elements. Such problems and their applications (for instance to *RF*-circuits, *PCB*-s problems and so on) are usually considered by means of various methods (slowly varying in time and space amplitudes and phases, numerical methods and so on, cf. [1]-[14]). We have developed (cf. [15]) a general approach for investigation of lossy transmission lines terminated by nonlinear loads without Heaviside condition $R/L = G/C$. From mathematical point of view in [15], we consider just linear hyperbolic systems. In [16] and [17], we have considered a Josephson superconductive transmission line system with sine type nonlinearities. Our main purpose here is to consider lossy transmission line with polynomial nonlinear distributed structure that leads to a nonlinear hyperbolic system. We extend Abolinya-Myshkis method (cf. reference of [16]) to attack the nonlinear boundary value problem and propose a new general approach to reduce the mixed problem for such nonlinear systems to an operator form in suitable function spaces. The arising nonlinearity is of polynomial type in view of distributed tunnel diode element. The nonlinear characteristics of the reactive elements generate nonlinear boundary conditions. We prove the existence of an approximated solution of the mixed problem and show a way to reach this solution by successive approximations.

We proceed from the circuit shown on **Figure 1**, where Z_0 and Z_1 are nonlinear reactive elements. We consider that a particular case Z_0 is a nonlinear capacitance, while Z_1 is a nonlinear inductance. In a similar way, it can be treated more complicated circuits (cf. [15]).

A lossy transmission line with distributed nonlinear resistive element can be prescribed by the following first order nonlinear hyperbolic system of partial differential equations (cf. [1]-[14]):

$$\begin{aligned} &C\frac{\partial u(x,t)}{\partial t}+\frac{\partial i(x,t)}{\partial x}+Gu(x,t)=\gamma\left(u(x,t)\right) \\ &L\frac{\partial i(x,t)}{\partial t}+\frac{\partial u(x,t)}{\partial x}+Ri(x,t)=0 \end{aligned} \tag{1}$$

$$(x,t)\in\Pi=\left\{(x,t)\in\mathrm{R}^2:(x,t)\in[0,\Lambda]\times[0,T]\right\}$$

where $u(x,t)$ and $i(x,t)$ are the unknown voltage and current, while L, C, R and G are inductance, capacitance, resistance and conductance per unit length; $\Lambda > 0$ is its length; and $\gamma(u)=\sum_{n=1}^{m} g_n u^n$ is a prescribed polynomial of arbitrary order with interval of negative resistance (in the applications most often of third order). For the above

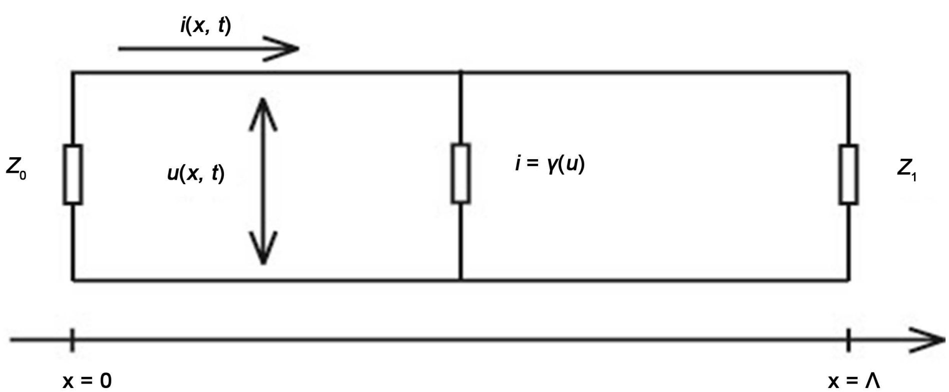

Figure 1. Lossy transmission line with distributed nonlinear resistive element with an interval of negative differential resistance in the characteristic.

system (1), one can formulate the following initial-boundary (or briefly mixed) problem: to find the unknown functions $u(x,t)$ and $i(x,t)$ in Π such that the following initial and boundary conditions are satisfied

$$u(x,0)=u_0(x),\ i(x,0)=i_0(x),\ x\in[0,\Lambda] \tag{2}$$

$$-i(0,t)=\frac{\mathrm{d}C_0(u(0,t))}{\mathrm{d}u}\frac{\mathrm{d}(u(0,t))}{\mathrm{d}t},\ \frac{\mathrm{d}L_1(i(\Lambda,t))}{\mathrm{d}i}\frac{\mathrm{d}(i(\Lambda,t))}{\mathrm{d}t}=u(\Lambda,t),\ t\in[0,T] \tag{3}$$

where $i_0(x)$ and $u_0(x)$ are prescribed initial functions the current and voltage at the initial instant; $i=C_0(u), u=L_1(i)$ are characteristics of the reactive elements Z_0, Z_1.

Rewrite the system (1) in the form

$$\begin{aligned}&\frac{\partial u(x,t)}{\partial t}+\frac{1}{C}\frac{\partial i(x,t)}{\partial x}+\frac{G}{C}u(x,t)=\frac{1}{C}\sum_{n=1}^{m}g_n u^n(x,t)\\&\frac{\partial i(x,t)}{\partial t}+\frac{1}{L}\frac{\partial u(x,t)}{\partial x}+\frac{R}{L}i(x,t)=0.\end{aligned} \tag{4}$$

2. Transformation of the Partial Differential System

First we present the system (4) in matrix form:

$$\begin{bmatrix}\partial u/\partial t\\ \partial i/\partial t\end{bmatrix}+\begin{bmatrix}0 & 1/C\\ 1/L & 0\end{bmatrix}\begin{bmatrix}\partial u/\partial x\\ \partial i/\partial x\end{bmatrix}+\begin{bmatrix}G/C & 0\\ 0 & R/L\end{bmatrix}\begin{bmatrix}u\\ i\end{bmatrix}\begin{bmatrix}u\\ i\end{bmatrix}=\begin{bmatrix}\frac{1}{C}\sum_{n=1}^{m}g_n u^n(x,t)\\ 0\end{bmatrix}.$$

Introducing denotations

$$A_1=\begin{bmatrix}0 & 1/C\\ 1/L & 0\end{bmatrix},\ A_2=\begin{bmatrix}G/C & 0\\ 0 & R/L\end{bmatrix},\ U=\begin{bmatrix}u\\ i\end{bmatrix},$$

$$\frac{\partial U}{\partial t}=\begin{bmatrix}\partial u/\partial t\\ \partial i/\partial t\end{bmatrix},\ \frac{\partial U}{\partial x}=\begin{bmatrix}\partial u/\partial x\\ \partial i/\partial x\end{bmatrix},\ \Gamma=\begin{bmatrix}(1/C)\sum_{n=1}^{m}g_n u^n(x,t)\\ 0\end{bmatrix}$$

we have

$$\frac{\partial U}{\partial t}+A_1\frac{\partial U}{\partial x}+A_2U=\Gamma. \tag{5}$$

To transform the matrix $A_1=\begin{bmatrix}0 & 1/C\\ 1/L & 0\end{bmatrix}$ in diagonal form we solve the characteristic equation $\begin{vmatrix}-\lambda & 1/C\\ 1/L & -\lambda\end{vmatrix}=0$. Its roots are $\lambda_1=1/\sqrt{LC}$, $\lambda_2=-(1/\sqrt{LC})$. The eigen-vectors are $(\xi_1^{(1)},\xi_2^{(1)})=(\sqrt{C},\sqrt{L})$, $(\xi_1^{(2)},\xi_2^{(2)})=(-\sqrt{C},\sqrt{L})$. We form the matrix by eigen-vectors $H=\begin{bmatrix}\sqrt{C} & \sqrt{L}\\ -\sqrt{C} & \sqrt{L}\end{bmatrix}$. Then $H^{-1}=\begin{bmatrix}1/2\sqrt{C} & -1/2\sqrt{C}\\ 1/2\sqrt{L} & 1/2\sqrt{L}\end{bmatrix}$ and $A^{\text{can}}=HA_1H^{-1}=\begin{bmatrix}1/\sqrt{LC} & 0\\ 0 & -1/\sqrt{LC}\end{bmatrix}$.

Introduce new variables $Z = HU, U = H^{-1}Z$, where $Z = \begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix}$. Therefore

$$\left| \begin{aligned} V(x,t) &= \sqrt{C}u(x,t) + \sqrt{L}i(x,t) \\ I(x,t) &= -\sqrt{C}u(x,t) + \sqrt{L}i(x,t) \end{aligned} \right. \quad \text{and} \quad \left| \begin{aligned} u(x,t) &= \left(1/2\sqrt{C}\right)V(x,t) - \left(1/2\sqrt{C}\right)I(x,t) \\ i(x,t) &= \left(1/2\sqrt{L}\right)V(x,t) + \left(1/2\sqrt{L}\right)I(x,t). \end{aligned} \right. \tag{6}$$

Substituting $U = H^{-1}Z$ in Equation (5) we obtain

$$\frac{\partial\left(H^{-1}Z\right)}{\partial t} + A_1 \frac{\partial\left(H^{-1}Z\right)}{\partial x} + A_2\left(H^{-1}Z\right) = \Gamma \Rightarrow H^{-1}\frac{\partial Z}{\partial t} + \left(A_1 H^{-1}\right)\frac{\partial Z}{\partial x} + \left(A_2 H^{-1}\right)Z = \Gamma$$

or

$$\frac{\partial Z}{\partial t} + \left(HA_1H^{-1}\right)\frac{\partial Z}{\partial x} + \left(HA_2H^{-1}\right)Z = H\Gamma. \tag{7}$$

But

$$HA_2H^{-1} = \begin{bmatrix} \frac{1}{2}\left(\frac{G}{C} + \frac{R}{L}\right) & \frac{1}{2}\left(-\frac{G}{C} + \frac{R}{L}\right) \\ \frac{1}{2}\left(-\frac{G}{C} + \frac{R}{L}\right) & \frac{1}{2}\left(\frac{G}{C} + \frac{R}{L}\right) \end{bmatrix}, \quad H\Gamma = \begin{bmatrix} \frac{1}{\sqrt{C}}\sum_{n=1}^{m} g_n\left(\frac{V(x,t) - I(x,t)}{2\sqrt{C}}\right)^n \\ -\frac{1}{\sqrt{C}}\sum_{n=1}^{m} g_n\left(\frac{V(x,t) - I(x,t)}{2\sqrt{C}}\right)^n \end{bmatrix}.$$

Then introducing denotations $\alpha = \frac{1}{2}\left(\frac{R}{L} - \frac{G}{C}\right), \beta = \frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right)$ we obtain from Equation (7)

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial V}{\partial x} &= -\beta V - \alpha I + \frac{1}{\sqrt{C}}\sum_{n=1}^{m} g_n\left(\frac{V-I}{2\sqrt{C}}\right)^n \\ \frac{\partial I}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial I}{\partial x} &= -\alpha V - \beta I - \frac{1}{\sqrt{C}}\sum_{n=1}^{m} g_n\left(\frac{V-I}{2\sqrt{C}}\right)^n. \end{aligned} \tag{8}$$

Introduce again new variables

$$V(x,t) = \mathrm{e}^{-\beta t}W(x,t), \quad I(x,t) = \mathrm{e}^{-\beta t}J(x,t) \tag{9}$$

and then the system (8) reduces to

$$\begin{aligned} \frac{\partial W}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial W}{\partial x} &= -\alpha J + \frac{\mathrm{e}^{\beta t}}{\sqrt{C}}\sum_{n=1}^{m} g_n\left(\mathrm{e}^{-\beta t}\frac{W-J}{2\sqrt{C}}\right)^n \\ \frac{\partial J}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial J}{\partial x} &= -\alpha W - \frac{\mathrm{e}^{\beta t}}{\sqrt{C}}\sum_{n=1}^{m} g_n\left(\mathrm{e}^{-\beta t}\frac{W-J}{2\sqrt{C}}\right)^n. \end{aligned}$$

The new transformation formulas are

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{C}}\mathrm{e}^{-\beta t}W(x,t) - \frac{1}{2\sqrt{C}}\mathrm{e}^{-\beta t}J(x,t) \\ i(x,t) &= \frac{1}{2\sqrt{L}}\mathrm{e}^{-\beta t}W(x,t) + \frac{1}{2\sqrt{L}}\mathrm{e}^{-\beta t}J(x,t). \end{aligned} \tag{10}$$

The new initial conditions we obtain from Equations (2), (6) and (9) for $x \in [0, \Lambda]$:

$$\begin{aligned} W(x,0) &= \sqrt{C}u(x,0) + \sqrt{L}i(x,0) = \sqrt{C}u_0(x) + \sqrt{L}i_0(x) \equiv W_0(x) \\ J(x,0) &= -\sqrt{C}u(x,0) + \sqrt{L}i(x,0) = -\sqrt{C}u_0(x) + \sqrt{L}i_0(x) \equiv J_0(x). \end{aligned}$$

The new boundary conditions we obtain from Equations (3):

$$\begin{aligned}
&-\frac{1}{2\sqrt{L}}\mathrm{e}^{-\beta t}W(0,t)-\frac{1}{2\sqrt{L}}\mathrm{e}^{-\beta t}J(0,t)\\
&=\frac{\mathrm{d}}{\mathrm{d}u}C_0\left(\frac{\mathrm{e}^{-\beta t}\left(W(0,t)-J(0,t)\right)}{2\sqrt{C}}\right)\frac{1}{2\sqrt{C}}\frac{\mathrm{d}\left(\mathrm{e}^{-\beta t}W(0,t)-\mathrm{e}^{-\beta t}J(0,t)\right)}{\mathrm{d}t},\ t\in[0,T]\\
&\frac{\mathrm{d}}{\mathrm{d}i}L_1\left(\frac{\mathrm{e}^{-\beta t}\left(W(\Lambda,t)+J(\Lambda,t)\right)}{2\sqrt{L}}\right)\frac{1}{2\sqrt{L}}\frac{\mathrm{d}\left(\mathrm{e}^{-\beta t}W(\Lambda,t)+\mathrm{e}^{-\beta t}J(\Lambda,t)\right)}{\mathrm{d}t}\\
&=\frac{\mathrm{e}^{-\beta t}\left(W(\Lambda,t)-J(\Lambda,t)\right)}{2\sqrt{C}},\ t\in[0,T].
\end{aligned}\tag{11}$$

In order to solve the last equations with respect to the derivatives we consider the properties of nonlinear capacitive and inductive elements. For the capacitive element (cf. [15]) we have $C_0(u)=c_0\Big/\sqrt[h]{1-(u/\Phi_0)}$, where $c_0>0,\Phi_0>0,h\in[2,3]$ are constants and $|u|\le\phi_0<\Phi_0$. If $u\in[-\phi_0,\phi_0]$, then $\mathrm{d}C_0(u)/\mathrm{d}u$ has strictly positive lower bound.

Indeed (cf. [15]), $\mathrm{d}C_0(u)/\mathrm{d}u=c_0\sqrt[h]{\Phi_0}\left[\Phi_0-((h-1)/h)u\right]\Big/(\Phi_0-u)^{(1/h)+1}$.

To obtain $\mathrm{d}C_0(u)/\mathrm{d}u=C_0^1>0$ we make

***Assumption* (*C*)** $|u|\le\phi_0<\Phi_0$.

If we choose $\phi_0<\Phi_0\min\{h/(2h+1),h/(h-1)\}=\Phi_0 h/(2h+1)$ it follows $\mathrm{d}C_0(u)/\mathrm{d}u>0$ and $\mathrm{d}^2C_0(u)/\mathrm{d}u^2>0$ for $u\in[-\phi_0,\phi_0]$ and therefore

$$\mathrm{d}C_0(u)/\mathrm{d}u>\mathrm{d}C_0(-\phi_0)/\mathrm{d}u=c_0\sqrt[h]{\Phi_0}\left[\Phi_0+((h-1)/h)\phi_0\right]\Big/(\Phi_0+\phi_0)^{\frac{1}{h}+1}=C_0^1>0.$$

Besides

$$\left|\mathrm{d}C_0(u)/\mathrm{d}u\right|\le\left(2c_0\sqrt[h]{\Phi_0}\right)\Big/(\Phi_0-\phi_0)^{\frac{1}{h}+2}\left[\Phi_0+((2h+1)/h)\phi_0\right]\Big/h\equiv\bar{C}_0^1.$$

The inductive element has *I-L* characteristic of polynomial type.

To solve the second equation (11) with respect to $\mathrm{d}i/\mathrm{d}t$ we make

***Assumptions* (L)** $\mathrm{d}L_1(i)/\mathrm{d}i\ge\hat{L}_1^1>0,|i|\le i_0$, $\left|\mathrm{d}L_1(i)(t)/\mathrm{d}i\right|\le\breve{L}_1^1,\left|\mathrm{d}^2L_1(i)(t)/\mathrm{d}i^2\right|\le\breve{L}_1^2$.

In view of $Z_0=\sqrt{L/C}$ we obtain

$$\frac{\mathrm{d}W(0,t)}{\mathrm{d}t}=\frac{\mathrm{d}J(0,t)}{\mathrm{d}t}+\beta\left(W(0,t)-J(0,t)\right)-\frac{1}{Z_0}\frac{W(0,t)+J(0,t)}{\mathrm{d}C_0(u)/\mathrm{d}u}$$

$$\frac{\mathrm{d}J(\Lambda,t)}{\mathrm{d}t}=-\frac{\mathrm{d}W(\Lambda,t)}{\mathrm{d}t}+Z_0\frac{W(\Lambda,t)-J(\Lambda,t)}{\mathrm{d}L_1(i)/\mathrm{d}i}+\beta\left(W(\Lambda,t)+J(\Lambda,t)\right).$$

We present the above relations in an integral form under

***Assumptions* (*CC*)** $W(0,0)=J(0,0),-W(\Lambda,0)=J(\Lambda,0)$,

$$W(0,t)=J(0,t)+\beta\int_0^t\left(W(\Lambda,s)-J(\Lambda,s)\right)\mathrm{d}s-\frac{1}{Z_0}\int_0^t\frac{W(0,s)+J(0,s)}{\mathrm{d}C_0(u)/\mathrm{d}u}\mathrm{d}s,$$

$$J(\Lambda,t)=-W(\Lambda,t)+Z_0\int_0^t\frac{W(\Lambda,s)-J(\Lambda,s)}{\mathrm{d}L_1(i)/\mathrm{d}i}\mathrm{d}s+\beta\int_0^t\left(W(\Lambda,s)+J(\Lambda,s)\right)\mathrm{d}s.$$

3. Operator Formulation of the Mixed Problem for the Transmission Line System

Now we are able to formulate the mixed problem with respect to the unknown functions (W,J): to find (W,J) satisfying the system and initial and boundary conditions

$$\frac{\partial W}{\partial t}+\frac{1}{\sqrt{LC}}\frac{\partial W}{\partial x}=-\alpha J+\frac{e^{\beta t}}{\sqrt{C}}\sum_{n=1}^{m}g_n\left(e^{-\beta t}\frac{W-J}{2\sqrt{C}}\right)^n,$$
$$\frac{\partial J}{\partial t}-\frac{1}{\sqrt{LC}}\frac{\partial J}{\partial x}=-\alpha W-\frac{e^{\beta t}}{\sqrt{C}}\sum_{n=1}^{m}g_n\left(e^{-\beta t}\frac{W-J}{2\sqrt{C}}\right)^n,$$
$$W(x,0)=W_0(x),\ J(x,0)=J_0(x),\ x\in[0,\Lambda]$$
$$W(0,t)=J(0,t)+\beta\int_0^t\big(W(0,s)-J(0,s)\big)\mathrm{d}s-\frac{1}{Z_0}\int_0^t\frac{W(0,s)+J(0,s)}{\mathrm{d}C_0(u)/\mathrm{d}u}\mathrm{d}s,t\in[0,T]$$
$$J(\Lambda,t)=-W(\Lambda,t)+Z_0\int_0^t\frac{\big(W(\Lambda,s)-J(\Lambda,s)\big)}{\mathrm{d}L_1(i)/\mathrm{d}i}\mathrm{d}s+\beta\int_0^t\big(W(\Lambda,s)+J(\Lambda,s)\big)\mathrm{d}s,t\in[0,T].\tag{12}$$

In what follows we give an operator representation of the above mixed problem (12).

Recall that $0\le t\le T$ and $v=1/\sqrt{LC}$ and $T=\Lambda/v=\Lambda\sqrt{LC}$. The ordinary differential equations (Cauchy problem) for the characteristics of the hyperbolic system are

$$\mathrm{d}\xi/\mathrm{d}\tau=v,\xi(t)=x\ \text{ for each }\ (x,t)\in\Pi\Rightarrow\varphi_W(\tau;x,t)=v\tau+x-vt\tag{13}$$
$$\mathrm{d}\xi/\mathrm{d}\tau=-v,\xi(t)=x\ \text{ for each }\ (x,t)\in\Pi\Rightarrow\varphi_J(\tau;x,t)=-v\tau+x+vt.\tag{14}$$

The functions $\lambda_W(x,t)=v>0$ and $\lambda_J(x,t)=-v<0$ are continuous ones. This implies that for every $(x_0,t_0)\in(0,\Lambda)\times(0,T)$ there is a unique (to the left from t_0) solution $x=\varphi_W(t;x_0,t_0)$ for $\mathrm{d}x/\mathrm{d}t=v$; $x(t_0)=x_0$, and respectively $x=\varphi_J(t;x_0,t_0)$ for $\mathrm{d}x/\mathrm{d}t=-v$; $x(t_0)=x_0$. Denote by $\chi_W(x,t)$ the smallest value of τ such that the solution $\varphi_W(\tau;x,t)=v\tau+x-vt$ of Equation (13) still belongs to Π and respectively the solution $\varphi_J(\tau;x,t)=-v\tau+x+vt$ of Equation (14) by $\chi_J(x,t)$. If $\chi_W(x,t)>0$ then $\varphi_W(\chi_W(x,t);x,t)=0$ or $\varphi_W(\chi_W(x,t);x,t)=\Lambda$ and respectively if $\chi_J(x,t)>0$ then $\varphi_J(\chi_J(x,t);x,t)=0$ or $\varphi_J(\chi_J(x,t);x,t)=\Lambda$. In our case

$$\chi_W(x,t)=\begin{cases}t-(x/v) & \text{for } vt-x>0\\ 0 & \text{for } vt-x\le 0\end{cases};\quad \chi_J(x,t)=\begin{cases}t-(\Lambda-x)/v & \text{for } vt+x-\Lambda>0\\ 0 & \text{for } vt+x-\Lambda\le 0.\end{cases}$$

Remark 1. We notice that $0\le\chi_W(x,t)<t,0\le\chi_J(x,t)<t$. It is easy to see that

$$\varphi_W(\tau;x,t)=v\tau+x-vt\Rightarrow\varphi_W(0;x,t)=x-vt;$$
$$\varphi_J(\tau;x,t)=-v\tau+x+vt\Rightarrow\varphi_J(0;x,t)=x+vt.$$

Introduce the sets:

$$\Pi_{in,W}=\{(x,t)\in\Pi:\chi_W(x,t)=0\}\equiv\{(x,t)\in\Pi:x-vt\ge 0\},$$
$$\Pi_{in,J}=\{(x,t)\in\Pi:\chi_J(x,t)=0\}\equiv\{(x,t)\in\Pi:x+vt-\Lambda\le 0\},$$

$$\Pi_{0W}=\left\{(x,t)\in\Pi:\chi_W(x,t)>0,\varphi_W\left(\chi_W(x,t);x,t\right)=v(vt-x)/v+x-vt=0\right\},$$

$$\Pi_{0J}=\left\{(x,t)\in\Pi:\chi_J(x,t)>0,\varphi_J\left(\chi_J(x,t);x,t\right)=-v(vt+x-\Lambda)/v+x+vt=0\right\}=\varnothing,$$

$$\Pi_{\Lambda W}=\left\{(x,t)\in\Pi:\chi_W(x,t)>0,\varphi_W\left(\chi_W(x,t);x,t\right)=v(vt-x)/v+x-vt=\Lambda\right\}=\varnothing,$$

$$\Pi_{\Lambda J}=\left\{(x,t)\in\Pi:\chi_J(x,t)>0,\varphi_J\left(\chi_J(x,t);x,t\right)=-v(vt+x-\Lambda)/v+x+vt=\Lambda\right\}.$$

Prior to present problem (12) in operator form we introduce

$$\Phi_W(W,J)(x,t)=\begin{cases}W_0\left(\varphi_W(0;x,t)\right),(x,t)\in\Pi_{in,W}\\ \Phi_{0W}(W,J)\left(\chi_W(x,t)\right),(x,t)\in\Pi_{0W}\\ \Phi_{\Lambda W}(W,J)\left(\chi_W(x,t)\right),(x,t)\in\Pi_{\Lambda W}\end{cases}$$
$$=\begin{cases}W_0(x-vt),\ (x,t)\in\Pi_{in,W}\\ \Phi_{0V}(W,J)\left(\chi_V(x,t)\right),(x,t)\in\Pi_{0W}\end{cases}$$

and

$$\Phi_J(W,J)(x,t)=\begin{cases}J_0\left(\varphi_J(0;x,t)\right),\ (x,t)\in\Pi_{in,J}\\ \Phi_{0J}(W,J)\left(\chi_J(x,t)\right),\ (x,t)\in\Pi_{0J}\\ \Phi_{\Lambda J}(V,I)\left(\chi_J(x,t)\right),\ (x,t)\in\Pi_{\Lambda J}\end{cases}$$
$$=\begin{cases}J_0(x+vt),\ (x,t)\in\Pi_{in,J}\\ \Phi_{\Lambda J}(W,J)\left(\chi_J(x,t)\right),\ (x,t)\in\Pi_{\Lambda J}\end{cases}$$

or

$$\Phi_W(W,J)(x,t)$$
$$=\begin{cases}W_0(x-vt),\ (x,t)\in\Pi_{in,W}\\ J\left(0,\chi_W(x,t)\right)+\beta\int_0^{\chi_W(x,t)}\left(W(0,s)-J(0,s)\right)\mathrm{d}s\\ -\dfrac{1}{Z_0}\int_0^{\chi_W(x,t)}\dfrac{W(0,s)+J(0,s)}{\mathrm{d}C_0(u)/\mathrm{d}u}\mathrm{d}s,(x,t)\in\Pi_{0W}\end{cases}$$

$$\Phi_J(W,J)(x,t)$$
$$=\begin{cases}J_0(x+vt),\ (x,t)\in\Pi_{in,J}\\ -W(\Lambda,\chi_J)+Z_0\int_0^{\chi_J}\dfrac{\left(W(\Lambda,s)-J(\Lambda,s)\right)}{\mathrm{d}L_1(i)/\mathrm{d}i}\mathrm{d}s+\beta\int_0^{\chi_J}\left(W(\Lambda,s)+J(\Lambda,s)\right)\mathrm{d}s,\ (x,t)\in\Pi_{\Lambda J}.\end{cases}$$

So we assign to the above mixed problem the following system of operator equations (cf. [16], [17]):

$$W(x,t)=\Phi_W(W,J)(x,t)-\alpha\int_{\chi_W(x,t)}^{t}J(x,s)\mathrm{d}s$$
$$+\frac{1}{\sqrt{C}}\int_{\chi_W(x,t)}^{t}\mathrm{e}^{\beta s}\sum_{n=1}^{m}g_n\cdot\left(\mathrm{e}^{-\beta s}\frac{W(x,s)-J(x,s)}{2\sqrt{C}}\right)^n\mathrm{d}s$$

$$J(x,t)=\Phi_J(W,J)(x,t)-\alpha\int_{\chi_J(x,t)}^{t}W(x,s)\,\mathrm{d}s$$
$$-\frac{1}{\sqrt{C}}\int_{\chi_J(x,t)}^{t}\mathrm{e}^{\beta s}\sum_{n=1}^{m}g_n\cdot\left(\mathrm{e}^{-\beta s}\frac{W(x,s)-J(x,s)}{2\sqrt{C}}\right)^n\mathrm{d}s.$$

4. Existence Theorem

In order to obtain a contractive operator we consider the mixed problem (12) on the subset $\Pi_\varepsilon=[\varepsilon,\Lambda-\varepsilon]\times[0,T]$. We introduce the sets

$$M_W=\left\{W\in C(\Pi_\varepsilon):|W(x,t)|\le W_0\mathrm{e}^{\mu t}\right\}\ \text{ and }\ M_J=\left\{J\in C(\Pi_\varepsilon):|J(x,t)|\le J_0\mathrm{e}^{\mu t}\right\},$$

where W_0, J_0 and μ are positive constants chosen below. It is easy to verify that $M_W\times M_J$ turns out into a complete metric space with respect to the metric

$$\rho_\mu\left((W,J),(\bar W,\bar J)\right)=\max\left\{\rho(W,\bar W),\rho(J,\bar J)\right\},$$

where

$$\rho(W,\bar W)=\operatorname{ess\,sup}\left\{\mathrm{e}^{-\mu t}\left|W(x,t)-\bar W(x,t)\right|:x\in[\varepsilon,\Lambda-\varepsilon],t\in[0,T]\right\},$$
$$\rho(J,\bar J)=\operatorname{ess\,sup}\left\{\mathrm{e}^{-\mu t}\left|J(x,t)-\bar J(x,t)\right|:x\in[\varepsilon,\Lambda-\varepsilon],t\in[0,T]\right\}.$$

Now we define an operator $B=(B_W,B_J):M_W\times M_J\to M_W\times M_J$ by the formulas

$$B_W(W,J)(x,t):=\begin{cases}W_0(x-vt),\ (x,t)\in\Pi_{in,W}\cap\Pi_\varepsilon\\ J(0,\chi_W(x,t))+\beta\displaystyle\int_0^{\chi_W(x,t)}(W(0,s)-J(0,s))\,\mathrm{d}s-\frac{1}{Z_0}\int_0^{\chi_W(x,t)}\frac{W(0,s)+J(0,s)}{\mathrm{d}C_0(u)/\mathrm{d}u}\mathrm{d}s\\ -\alpha\displaystyle\int_{\chi_W(x,t)}^{t}J(x,s)\,\mathrm{d}s+\frac{1}{\sqrt{C}}\int_{\chi_W(x,t)}^{t}\mathrm{e}^{\beta s}\sum_{n=1}^{m}g_n\cdot\left(\mathrm{e}^{-\beta s}\frac{W(x,s)-J(x,s)}{2\sqrt{C}}\right)^n\mathrm{d}s,\ (x,t)\in\Pi_{0W}\cap\Pi_\varepsilon;\end{cases}$$

$$B_J(W,J)(x,t)=\begin{cases}J_0(x+vt),\ (x,t)\in\Pi_{in,J}\cap\Pi_\varepsilon\\ -W(\Lambda,\chi_J)+Z_0\displaystyle\int_0^{\chi_J}\frac{(W(\Lambda,s)-J(\Lambda,s))}{\mathrm{d}L_1(i)/\mathrm{d}i}\mathrm{d}s+\beta\int_0^{\chi_J}(W(\Lambda,s)+J(\Lambda,s))\,\mathrm{d}s\\ -\alpha\displaystyle\int_{\chi_J(x,t)}^{t}W(x,s)\,\mathrm{d}s-\frac{1}{\sqrt{C}}\int_{\chi_J(x,t)}^{t}\mathrm{e}^{\beta s}\sum_{n=1}^{m}g_n\cdot\left(\mathrm{e}^{-\beta s}\frac{W(x,s)-J(x,s)}{2\sqrt{C}}\right)^n\mathrm{d}s,\ (x,t)\in\Pi_{\Lambda J}\cap\Pi_\varepsilon.\end{cases}$$

Remark 2. *Assumption* (*C*) and *Assumptions* (*L*) in view of Equations (10) imply

$$|u(x,t)|\le(W_0+J_0)/(2\sqrt{C})\le\phi_0;\ \ |i(x,t)|\le(W_0+J_0)/(2\sqrt{L})\le i_0.$$

Theorem 1. Let the following conditions be fulfilled:

1) *Assumption* (C), *Assumptions* (L), *Assumption* (CC) and $|W_0(x)|\le W_{00}$, $|J_0(x)|\le J_{00}$ for $x\in[\varepsilon,\Lambda-\varepsilon]$ as W_{00},J_{00} are sufficiently small while $\mu>0$ is sufficiently large;

2) $\mathrm{e}^{-\mu\frac{\varepsilon}{v}}J_0+\mathrm{e}^{-\mu\frac{\varepsilon}{v}}\left(|\beta|+\frac{1}{Z_0C_0^1}\right)\frac{W_0+J_0}{\mu}+\frac{|\alpha|J_0}{\mu}+\frac{1}{\mu\sqrt{C}}\sum_{n=1}^{m}|g_n|\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^n\mathrm{e}^{(n-1)(\mu-\beta)T}\le W_0$;

3) $e^{-\frac{\mu}{v}\varepsilon}W_0+e^{-\frac{\mu}{v}\varepsilon}\frac{W_0+J_0}{\mu}\left(\frac{Z_0}{\widehat{L}_1^1}+|\beta|\right)+\frac{|\alpha|W_0}{\mu}+\frac{1}{\mu\sqrt{C}}\sum_{n=1}^{m}|g_n|\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^n e^{(n-1)(\mu-\beta)T}\le J_0$;

4) $K_W=e^{-\mu\frac{\varepsilon}{v}}+\left(2|\beta|+\frac{2}{Z_0C_0^1}+|\alpha|\right)\frac{1}{\mu}+\frac{1}{\mu\sqrt{C}}\sum_{n=1}^{m}\frac{n|g_n|(W_0+J_0)^{n-1}}{\left(2\sqrt{C}\right)^n}e^{(n-1)(\mu-\beta)T}<1$;

5) $K_J=e^{-\frac{\mu}{v}\varepsilon}+\left(\frac{2}{\widehat{L}_1^1}+2|\beta|+|\alpha|\right)\frac{1}{\mu}+\frac{1}{\mu\sqrt{C}}\sum_{n=1}^{m}n|g_n|\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^{n-1}e^{(n-1)(\mu-\beta)T}<1$.

Then there exists a unique solution of the problem (12).

Proof. We establish that the operator B maps the set $M_W\times M_J$ into itself.

First we notice that $B_W(x,t)$ and $B_J(x,t)$ are continuous functions. We show

$$|B_W(x,t)|\le W_0e^{\mu t},\quad |B_J(x,t)|\le J_0e^{\mu t}.$$

Indeed, for sufficiently small $W_{00}>0$ and in view of $\chi_W(x,t)\le t-(x/v)\le t-(\varepsilon/v)$ and $t-\chi_W(x,t)\le x/v\le(\Lambda-\varepsilon)/v$ we have

$$\begin{aligned}
&|\Phi_W(x,t)|\\
&\le\begin{cases}|W_0(x-vt)|\\ |J(0,\chi_W(x,t))|+|\beta|\int_0^{\chi_W(x,t)}|W(0,s)-J(0,s)|ds+\frac{1}{Z_0}\int_0^{\chi_W(x,t)}\frac{|W(0,s)+J(0,s)|}{|dC_0(u)/du|}ds\end{cases}\\
&\le e^{\mu\chi_W}\max\left\{W_{00};J_0+\left(|\beta|+\frac{1}{Z_0C_0^1}\right)\frac{W_0+J_0}{\mu}\right\}\le e^{\mu\left(t-\frac{\varepsilon}{v}\right)}\left[J_0+\left(|\beta|+\frac{1}{Z_0C_0^1}\right)\frac{W_0+J_0}{\mu}\right].
\end{aligned}$$

Then for the first component we have

$$\begin{aligned}
|B_W(W,J)(x,t)|&\le e^{\mu\left(t-\frac{\varepsilon}{v}\right)}\left[J_0+\left(|\beta|+\frac{1}{Z_0C_0^1}\right)\frac{W_0+J_0}{\mu}\right]+|\alpha|\int_{\chi_W(x,t)}^{t}|J(x,s)|ds\\
&\quad+\frac{1}{\sqrt{C}}\int_{\chi_W(x,t)}^{t}e^{\beta s}\sum_{n=1}^{m}|g_n|\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^n e^{n\mu s}e^{-n\beta s}ds\\
&\le e^{\mu\left(t-\frac{\varepsilon}{v}\right)}\left[J_0+\left(|\beta|+\frac{1}{Z_0C_0^1}\right)\frac{W_0+J_0}{\mu}\right]+\frac{e^{\mu t}|\alpha|J_0}{\mu}\\
&\quad+\frac{1}{\sqrt{C}}\sum_{n=1}^{m}|g_n|\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^n\int_{\chi_W(x,t)}^{t}e^{(n-1)(\mu-\beta)s}e^{\mu s}ds\\
&\le e^{\mu t}\left[e^{-\mu\frac{\varepsilon}{v}}J_0+e^{-\mu\frac{\varepsilon}{v}}\left(|\beta|+\frac{1}{Z_0C_0^1}\right)\frac{W_0+J_0}{\mu}+\frac{|\alpha|J_0}{\mu}\right.\\
&\quad\left.+\frac{1}{\sqrt{C}}\sum_{n=1}^{m}|g_n|\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^n\frac{e^{(n-1)(\mu-\beta)T}}{\mu}\right]\\
&\le e^{\mu t}W_0.
\end{aligned}$$

In view of

$$\chi_J(x,t)\le t+(x-\Lambda)/v\le t+(\Lambda-\varepsilon-\Lambda)/v=t-(\varepsilon/v)$$

and

$$t-\chi_J(x,t)\le t-t-(x-\Lambda)/v\le(\Lambda-x)/v\le(\Lambda-\varepsilon)/v$$

for sufficiently small J_{00} for the second component we obtain:

$$\left|B_J(W,J)(x,t)\right|\le\begin{cases}J_{00}\\ \left|W(\Lambda,\chi_J)\right|+Z_0\int_0^{\chi_J}\frac{\left|W(\Lambda,s)-J(\Lambda,s)\right|}{\left|\mathrm{d}L_1(i)/\mathrm{d}i\right|}\mathrm{d}s+|\beta|\int_0^{\chi_J}\left|W(\Lambda,s)+J(\Lambda,s)\right|\mathrm{d}s\end{cases}$$

$$+|\alpha|\int_{\chi_J(x,t)}^{t}\left|W(x,s)\right|\mathrm{d}s+\frac{1}{\sqrt{C}}\int_{\chi_J(x,t)}^{t}\mathrm{e}^{\beta s}\sum_{n=1}^{m}|g_n|\cdot\left(\frac{W_0+J_0}{2\sqrt{C}}\mathrm{e}^{\mu s}\mathrm{e}^{-\beta s}\right)^n\mathrm{d}s$$

$$\le W_0\mathrm{e}^{\mu\chi_J}+\frac{Z_0(W_0+J_0)}{\hat{L}_1^1}\int_0^{\chi_J}\mathrm{e}^{\mu s}\mathrm{d}s+|\beta|(W_0+J_0)\int_0^{\chi_J}\mathrm{e}^{\mu s}\mathrm{d}s$$

$$+|\alpha|W_0\int_{\chi_J(x,t)}^{t}\mathrm{e}^{\mu s}\mathrm{d}s+\frac{1}{\sqrt{C}}\sum_{n=1}^{m}|g_n|\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^n\int_{\chi_J(x,t)}^{t}\mathrm{e}^{(n-1)(\mu-\beta)s}\mathrm{e}^{\mu s}\mathrm{d}s$$

$$\le\mathrm{e}^{\mu t}\left[\mathrm{e}^{-\frac{\mu}{v}\varepsilon}W_0+\mathrm{e}^{-\frac{\mu}{v}\varepsilon}\frac{W_0+J_0}{\mu}\left(\frac{Z_0}{\hat{L}_1^1}+|\beta|\right)+\frac{|\alpha|W_0}{\mu}+\frac{1}{\sqrt{C}}\sum_{n=1}^{m}|g_n|\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^n\frac{\mathrm{e}^{(n-1)(\mu-\beta)T}}{\mu}\right]$$

$$\le\mathrm{e}^{\mu t}J_0.$$

Now we show that B is a contractive operator.

Indeed, for the first component we obtain:

$$\left|B_W(W,J)(x,t)-B_W(\bar{W},\bar{J})(x,t)\right|$$

$$\le\left|J(0,\chi_W(x,t))-\bar{J}(0,\chi_W(x,t))\right|+|\beta|\int_0^{\chi_W(x,t)}\left(\left|W(0,s)-\bar{W}(0,s)\right|+\left|J(0,s)-\bar{J}(0,s)\right|\right)\mathrm{d}s$$

$$+\frac{1}{Z_0C_0^1}\int_0^{\chi_W(x,t)}\left(\left|W(0,s)-\bar{W}(0,s)\right|+\left|J(0,s)-\bar{J}(0,s)\right|\right)\mathrm{d}s+|\alpha|\int_{\chi_W(x,t)}^{t}\left|J(x,s)-\bar{J}(x,s)\right|\mathrm{d}s$$

$$+\frac{1}{\sqrt{C}}\int_{\chi_W(x,t)}^{t}\mathrm{e}^{\beta s}\sum_{n=1}^{m}\frac{|g_n|\mathrm{e}^{-n\beta s}}{(2\sqrt{C})^n}\cdot\left|\left(W(x,s)-J(x,s)\right)^n-\left(\bar{W}(x,s)-\bar{J}(x,s)\right)^n\right|\mathrm{d}s$$

$$\le\rho(J,\bar{J})\mathrm{e}^{\mu\chi_W}+\left(\rho(W,\bar{W})+\rho(J,\bar{J})\right)\left(|h|+\frac{1}{Z_0C_0^1}\right)\int_0^{\chi_W(x,t)}\mathrm{e}^{\mu s}\mathrm{d}s+|\alpha|\rho(J,\bar{J})\int_{\chi_W(x,t)}^{t}\mathrm{e}^{\mu s}\mathrm{d}s$$

$$+\frac{1}{\sqrt{C}}\int_{\chi_W(x,t)}^{t}\sum_{n=1}^{m}\frac{|g_n|n\mathrm{e}^{-(n-1)\beta s}}{(2\sqrt{C})^n}(W_0+J_0)^{n-1}\mathrm{e}^{(n-1)\mu s}\left|W(x,s)-J(x,s)-\bar{W}(x,s)+\bar{J}(x,s)\right|\mathrm{d}s$$

$$\le\rho_\mu\left((W,J),(\bar{W},\bar{J})\right)\left[\mathrm{e}^{\mu\chi_W}+\left(2|\beta|+\frac{2}{Z_0C_0^1}\right)\frac{\mathrm{e}^{\mu t}}{\mu}+|\alpha|\frac{\mathrm{e}^{\mu t}}{\mu}+\frac{1}{\sqrt{C}}\sum_{n=1}^{m}\frac{|g_n|n}{(2\sqrt{C})^n}(W_0+J_0)^{n-1}\int_{\chi_W(x,t)}^{t}\mathrm{e}^{(n-1)(\mu-\beta)s}\mathrm{e}^{\mu s}\mathrm{d}s\right]$$

$$\le\mathrm{e}^{\mu t}\rho_\mu\left((W,J),(\bar{W},\bar{J})\right)\left[\mathrm{e}^{-\mu\frac{\varepsilon}{v}}+\left(2|\beta|+\frac{2}{Z_0C_0^1}+|\alpha|\right)\frac{1}{\mu}+\frac{1}{\mu\sqrt{C}}\sum_{n=1}^{m}\frac{n|g_n|(W_0+J_0)^{n-1}}{(2\sqrt{C})^n}\mathrm{e}^{(n-1)(\mu-\beta)T}\right]$$

$$\equiv\mathrm{e}^{\mu t}\rho_\mu\left((W,J),(\bar{W},\bar{J})\right)K_W$$

$$\Rightarrow\rho\left(B_W(W,J),B_W(\bar{W},\bar{J})\right)\le K_W\rho_\mu\left((W,J),(\bar{W},\bar{J})\right).$$

Similarly for the second component we obtain

$$
\begin{aligned}
&\left|B_J\left(W,J\right)\left(x,t\right)-B_J\left(\bar{W},\bar{J}\right)\left(x,t\right)\right| \\
&\le \left|W\left(\Lambda,\chi_J\right)-\bar{W}\left(\Lambda,\chi_J\right)\right|+Z_0\int_0^{\chi_J}\frac{\left(\left|W\left(\Lambda,s\right)-\bar{W}\left(\Lambda,s\right)\right|+\left|J\left(\Lambda,s\right)-\bar{J}\left(\Lambda,s\right)\right|\right)}{\left|\mathrm{d}L_1\left(i\right)/\mathrm{d}i\right|}\mathrm{d}s \\
&\quad+\left|\beta\right|\int_0^{\chi_J}\left(\left|W\left(\Lambda,s\right)-\bar{W}\left(\Lambda,s\right)\right|+\left|J\left(\Lambda,s\right)-\bar{J}\left(\Lambda,s\right)\right|\right)\mathrm{d}s+\left|\alpha\right|\int_{\chi_J}^{t}\left|W\left(x,s\right)-\bar{W}\left(x,s\right)\right|\mathrm{d}s \\
&\quad+\frac{1}{\sqrt{C}}\int_{\chi_J}^{t}\sum_{n=1}^{m}\left|g_n\right|\mathrm{e}^{(1-n)\beta s}\left|\left(\frac{W\left(x,s\right)-J\left(x,s\right)}{2\sqrt{C}}\right)^n-\left(\frac{\bar{W}\left(x,s\right)-\bar{J}\left(x,s\right)}{2\sqrt{C}}\right)^n\right|\mathrm{d}s \\
&\le \rho\left(W,\bar{W}\right)\mathrm{e}^{\mu\chi_J}+\frac{\rho\left(W,\bar{W}\right)+\rho\left(J,\bar{J}\right)}{\hat{L}_1^1}\int_0^{\chi_J}\mathrm{e}^{\mu s}\mathrm{d}s+\left|h\right|\left(\rho\left(W,\bar{W}\right)+\rho\left(J,\bar{J}\right)\right)\int_0^{\chi_J}\mathrm{e}^{\mu s}\mathrm{d}s \\
&\quad+\rho\left(W,\bar{W}\right)\left|\alpha\right|\int_0^{\chi_J}\mathrm{e}^{\mu s}\mathrm{d}s+\frac{\left(\rho\left(W,\bar{W}\right)+\rho\left(J,\bar{J}\right)\right)}{\sqrt{C}}\int_{\chi_J(x,t)}^{t}\mathrm{e}^{\beta s}\sum_{n=1}^{m}n\left|g_n\right|\mathrm{e}^{-n\beta s}\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^{n-1}\mathrm{e}^{n\mu s}\mathrm{d}s \\
&\le \rho_\mu\left(\left(W,J\right),\left(\bar{W},\bar{J}\right)\right)\left[\mathrm{e}^{\mu\chi_J}+\frac{2}{\hat{L}_1^1}\frac{\mathrm{e}^{\mu t}}{\mu}+2\left|\beta\right|\frac{\mathrm{e}^{\mu t}}{\mu}+\left|\alpha\right|\frac{\mathrm{e}^{\mu t}}{\mu}+\frac{1}{C}\sum_{n=1}^{m}n\left|g_n\right|\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^{n-1}\int_{\chi_J(x,t)}^{t}\mathrm{e}^{(n-1)(\mu-\beta)s}\mathrm{e}^{\mu s}\mathrm{d}s\right] \\
&\le \mathrm{e}^{\mu t}\rho_\mu\left(\left(W,J\right),\left(\bar{W},\bar{J}\right)\right)\left[\mathrm{e}^{-\frac{\mu}{\nu}\varepsilon}+\left(\frac{2}{\hat{L}_1^1}+2\left|\beta\right|+\left|\alpha\right|\right)\frac{1}{\mu}+\frac{1}{\mu C}\sum_{n=1}^{m}n\left|g_n\right|\left(\frac{W_0+J_0}{2\sqrt{C}}\right)^{n-1}\mathrm{e}^{(n-1)(\mu-\beta)T}\right] \\
&\equiv \mathrm{e}^{\mu t}K_J\rho_\mu\left(\left(W,J\right),\left(\bar{W},\bar{J}\right)\right) \\
&\Rightarrow \rho\left(B_J\left(W,J\right),B_J\left(\bar{W},\bar{J}\right)\right)\le K_J\rho_\mu\left(\left(W,J\right),\left(\bar{W},\bar{J}\right)\right).
\end{aligned}
$$

Therefore

$$\rho_\mu\left(\left(B_W\left(W,J\right),B_J\left(W,J\right)\right),\left(B_W\left(\bar{W},\bar{J}\right),B_J\left(\bar{W},\bar{J}\right)\right)\right)\le\max\left\{K_W,K_J\right\}\rho_\mu\left(\left(W,J\right),\left(\bar{W},\bar{J}\right)\right)$$

and the operator B has a unique fixed point which is a solution of the mixed problem above formulated in the set $\Pi_\varepsilon=\left[\varepsilon,\Lambda-\varepsilon\right]\times\left[0,T\right]$.

Theorem 1 is thus proved.

Remark 3. We point out that for every $\varepsilon>0$ there is a unique solution $\left(W_\varepsilon,J_\varepsilon\right)$ in $\Pi_\varepsilon=\left[\varepsilon,\Lambda-\varepsilon\right]\times\left[0,T\right]$. The sequence $\left(W_\varepsilon,J_\varepsilon\right)$ is not necessary convergent when $\varepsilon\to0$. To find a convergent subsequence we proceed as in [17]. Extending the solution on $\Pi=\left[0,\Lambda\right]\times\left[0,T\right]$ we can choose a convergent subsequence. The first approximation can be chosen, for instance, as a solution of the linearized system (12).

5. Conclusion Remarks

1) We note that the interval $\left[0,T\right]$ is not sufficiently small.

2) We show a simple verification of all inequalities of the main theorem for soft nonlinearity $m=3$ (cf. [1]). Consider a lossy transmission line (cf. [1]-[15]) satisfying the Heaviside condition with specific parameters:

$$\Lambda=\left(1/4\right)\text{ m};\ L=0.2\ \mu\text{H/m};\ C=36\text{ pF/m};R=0.1\Omega/\text{m};G=18\times10^{-6}\ \Omega/\text{m};$$

$$\alpha=\frac{1}{2}\left(\frac{R}{L}-\frac{G}{C}\right)=\frac{1}{2}\left(\frac{0.1}{0.2\times10^{-6}}-\frac{18\times10^{-6}}{36\times10^{-12}}\right)=0;\quad \beta=R/L\approx5\times10^{5};$$

$$\sqrt{LC}=\sqrt{0.2\times10^{-6}\times36\times10^{-12}}=2.7\times10^{-9};$$

$$Z_0=\sqrt{L/C}=\sqrt{0.2\times10^{-6}/36\times10^{-12}}\approx75\,\Omega;$$

$$T=\Lambda\sqrt{LC}=6.75\times10^{-10}\text{ s}.$$

Let us choose a polynomial $\gamma(u)=0.2u-0.5u^2+0.1u^3$ with interval of negative differential resistance, $\mu=10^{10}$ and $W_0\approx J_0\cong10^{-10}$. Then $\mathrm{e}^{(\mu\varepsilon)/v}=\mathrm{e}^{40\varepsilon}$; $\mathrm{e}^{(\mu-\beta)T}\approx\mathrm{e}^{\mu T}\approx\mathrm{e}^{6.75}\approx854$. The *pn*-junction capacity is $c_0=50\text{ pF}=5\times10^{-11}\text{ F}$, while the *pn*-junction potential $\Phi=0.4$. For $h=2$ and $\phi_0=0.3$ the minimal value of $\mathrm{d}C_0(.)/\mathrm{d}u$ is $\mathrm{d}C_0(u)/\mathrm{d}u\ge C_0^1\approx3\times10^{-11}>0$.

We choose $i_0=I_0=0.01<\sqrt{10}$ such that $\mathrm{d}L_1(i)/\mathrm{d}i=2-0.2i>2-0.2i_0=\widehat{L}_1^1\approx2$.

Then the inequalities from Remark 3 and two of inequalities from Theorem 1 become

$$(W_0+J_0)/(2\sqrt{C})\le\phi_0;(W_0+J_0)/(2\sqrt{L})\le i_0\Rightarrow(10^{-2})/6\le0.3;\ 10^{-8}/\sqrt{0.2\times10^{-6}}\le0.01;$$

$$\mathrm{e}^{-40\varepsilon}+\mathrm{e}^{-40\varepsilon}\left(5\times10^5+\frac{1}{75.2}\right)\frac{2}{10^{10}}+\frac{1}{10^{10}}\left[\frac{0.1}{18\times10^{-12}}+\frac{W_0}{4.36^{3/2}\times\left(10^{-12}\right)^{3/2}}+\frac{0.1\times W_0^2}{2.36^2\times\left(10^{-12}\right)^2}\right]\le1;$$

$$K_W=\mathrm{e}^{-40\varepsilon}+\left(5\times10^5+\frac{1}{75.2}\right)\frac{2}{10^{10}}+\frac{1}{10^{10}}\left[\frac{0.1}{18\times10^{-12}}+\frac{W_0}{4.36^{3/2}\times\left(10^{-12}\right)^{3/2}}+\frac{0.1\times W_0^2}{2.36^2\times\left(10^{-12}\right)^2}\right]<1.$$

References

[1] Utkin, G.M. (1978) Auto-Oscillating Systems and Wave Amplifiers. Sovetskoe Radio, Moscow.

[2] Holt, C. (1967) Introduction in Electromagnetic Fields and Waves. John Wiley & Sons, New York.

[3] Jordan, E.C. and Balmain, K.G. (1968) Electromagnetic Waves and Radiating Systems. Prentice-Hall, Inc., Englewood Cliffs.

[4] Ishimaru, A. (1991) Electromagnetic Wave Propagation Radiation and Scattering. Prentice-Hall, Inc., New Jersey.

[5] Pozar, D. (1998) Microwave Engineering. John Wiley & Sons, New York.

[6] Paul, C.R. (1994) Analysis of Multi-Conductor Transmission Lines. Wiley-Inter Science Publication, John Wiley & Sons, New York.

[7] Ramo, S., Whinnery, J.R. and van Duzer, T. (1994) Fields and Waves in Communication Electronics. John Wiley & Sons, Inc., New York.

[8] Vizmuller, P. (1995) RF Design Guide Systems, Circuits and Equations. Artech House, Inc., Boston, London.

[9] Magnusson, P.C., Alexander, G.C. and Tripathi, V.K. (1992) Transmission Lines and Wave Propagation. 3rd Edition, CRC Press, Boca Raton.

[10] Dunlop, J. and Smith, D.G. (1994) Telecommunications Engineering. Chapman & Hall, London. http://dx.doi.org/10.1007/978-1-4899-2929-7

[11] Maas, S.A. (2003) Nonlinear Microwave and RF Circuits. 2nd Edition, Artech House, Boston, London.

[12] Misra, D.K. (2004) Radio-Frequency and Microwave Communication Circuits. Analysis and Design. 2nd Edition, University of Wisconsin-Milwaukee, John Wiley & Sons, Inc., Publication. http://dx.doi.org/10.1002/0471653764

[13] Miano, G. and Maffucci, A. (2001) Transmission Lines and Lumped Circuits. Academic Press, New York.

[14] Scott, A. (1970) Active and Nonlinear Wave Propagation in Electronics. John Wiley & Sons, New York.

[15] Angelov, V.G. (2014) A Method for Analysis of Transmission Lines Terminated by Nonlinear Loads. Nova Science, New York.

[16] Angelov, V.G. (2015) Josephson Lossless Transmission Lines with Nonlinear R-Element. *International Research Journal of Natural Sciences*, **3**, 59-79.

[17] Angelov, V.G. (2016) Lossy Transmission Lines with Josephson Junction—Continuous Generalized Solutions. *Communication in Applied Analysis*, **20**, 91-106.