Log-Concavity of Centered Polygonal Figurate Number Sequences

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Abstract
This paper investigates the log-concavity of the centered $m$-gonal figurate number sequences. The author proves that for $m \geq 3$, the sequence $(C_n(m))$ of centered $m$-gonal figurate numbers is a log-concave.

Keywords
Log-Concavity, Figurate Numbers, Centered Polygonal, Number Sequences

Subject Areas: Discrete Mathematics, Combinatorial Sequences, Recurrences

1. Introduction
For $n \geq 1$ and $m \geq 3$, let $C_n(m)$ denote the $n^{th}$ term of the centered $m$-gonal figurate number sequence. E. Deza and M. Deza [1] stated that $C_n(m)$ could be defined by the following recurrence relation:

$$C_{n+1}(m) = C_n(m) + mn$$

where $C_1(m) = 1$. E. Deza and M. Deza [1] also gave different properties of $C_n(m)$ and obtained

$$C_n(m) = 1 + \frac{m(n-1)n}{2} = \frac{mn^2 - mn + 2}{2}$$

where $n \geq 1$ and $m \geq 3$. For $m \geq 3$, some terms of the sequence $(C_n(m))$ are as follows:

$1,1 + m,1 + 3m,1 + 6m,1 + 10m,1 + 15m,1 + 21m,1 + 28m,\ldots$

Some scholars have been studying the log-concavity (or log-convexity) of different numbers sequences such as Fibonacci & Hyperfibonacci numbers, Lucas & Hyperlucas numbers, Bell numbers, Hyperpell numbers, Motzkin numbers, Fine numbers, Franel numbers of order 3 & 4, Apéry numbers, Large Schröder numbers,
Central Delannoy numbers, Catalan-Larcombe-French numbers sequences, and so on (see for instance [2]-[9]).

To the best of the author’s knowledge, among all the aforementioned works on the log-concavity and log-convexity of number sequences, no one has studied the log-concavity (or log-convexity) of centered \( m \)-gonal figurate number sequences. In [1] [10] [11], some properties of centered figurate numbers are given. The main aim of this paper is to discuss properties related to the sequence \( \{C_n(m)\}_{n \geq 0} \). Now we recall some definitions involved in this paper.

**Definition 1.** Let \( \{s_n\}_{n \geq 0} \) be a sequence of positive numbers. If for all \( i \geq 1, s_i^2 \geq s_{i-1} s_{i+1} \), the sequence \( \{s_n\}_{n \geq 0} \) is called log-concave.

**Definition 2.** Let \( \{s_n\}_{n \geq 0} \) be a sequence of positive numbers. If for all \( i \geq 1, s_i^2 \leq s_{i-1} s_{i+1} \), the sequence \( \{s_n\}_{n \geq 0} \) is called log-convex. In case of equality, \( s_i^2 = s_{i-1} s_{i+1}, i \geq 1 \), we call the sequence \( \{s_n\}_{n \geq 0} \) geometric or log-straight.

**Definition 3.** Let \( \{s_n\}_{n \geq 0} \) be a sequence of positive numbers. The sequence \( \{s_n\}_{n \geq 0} \) is log-concave (log-convex) if and only if its quotient sequence \( \left\{ \frac{s_{n+1}}{s_n} \right\}_{n \geq 0} \) is non-increasing (non-decreasing).

Log-concavity and log-convexity are important properties of combinatorial sequences and they play a crucial role in many fields, for instance economics, probability, mathematical biology, quantum physics and white noise theory [2] [12]-[18].

### 2. Log-Concavity of Centered \( m \)-gonal Figurate Number Sequences

In this section, we state and prove the main results of this paper.

**Theorem 4.** For \( m \geq 3 \) and \( n \geq 3 \), the following recurrence formulas for centered \( m \)-gonal number sequences hold:

\[
C_n(m) = R(n)C_{n-1}(m) + S(n)C_{n-2}(m)
\]

with the initial conditions \( C_1(m) = 1, C_2(m) = 1 + m \) and the recurrence of its quotient sequence is given by

\[
x_{n+1} = R(n) + \frac{S(n)}{x_{n-2}}
\]

with the initial condition \( x_1 = 1 + m \). 

*Proof.* By (1), we have

\[
C_{n+1}(m) = C_n(m) + mn
\]

It follows that

\[
C_{n+2}(m) = C_{n+1}(m) + m(n+1)
\]

Rewriting (5) and (6) for \( n \geq 3 \), we have

\[
C_{n-1}(m) = C_{n-2}(m) + m(n-2)
\]

\[
C_n(m) = C_{n-1}(m) + m(n-1)
\]

Multiplying (7) by \( m(n-1) \) and (8) by \( m(n-2) \), and subtracting as to cancel the non homogeneous part, one can obtain the homogeneous second-order linear recurrence for \( C_n(m) \):

\[
C_n(m) = \left[ \frac{2n-3}{n-2} \right] C_{n-1}(m) - \left[ \frac{n-1}{n-2} \right] C_{n-2}(m), \forall n, m \geq 3.
\]

By denoting

\[
\frac{2n-3}{n-2} = R(n)
\]

and
\( \frac{n-1}{n-2} = S(n), \)

one can obtain

\[ C_n(m) = R(n)C_{n-1}(m) + S(n)C_{n-2}(m), \forall n, m \geq 3 \]  \hspace{1cm} (10)

with given initial conditions \( C_1(m) = 1 \) and \( C_2(m) = 1 + m \).

By dividing (10) through by \( C_{n-1}(m) \), one can also get the recurrence of its quotient sequence \( x_{n-1} \) as

\[ x_{n-1} = R(n) + \frac{S(n)}{x_{n-2}}, n \geq 3 \]  \hspace{1cm} (11)

with initial condition \( x_1 = 1 + m \).

\textbf{Lemma 5.} For the centered m-gonal figurate number sequence \( \{C_n(m)\}_{n \geq 1} \), let \( x_n = \frac{C_{n+1}(m)}{C_n(m)} \) for \( n \geq 1 \) and \( m \geq 3 \). Then we have \( 1 < x_n \leq 1 + m \) for \( n \geq 1 \).

\textit{Proof.} Assume \( x_n \neq 1 \) for \( n \geq 1 \) and \( m \geq 3 \). Otherwise, \( 1 = x_n = \frac{C_{n+1}(m)}{C_n(m)} = \frac{2 + mn(n+1)}{2 + mn(n-1)}. \) It follows that \( -1 = 1 \) which not true. Now it is clear that \( x_n \neq 1 \) and

\[ x_1 = 1 + m, x_2 = 3 - \frac{2}{1+m}, x_3 = 2 - \frac{1}{1+3m} > 1, \text{ for } m \geq 3. \]  \hspace{1cm} (13)

Assume that \( x_n > 1 \) for all \( n \geq 3 \). It follows from (11) that

\[ x_n = \frac{2n-1}{n-1} \frac{n}{(n-1)x_{n-1}}, n \geq 2 \]  \hspace{1cm} (14)

For \( n \geq 3 \), by (14), we have

\[ x_{n+1} - 1 = \frac{n+1}{n} - \frac{n+1}{nx_n} \]  \hspace{1cm} (15)

\[ = \frac{(n+1)x_n - (n+1)}{nx_n} \]  \hspace{1cm} (16)

\[ = \frac{(n+1)(x_n - 1)}{nx_n} \]  \hspace{1cm} (17)

\[ > 0 \text{ for } m \geq 3. \]

Hence \( x_n > 1 \) for \( n \geq 1 \) and \( m \geq 3 \).

Similarly, it is known that

\[ x_1 = 1 + m, x_2 = 3 - \frac{2}{1+m}, x_3 = 2 - \frac{1}{1+3m} < 1 + m, \text{ for } m \geq 3. \]  \hspace{1cm} (18)

Assume that \( x_n \leq 1 + m \) for all \( n \geq 3 \). It follows from (11) that

\[ x_n = \frac{2n-1}{n-1} \frac{n}{(n-1)x_{n-1}}, n \geq 2 \]  \hspace{1cm} (19)

For \( n \geq 3 \), by (19), we have

\[ x_{n+1} - (1 + m) = \frac{n+1 - mn}{n} - \frac{n+1}{nx_n} \]  \hspace{1cm} (20)
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\[ (n+1-mn)x_n - (n+1) \]
\[ \frac{nx_n}{2n-1} \]

Hence \( x_n \leq 1+m \) for \( n \geq 1 \) and \( m \geq 3 \).

Thus, in general, from the above two cases it follows that \( 1 < x_n \leq 1+m \) for \( n \geq 1 \) and \( m \geq 3 \).

Lemma 6. For the centered \( m \)-gonal figurate number sequence \( \{C_n(m)\}_{n \geq 1} \), the quotient sequence \( \{x_n\}_{n \geq 1} \), given in (4), is a decreasing sequence for \( m \geq 3 \).

Proof. Let \( \{x_n\}_{n \geq 1} \) be a quotient sequence given in (4). We prove by induction that the sequence \( \{x_n\}_{n \geq 1} \) is decreasing. Indeed, since \( x_1 = 1+m, x_2 = 3- \frac{2}{1+m}, x_3 = 2 - \frac{1}{1+3m} \), we have \( x_1 > x_2 > x_3 \). Next we assume that \( x_n < x_{n-1} \).

By using (11), one can obtain

\[ x_n = \frac{2n-1}{n-1} - \frac{n}{(n-1)x_{n-1}}, n \geq 2 \]

with initial condition \( x_1 = 1+m \).

For \( n \geq 3 \), by (22), we get

\[ x_{n+1} - x_n = \frac{2n+1}{n} \frac{n+1}{n} \frac{2n-1}{n} \frac{n-1}{(n-1)x_{n-1}} \]

\[ = \frac{2n+1}{n} - \frac{2n-1}{n} + \frac{1}{n} \left[ \frac{n}{n-1} - \frac{n+1}{n} \right] + \frac{n}{n-1} \left[ \frac{1}{n-1} \frac{1}{x_{n-1}} \right] \]

\[ = \frac{x_n - 1}{n(n-1)x_n} + \frac{n}{n-1} \left[ \frac{1}{x_{n-1}} \frac{1}{x_n} \right] < 0. \]

By Lemma 5 and induction assumption, one can get \( x_{n+1} - x_n < 0 \) for \( n \geq 3 \).

Thus, the sequence \( \{x_n\}_{n \geq 1} \) is decreasing for \( m \geq 3 \).

Theorem 7 For \( m \geq 3 \), the sequence \( \{C_n(m)\}_{n \geq 1} \) of centered \( m \)-gonal figurate numbers is a log-concave.

Proof. Let \( \{C_n(m)\}_{n \geq 1} \) be a sequence of centered \( m \)-gonal figurate numbers and \( \{x_n\}_{n \geq 1} \) its quotient sequence, given by (4). To prove the log-concavity of \( \{C_n(m)\}_{n \geq 1} \) for all \( m \geq 3 \), it suffices to show that the quotient sequence \( \{x_n\}_{n \geq 1} \) is decreasing.

By Lemma 6, the quotient sequence \( \{x_n\}_{n \geq 1} \) is decreasing. Thus, by definition 3, the sequence \( \{C_n(m)\}_{n \geq 1} \) of centered \( m \)-gonal figurate numbers is a log-concave for \( m \geq 3 \). This completes the proof of the theorem.

3. Conclusion

In this paper, we have discussed the log-behavior of centered \( m \)-gonal figurate number sequences. We have also proved that for \( m \geq 3 \), the sequence \( \{C_n(m)\}_{n \geq 1} \) of centered \( m \)-gonal figurate numbers is a log-concave.

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References


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