The Proof of Hilbert's Seventh Problem about Transcendence of *e* + *π*

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Abstract

We prove that *e* + *π* is a transcendental number. We use proof by contradiction. The key to solve the problem is to establish a function that doesn't satisfy the relational expression that we derive, thereby produce a conflicting result which can verify our assumption is incorrect.

Keywords

Hilbert's Conjecture, Transcendental Number, The Transcendence of *e* + *π*

Subject Areas: **Algebra, Algebraic Geometry**

1. Introduction

Hilbert's seventh problem is about transcendental number. The proof of transcendental number is not very easy. We have proved the transcendence of "*e*" and "*π*". However, for over a hundred years, no one can prove the transcendence of "*e* + *π*" [1]. The purpose of this article is to solve this problem and prove that *e* + *π* is a transcendental number.

2. Proof

1) Assuming $f(x)$ is any one polynomial of degree *n*. $f(x)=c_0+c_1x+c_2x^2+\cdots+c_nx^n$, $c_0\neq 0$, Let $F(x)=\frac{f(x)}{\ln(e+\pi)}+\frac{f'(x)}{\ln^2(e+\pi)}+\frac{f''(x)}{\ln^3(e+\pi)}+\ldots+\frac{f^{(n)}(x)}{\ln^{n+1}(e+\pi)}$.

Now we consider this integral: $\int_0^b f(x)(e+\pi)^{-x}\,\mathrm{d}x$. By integrability by parts, we can get the following Formula (2.1):

$$\int_0^b f(x)(e+\pi)^{-x}\,\mathrm{d}x=-(e+\pi)^{-b}F(b)+F(0) \tag{2.1}$$

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2) Assuming $e+\pi$ is a algebraic number, so it should satisfy some one algebraic equation with integral coefficients: $c_0+c_1x+c_2x^2+\cdots+c_nx^n=0$, $c_0\neq 0$.

According to Formula (2.1), using $(e+\pi)^{-b}$ multiplies both sides of Formula (2.1) and let be separately equal to $0,1,2,\cdots,n$. We get the following result.

$$\begin{aligned}\sum_{k=0}^{n}c_kF(k)&=F(0)\sum_{k=0}^{n}c_k(e+\pi)^k-\sum_{k=0}^{n}c_k(e+\pi)^k\int_0^k f(x)(e+\pi)^{-x}\,\mathrm{d}x\\&=-\sum\nolimits_{k=0}^{n}c_k(e+\pi)^k\int_0^k f(x)(e+\pi)^{-x}\,\mathrm{d}x\end{aligned}\tag{2.2}$$

So, all we need to do or the key to solve the problem is to find a suitable $f(x)$ that it doesn't satisfy the Formula (2.2) above.

3) So we let $f(x)=\dfrac{x^{p-1}(x-1)^p(x-2)^p\cdots(x-n)^p}{(p-1)!}$ [2], $b>n$, $b>c_0$ and *b* is a prime number Because of $(x-i)^p\mid f(x)$, $i=1,2,\cdots,n$, so $f(x),\cdots,f^{(p-1)}(x)$ can be divisible by $(x-i)$ and when $x=1,2,\cdots,n$, all of $f(x),\cdots,f^{(p-1)}(x)$ equal zero.

Furthermore, we consider x^k whose (*p* + *a*)-th derivative ($a\geq 0$); when $k<p+a$, the derivative is zero. And when $k\geq p+a$, the derivative is $k(k-1)(k-2)\cdots(k-(p+a)+1)x^{k-(p+a)}$. What's more, the coefficient of x^k is a multiple of (*p* + *a*)!, so it's alse a multiple of (*p* − 1)! and *p*.

By the analysis above, we can know that $F(1),F(2),\cdots,F(n)$ are multiples of *p*.

Now we see $F(0)$; we know,

$$F(0)=\frac{f(0)}{\ln(e+\pi)}+\frac{f'(0)}{\ln^2(e+\pi)}+\cdots+\frac{f^{(P-2)}(0)}{\ln^{P-1}(e+\pi)}+\frac{f^{(P-1)}(0)}{\ln^{P}(e+\pi)}+\frac{f^{(P)}(0)}{\ln^{P+1}(e+\pi)}+\cdots+\frac{f^{((n+1)P-1)}(0)}{\ln^{(n+1)}(e+\pi)}$$

and its the sum of the first *p* − 1 item is zero (because the degree of each term of $f(x)$ is not lower than $p-1$). All from the (*p* + 1)-th item to the end are multiples of *p*. But the *p*-th item $\left(\dfrac{f^{(P-1)}(0)}{\ln^P(e+\pi)}\right)$ is the (*p* − 1)-th derivative of $\dfrac{\left((-1)^m m!\right)^p x^{p-1}}{(p-1)!\ln^P(e+\pi)}$. So, $\dfrac{f^{(P-1)}(0)}{\ln^P(e+\pi)}=\dfrac{\left((-1)^m m!\right)^p}{\ln^P(e+\pi)}$, and $F(0)$ and $\dfrac{\left((-1)^m m!\right)^p}{\ln^P(e+\pi)}$ are congruence, written $F(0)\equiv\dfrac{\left((-1)^m m!\right)^p}{\ln^P(e+\pi)}(\mathrm{mod}\ p)$. Thereby, $\sum_{k=0}^{n}c_kF(k)\equiv c_0F(0)\equiv c_0\dfrac{\left((-1)^m m!\right)^p}{\ln^P(e+\pi)}(\mathrm{mod}\ p)$, but $b>n$, $b>c_0$, and *b* is a prime number, so

$$p\nmid c_0\frac{\left((-1)^m m!\right)^p}{\ln^P(e+\pi)},\quad \sum\nolimits_{k=0}^{n}c_kF(k)\not\equiv 0(\mathrm{mod}\ p)\tag{2.3}$$

4) Next, we need to prove that $\left|-\sum_{k=0}^{n}c_k(e+\pi)^k\int_0^k f(x)(e+\pi)^{-x}\,\mathrm{d}x\right|<1$ when *p* tends to be sufficiently large.

When *x* changes from 0 to *n*, the absolute value of each factor $x-i\,(i=0,1,\cdots,n)$ of $f(x)$ is not more than n, so $|f(x)|\leq\dfrac{n(n+1)^{p-1}}{(p-1)!}$, $0\leq x\leq n$.

So by integral property: when $0\leq k\leq n$,

$$\left|\int_0^k f(x)(e+\pi)^{-x}\,\mathrm{d}x\right| \le \int_0^k \left|f(x)\right|(e+\pi)^{-x}\,\mathrm{d}x \le \frac{n^{(n+1)p-1}}{(p-1)!}\int_0^k (e+\pi)^{-x}\,\mathrm{d}x < \frac{n^{(n+1)p-1}}{(p-1)!}.$$

Let M equal $|c_0|+|c_1|+\cdots+|c_n|$,

thus,

$$\left|-\sum_{k=0}^n c_k (e+\pi)^k \int_0^k f(x)(e+\pi)^{-x}\,\mathrm{d}x\right| \le \sum_{k=0}^n \left|c_k (e+\pi)^k \int_0^k f(x)(e+\pi)^{-x}\,\mathrm{d}x\right|$$
$$< \left(\sum_{k=0}^n |c_k|\right)(e+\pi)^n \frac{n^{(n+1)p-1}}{(p-1)!} = M(e+\pi)^n \frac{n^{(n+1)p-1}}{(p-1)!}$$

When $p \to \infty, M(e+\pi)^n \frac{n^{(n+1)p-1}}{(p-1)!} \to 0$. So, $\left|-\sum_{k=0}^n c_k (e+\pi)^k \int_0^k f(x)(e+\pi)^{-x}\,\mathrm{d}x\right| < 1$ (2.4)

Finally, according to (2.3) and (2.4), we know (2.2) is incorrect. So, $e + \pi$ is a transcendental number.

3. Conjecture

By the proof above, we conclude that $e + \pi$ is a transcendental number. Besides, I suppose $\ln(e+\pi)$ is also a transcendental number. What's more, when a and b are two real numbers, and $\frac{b}{1-a} \ge \frac{e}{\pi}$, I suppose that $ae + b\pi$ is a transcendental number.

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References

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[2] Min, S.H. and Yan, S.J. (2003) Elementary Number Theory. 3rd Edition, Higher Education Press, Beijing.