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Constructing a Subsequence of $\left(\mathrm{Exp}(\mathrm{i}n)\right)_{n\in\mathbb{N}}$ Converging towards $\mathrm{Exp}(\mathrm{i}\alpha)$ for a Given $\alpha\in\mathbb{R}$

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Abstract

For a given positive irrational ζ and a real $t\in[0,1)$, the explicit construction of a sequence $j\mapsto N(\zeta,t,j)$ of positive integers, such that the sequence of fractional parts of products $\zeta N(\zeta,t,j)$ converges towards t, is given. Moreover, a constructive and quantitative demonstration of the well known fact, that the ranges of the functions cos and sin are dense in the interval $[-1,1]$, is presented. More precisely, for any $\alpha\in\mathbb{R}$, a sequence $j\mapsto\nu(\alpha,j)$ of positive integers is constructed explicitly in such a way that the estimate

$$\left|\mathrm{e}^{\mathrm{i}\nu(\alpha,j)}-\mathrm{e}^{\mathrm{i}\alpha}\right|<2^{-j-1/2}\quad\left(\mathrm{i}^2=-1\right)$$

holds true for any $j\in\mathbb{N}$. The technique used in the paper can give more general results, e.g. by replacing sine or cosine with continuous function $f:\mathbb{R}\to\mathbb{R}$ having an irrational period.

Keywords

Convergence, Dense, Estimate, Exponential, Fractional Part, Integer Part, Irrational, Limit Point, Sequence

Subject Areas: **Mathematical Analysis, Number Theory, Numerical Mathematics**

1. Introduction

There are several arguments known showing that the ranges of the sequences $\left(\cos(n)\right)_{n\in\mathbb{N}}$ and $\left(\sin(n)\right)_{n\in\mathbb{N}}$ to

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be dense in the interval $[-1,1]$ (see for example [1], Problem 4.22, p. 33 and [2], Problem 1.4.26, p. 45). In [3] the authors considered the limit points of the sequence $(\sin(n))_{n\in\mathbb{N}}$ rather constructively. Subsequently Ogilvy [4] presented more elegant but less direct analysis. In [5] it was demonstrated, on the basis of continued fraction theory, that the set $\{(\cos(n))^n \mid n\in\mathbb{N}\}$ is dense too in $[-1,1]$. Recently, these results were generalized in some directions in [6], considering instead of cosine and sine, a continuous function $f:\mathbb{R}\to\mathbb{R}$ having an irrational period. As a corollary the authors obtained that the sets $\{\cos(n) \mid n\in\mathbb{N}\}$ and $\{\sin(n) \mid n\in\mathbb{Z}\}$ are dense in $[-1,1]$. However, it was not confirmed in [6] that $\{\sin(n) \mid n\in\mathbb{N}\}$ is dense too in $[-1,1]$. The technique used in the above cited literature is more or less constructive or quantitative. To the best knowledge of the author, the most constructive approach to the problem of denseness of the sequence $(\sin(n))_{n\in\mathbb{N}}$ or more general of the sequence $(f(n))_{n\in\mathbb{N}}$ for a continuous function $f:\mathbb{R}\to\mathbb{R}$ having an irrational period can be found in [7].

We offer a concrete—direct, constructive and also quantitative (computational) approach to the limit points of the sequences $(\cos(n))_{n\in\mathbb{N}}$ and $(\sin(n))_{n\in\mathbb{N}}$, *i.e.* to the limit points of complex-valued sequence $(\mathrm{e}^{\mathrm{i}n})_{n\in\mathbb{N}}$.

The idea of continued fraction representation of a number suggests how to construct an algorithm producing a sequence of positive integers such that by applying the functions sin and cos we obtain two convergent sequences with prescribed limits in the interval $[-1,1]$. Crucial is the well-known fact that for any irrational number ζ the fractional parts $\delta(n\zeta)$, $n\in\mathbb{N}$, are dense in $[0,1]$. The purpose of the paper is to construct explicitly, for any positive irrational ζ and any $t\in[0,1)$, the sequence $j\mapsto N(\zeta,t,j)$ of positive integers such that the sequence of fractional parts of products $\zeta N(\zeta,t,j)$ converges towards *t*, and consequently, to construct explicitly, for any $\alpha\in\mathbb{R}$, the sequence of positive integers ν_j such that the estimate

$$\left|\mathrm{e}^{\mathrm{i}\nu_j}-\mathrm{e}^{\mathrm{i}\alpha}\right|<2^{-j-1/2}\quad(\mathrm{i}^2=-1)$$

holds true for any $j\in\mathbb{N}$.

2. Preliminaries

We begin with formal definition making possible to construct the desired sequence.

Definition 2.1 *For any* $x\in\mathbb{R}$ *the integer part or floor* $\lfloor x\rfloor$ *and the fractional part* $\delta(x)$ *of x are defined as follows*[1]:

$$\lfloor x\rfloor := \max\{n\in\mathbb{Z} \mid n\le x\} \text{ and } \delta(x):=x-[x].$$

As an immediate consequence of this definition we have, for any $x\in\mathbb{R}$:

1) $\lfloor x\rfloor = k \Leftrightarrow (k\in\mathbb{Z} \text{ and } x-1<k\le x)$ (1)

2) $0\le\delta(x)<1.$ (2)

Moreover, for any positive irrational number ζ and any positive integer n there exist (only one) non-negative number *k* and (only one) $\delta\in(0,\zeta)$ such that

$$n=k\cdot\zeta+\delta. \tag{3}$$

Indeed, considering Definition above, the numbers $k:=\left\lfloor \frac{n}{\zeta}\right\rfloor$ and $\delta:=n-k\cdot\zeta$ confirm the assertion. Namely, using (1), we have $k\in\mathbb{N}\cup\{0\}$ and $\frac{n}{\zeta}-1<k\le\frac{n}{\zeta}$, *i.e.* $0\le n-k\cdot\zeta<\zeta$ with $n-k\cdot\zeta$ irrational.

The crucial role is played by the following lemma.

Lemma 2.2. *Let* $\zeta\in\mathbb{R}^+\setminus\mathbb{Q}$, $n\in\mathbb{N}$, $k\in\mathbb{N}\cup\{0\}$ *and let* $\delta\in(0,\zeta)$ *be such that* $n=k\cdot\zeta+\delta$. *Then there exist* $n',k'\in\mathbb{N}$ *and* $\delta'\in\mathbb{R}$ *such that* $n'=k'\cdot\zeta+\delta'$, $0<\delta'\le\frac{1}{2}\delta$, $k'\ge 2k+1$ *and* $n'\ge 2n$. *Constructively, letting*

[1]The literature usually uses for fractional part of *x* different notations such as for example $frac(x)$ or $\langle x\rangle$ or even $\{x\}$. The last one symbol is not suitable due to possible confusion with the singleton containing the only element *x*.

$$k := \left\lfloor \frac{n}{\zeta} \right\rfloor,\ \delta := n - k\cdot\zeta,\ q := \left\lfloor \frac{\zeta}{\delta} \right\rfloor,\ \delta^* := (q+1)\delta - \zeta,\ p := -\left\lfloor -\frac{\delta}{2(\delta-\delta^*)} \right\rfloor,$$

the numbers

$$n' := \begin{cases} (q+1)n & \text{if } \delta^* \le \frac{\delta}{2}, \\ (pq+1)n & \text{if } \delta^* > \frac{\delta}{2} \end{cases} \quad \text{and} \quad k' := \left\lfloor \frac{n'}{\zeta} \right\rfloor \quad \text{and} \quad \delta' := n' - k\cdot\zeta$$

verify the statement.

Proof. Let us suppose that

$$\zeta \in \mathbb{R}^+ \setminus \mathbb{Q},\ n = k\cdot\zeta + \delta,\ n \in \mathbb{N},\ k \in \mathbb{N},\ \delta \in (0,\zeta). \tag{4}$$

Then $\frac{\zeta}{\delta} > 1$. Hence, the integer $q := \left\lfloor \frac{\zeta}{\delta} \right\rfloor \ge 1$ and, considering (1), $\frac{\zeta}{\delta} - 1 < q \le \frac{\zeta}{\delta}$, *i.e.*

$$q\cdot\delta \le \zeta < (q+1)\cdot\delta. \tag{5}$$

Moreover,

$$\begin{aligned} (q+1)n &= (q+1)(k\cdot\zeta+\delta) = k(q+1)\cdot\zeta + (q+1)\cdot\delta \\ &= (k(q+1)+1)\cdot\zeta + ((q+1)\cdot\delta - \zeta) = k^*\cdot\zeta + \delta^*, \end{aligned} \tag{6}$$

where $k^* := (k(q+1)+1) \in \mathbb{N}$ and

$$\delta^* := (q+1)\delta - \zeta = \delta - (\zeta - q\delta) \overset{(5)}{\in} (0,\delta). \tag{7}$$

Consequently,

$$\delta - \delta^* = (\zeta - q\delta) \in (0,\delta). \tag{8}$$

Now, we distinguish two cases: **(A)** $\boxed{\delta^* \le \delta/2}$ and **(B)** $\boxed{\delta^* > \delta/2}$.

(A) In this case we can set in Lemma 2.2 the integers $n' = (q+1)n \ge 2n$ and $k' = k^* = k(q+1)+1 \ge 2k+1$, and the fractional part $\delta' = \delta^*$.

(B) In this case we have the difference

$$\frac{\delta}{\delta-\delta^*} - \frac{\delta}{2(\delta-\delta^*)} = \frac{\delta}{2(\delta-\delta^*)} \overset{(\mathrm{B})}{>} \frac{\delta}{2\delta/2} = 1. \tag{9}$$

Therefore, there exists an integer $p \in \left[\frac{\delta}{2(\delta-\delta^*)}, \frac{\delta}{\delta-\delta^*}\right)$ such that the inequality

$$\frac{\delta}{2} \le p(\delta-\delta^*) < \delta \tag{10}$$

holds. Now, referring to (4), (6) and (8), we have

$$(pq+1)n \overset{(4)}{=} (pq+1)(k\cdot\zeta+\delta) = \underbrace{(pq+1)k\cdot\zeta + p\cdot\zeta}_{=k^{**}\cdot\zeta}\underbrace{-p\cdot\zeta + pq\delta + \delta}_{=\delta^{**}}.$$

Hence,

$$(pq+1)n = k^{**}\cdot\zeta + \delta^{**}, \tag{11}$$

where

$$k^{**} := \left((pq+1)k + p \right) \in \mathbb{N} \tag{12}$$

and, according to (8),

$$\delta^{**} := -p\cdot\zeta + pq\delta + \delta = \delta - p(\zeta - q\delta) \overset{(8)}{=} \delta - p\left(\delta - \delta^{*}\right). \tag{13}$$

Since $0 < \delta^{**} \le \frac{\delta}{2}$, due to (10) and (13), we can take in Lemma 2.2 the integers $n' = (pq+1)n$, $k' = k^{**}$ and $\delta' = \delta^{**}$.

We also note that the integer $p_0 := -\left\lfloor -\frac{\delta}{2\left(\delta-\delta^{*}\right)} \right\rfloor$ satisfies the estimate

$$-\frac{\delta}{2\left(\delta-\delta^{*}\right)} - 1 < -p_0 \le -\frac{\delta}{2\left(\delta-\delta^{*}\right)},$$

i.e., referring to (9), we have

$$\frac{\delta}{2\left(\delta-\delta^{*}\right)} \le p_0 < \frac{\delta}{2\left(\delta-\delta^{*}\right)} + 1 \overset{(9)}{<} \frac{\delta}{2\left(\delta-\delta^{*}\right)} + \frac{\delta}{2\left(\delta-\delta^{*}\right)} = \frac{\delta}{\delta-\delta^{*}}.$$

Thus, p_0 satisfies (10) and $p \ge p_0$ for every p satisfying (10). Moreover, in case (B), we have $\delta^{*} > \frac{\delta}{2}$. But, this estimate implies the inequality $\frac{\delta}{2\left(\delta-\delta^{*}\right)} > 1$. Consequently, $p_0 \ge 1$, *i.e.* we have $p \ge p_0 \ge 1$. Hence, in case (B), we estimate $n' = (pq+1)n \ge (1+1)n = 2n$ and $k' = (pq+1)k + p \ge 2k+1$.

Corollary 2.3. *Let* ζ *be any positive irrational,* $n(\zeta,1)$ *any positive integer and for every* $j \in \mathbb{N}$ *let us define*

1) $k(\zeta,j) := \left\lfloor \frac{n(\zeta,j)}{\zeta} \right\rfloor$

2) $d(\zeta,j) := n(\zeta,j) - k(\zeta,j)\cdot\zeta$

3) $q(\zeta,j) := \left\lfloor \frac{\zeta}{d(\zeta,j)} \right\rfloor$

4) $d^{*}(\zeta,j) := \left(q(\zeta,j)+1\right)\cdot d(\zeta,j) - \zeta$

5) $p(\zeta,j) := -\left\lfloor -\frac{d(\zeta,j)}{2\left(d(\zeta,j) - d^{*}(\zeta,j)\right)} \right\rfloor$

6) $n(\zeta,j+1) := \begin{cases} \left(q(\zeta,j)+1\right)n(\zeta,j) & \text{if } d^{*}(\zeta,j) \le \frac{1}{2}d(\zeta,j), \\ \left(p(\zeta,j)q(\zeta,j)+1\right)n(\zeta,j) & \text{if } d^{*}(\zeta,j) > \frac{1}{2}d(\zeta,j). \end{cases}$

In this way we obtain the sequences $\left(k(\zeta,j)\right)_{j\in\mathbb{N}}$ and $\left(n(\zeta,j)\right)_{j\in\mathbb{N}}$ of positive integers and the sequence $\left(d(\zeta,j)\right)_{j\in\mathbb{N}}$ such that for any $j \in \mathbb{N}$ there hold the following relations

i) $n(\zeta,j) \equiv k(\zeta,j)\cdot\zeta + d(\zeta,j)$

ii) $k(\zeta,j+1) \ge 2k(\zeta,j)+1$ and $n(\zeta,j+1) \ge 2n(\zeta,j)$

iii) $d(\zeta,j+1) \le \frac{1}{2}\cdot d(\zeta,j)$.

iv) $0 < d(\zeta, j) \le d(\zeta, 1) \cdot 2^{1-j}$.

Proof. For the sequences, which are given inductively, we can apply the preceding Lemma 2.2 to verify the assertions i)-iii) of the Corollary 2.3. Concerning the estimate iv), it is certainly true for $j = 1$ and, if $d(\zeta, j) \le d(\zeta, 1) \cdot 2^{1-j}$ for some $j \in \mathbb{N}$, then, using iii), we have

$$d(\zeta, j+1) \le \frac{1}{2} \cdot d(\zeta, j) \le \frac{1}{2} \cdot d(\zeta, 1) \cdot 2^{1-j} = d(\zeta, 1) \cdot 2^{-j}.$$

Remark 2.4. *The estimate iii) in Corollary* 2.3 *is rather sharp as is illustrated*[2] *in* **Figure 1** *where the graph of the sequence* $j \mapsto d(2\pi, j+1)/d(2\pi, j)$ *is depicted using* $n(2\pi, 1) = 1$.

Remark 2.5. *The estimate iv) in Corollary* 2.3 *seems to be rather rough as it is evident from* **Figure 2** *showing the graph of the sequence* $j \mapsto 5^j \times d(2\pi, j)$.

Remark 2.6. *Given positive irrational* ζ, *smaller is the factor* $d(\zeta, 1)$ *in Corollary* 2.3 *iv) faster is the convergence* $d(\zeta, j) \to 0$ *as* $j \to \infty$. *Therefore, for the initial number* $n(\zeta, 1)$ *in Corollary* 2.3 *a positive integer m should be chosen in such a way that the number* $\tilde{d}(\zeta, m) := m - \left\lfloor \frac{m}{\zeta} \right\rfloor \cdot \zeta$ *should be as small as possible. The* **Table 1** *illustrates the dynamics of the sequence* $m \mapsto \tilde{d}(2\pi, m)$.

Remark 2.7. *The* **Table 2** *shows, for* $n(2\pi, 1) = 13$, *the dynamics of the sequences* $j \mapsto d(2\pi, j)$ *and* $j \mapsto n(2\pi, j)$. *The latter grows very fast. However, if we put, for example,* $n(2\pi, 1) = 19$ *in Corollary* 2.3, *we would get a sequence that would grow a bit more slowly. By experimenting with Mathematica* [8] *we come to the conjecture that*

$$(j!)^{\sqrt{j}} < n(2\pi, j) < (j!)^j$$

for $n(2\pi, 1) = 13$ *and* $j \ge 8$. *However, this is only a hypothesis.*

3. Denseness

Theorem 3.1. *For* ζ *being any positive irrational,* $t \in [0, 1)$ *and, using the sequences* $n(1/\zeta, j)$ *and* $d(1/\zeta, j)$ *from Corollary* 2.3, *let us define*

1) $m(\zeta, t, j) := \left\lfloor \frac{t/\zeta + d(1/\zeta, 1) \cdot 2^{1-j}}{d(1/\zeta, j)} \right\rfloor \quad (j \in \mathbb{N})$,

2) $N(\zeta, t, j) := m(\zeta, t, j) \cdot n(1/\zeta, j) \quad (j \in \mathbb{N})$.

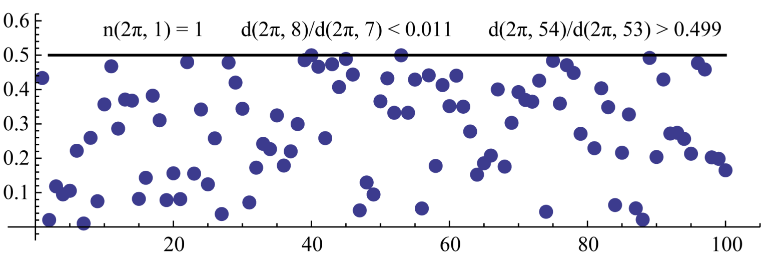

Figure 1. The graph of the sequence $j \mapsto d(2\pi, j+1)/d(2\pi, j)$.

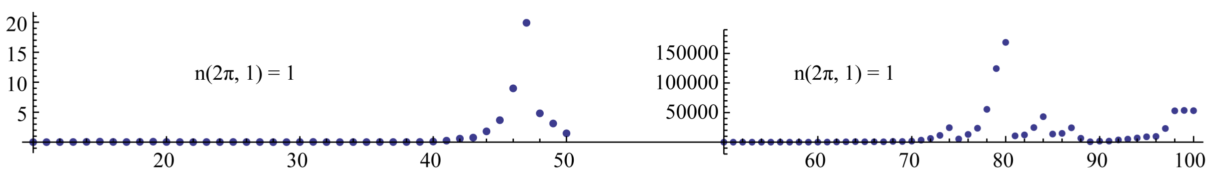

Figure 2. The graph of the sequence $j \mapsto 5^j \times d(2\pi, j)$.

[2]In this article all figures are produced using Mathematica [8].

Table 1. Dynamics of the sequence $m \mapsto \tilde{d}(2\pi, m)$.

m	1	6	7	12	13	19	44	710
$\tilde{d}(2\pi, m)$	1	6	$0.71\cdots$	$5.71\cdots$	$0.43\cdots$	$0.15\cdots$	$0.017\cdots$	$6.02\cdots\times 10^{-5}$

Table 2. Dynamics of the sequences $j \mapsto d(2\pi, j)$ and $j \mapsto n(2\pi, j)$.

j	2	5	10	14	51	74	100
$d(2\pi, j)$	$<10^{-2}$	$<10^{-4}$	$<10^{-9}$	$<10^{-11}$	$<10^{-35}$	$<10^{-48}$	$<10^{-65}$
$n(2\pi, j)$	$>10^{2}$	$>10^{14}$	$>10^{55}$	$>10^{102}$	$>10^{1031}$	$>10^{2013}$	$>10^{3523}$

Then the sequence $j \mapsto T(\zeta, t, j) := \zeta \cdot N(\zeta, t, j) - \lfloor \zeta \cdot N(\zeta, t, j) \rfloor$ *converges towards t as* $j \to \infty$. *Hence, the sequence of fractional parts of products* $\zeta N(\zeta, t, j)$, $j \in \mathbb{N}$, *is dense in the interval* $[0,1]$.

For several $t \in [0,1)$, **Figure 3**, **Figure 4** on page 7 illustrate convergence of the sequence $j \mapsto T(\pi, t, j)$ towards *t*, using $n(1/\pi, 1) = 1$.

Proof. Let us take $\zeta \in \mathbb{R}^+ \setminus \mathbb{Q}$ and $t \in [0,1)$. Now, considering Corollary 2.3 iv), we estimate

$$\frac{t/\zeta + d(1/\zeta, 1) \cdot 2^{1-j}}{d(1/\zeta, j)} \geq 1.$$

Therefore, according to the definition i) of $m(\zeta, t, j)$ and considering the equivalence (1) on page 2, we have $m(\zeta, t, j) \in \mathbb{N}$ and

$$\frac{t/\zeta + d(1/\zeta, 1) \cdot 2^{1-j}}{d(1/\zeta, j)} - 1 < m(\zeta, t, j) \leq \frac{t/\zeta + d(1/\zeta, 1) \cdot 2^{1-j}}{d(1/\zeta, j)}.$$

Consequently, again thanks to Corollary 2.3 iv),

$$t/\zeta < m(\zeta, t, j) \cdot d(1/\zeta, j) \leq t/\zeta + d(1/\zeta, 1) \cdot 2^{1-j}$$

or

$$t < \zeta \cdot m(\zeta, t, j) \cdot d(1/\zeta, j) \leq t + \zeta \cdot d(1/\zeta, 1) \cdot 2^{1-j} \quad (j \in \mathbb{N}). \tag{14}$$

Now, for $t < 1$, using the definition ii) of $N(\zeta, t, j)$ and considering Corollary 2.3 i), we have

$$N(\zeta, t, j) = m(\zeta, t, j)\left(k(1/\zeta, j) \cdot \frac{1}{\zeta} + d(1/\zeta, j)\right)$$

or

$$\zeta N(\zeta, t, j) = m(\zeta, t, j) \cdot k(1/\zeta, j) + \zeta \cdot m(\zeta, t, j) \cdot d(1/\zeta, j). \tag{15}$$

Also, using (14),

$$0 < \zeta \cdot m(\zeta, t, j) \cdot d(1/\zeta, j) < 1$$

holds for

$$j > j^*(\zeta, t) := \frac{1}{\ln 2} \cdot \ln \frac{2\zeta \cdot d(1/\zeta, 1)}{1 - t}.$$

Thus, according to (15), the fractional part of $j > j^*(\zeta, t)$ is equal to $\zeta \cdot m(\zeta, t, j) \cdot d(1/\zeta, j)$, for $j > j^*(\zeta, t)$, and, thanks to (14), converges towards *t* as $j \to \infty$. □

Theorem 3.2. *The closures of the sets* $\{\cos(n) \mid n \in \mathbb{N}\}$ *and* $\{\sin(n) \mid n \in \mathbb{N}\}$ *are equal to the interval* $[-1,1]$. *More constructively, setting* $n(2\pi, 1) = 19$ *in Corollary* 2.3, *and considering the sequences* $(n(2\pi, j))_{j \in \mathbb{N}}$ *and*

$\left(d\left(2\pi, j\right)\right)_{j\in\mathbb{N}}$ *defined in Corollary* 2.3, *let us define, for any* $\alpha\in\mathbb{R}$ *and* $j\in\mathbb{N}$,

1) $\mu(\alpha, j) := \left\lfloor \dfrac{\alpha+2^{-j-1}}{d(2\pi, j)} \right\rfloor$

2) $\nu(\alpha, j) := \mu(\alpha, j)\cdot n(2\pi, j)$.

Then

$$\left|\mathrm{e}^{\mathrm{i}\nu(\alpha,j)}-\mathrm{e}^{\mathrm{i}\alpha}\right| < 2^{-j-1/2} \quad \left(j\in\mathbb{N}\right). \tag{16}$$

The estimate (16) is illustrated on the **Figure 5** where is plotted the graph of the sequence $j\mapsto\left|\mathrm{e}^{\mathrm{i}\nu(\pi/4,j)}-\mathrm{e}^{\mathrm{i}\pi/4}\right|$ together with the graph of continuous function $j\mapsto 2^{-j-1/2}$.

Proof. Assume that all the suppositions of Theorem 3.2 are fulfilled. Then, since $d(2\pi,1)=19-\left\lfloor\dfrac{19}{2\pi}\right\rfloor\cdot 2\pi\in\left(0,\dfrac{1}{4}\right)$ we have, considering Corollary 2.3 iv), the estimate $d(2\pi, j)<\dfrac{1}{4}\cdot 2^{1-j}$, *i.e.*

$$2^{-j-1}-d(2\pi, j)>0, \quad \text{for } j\in\mathbb{N}. \tag{17}$$

Moreover, referring to the definition of $\mu(\alpha, j)$,

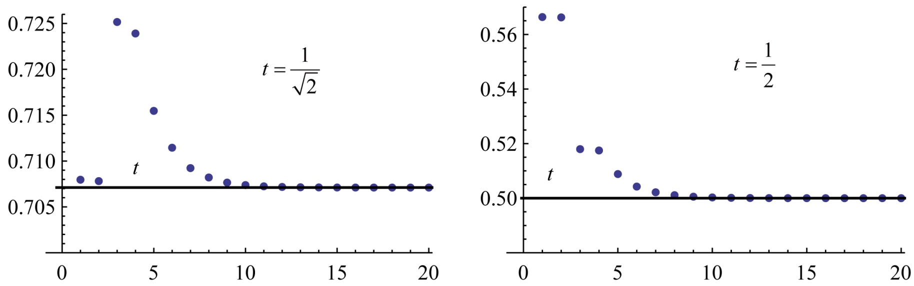

Figure 3. The graphs of the sequences $j\mapsto T(\pi,t,j)$ using $n(1/\pi,1)=1$.

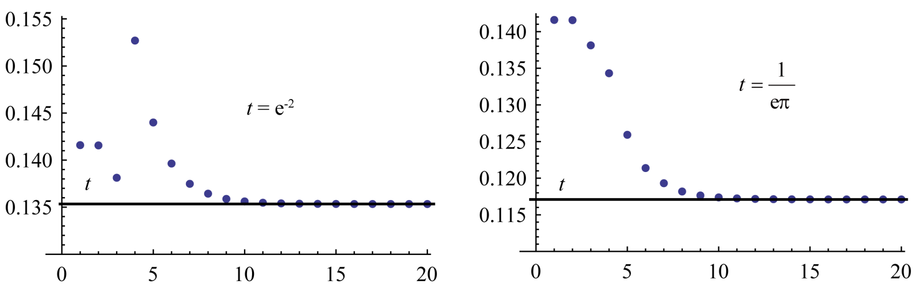

Figure 4. The graphs of the sequences $j\mapsto T(\pi,t,j)$ using $n(1/\pi,1)=1$.

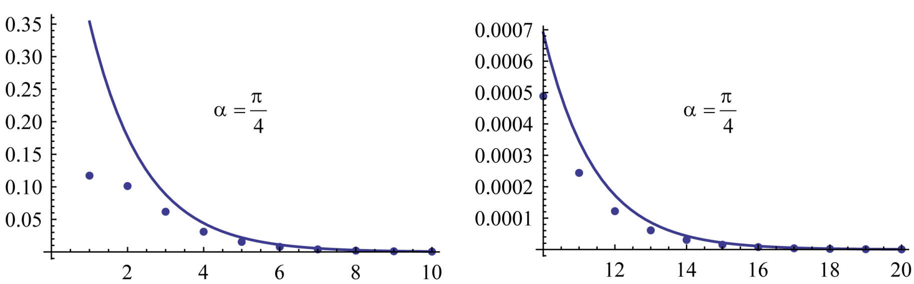

Figure 5. The graph of the sequence $j\mapsto\left|\mathrm{e}^{\mathrm{i}\nu(\pi/4,j)}-\mathrm{e}^{\mathrm{i}\pi/4}\right|$ and the graph of continuous function $j\mapsto 2^{-j-1/2}$.

$$\frac{\alpha+2^{-j-1}}{d(2\pi,j)}-1<\mu(\alpha,j)\le\frac{\alpha+2^{-j-1}}{d(2\pi,j)}.$$

That is, considering (17), we estimate

$$\alpha<\mu(\alpha,j)\cdot d(2\pi,j)\le\alpha+2^{-j-1}. \tag{18}$$

Now, according to Corollary 2.3, we have

$$\nu(\alpha,j)=\mu(\alpha,j)\cdot n(2\pi,j)=\underbrace{\big(\mu(\alpha,j)k(2\pi,j)\big)}_{\in\mathbb{Z}}\cdot 2\pi+\mu(\alpha,j)\cdot d(2\pi,j).$$

Hence,

$$\begin{aligned}\sin\big(\nu(\alpha,j)\big)&\equiv\sin\big(\mu(\alpha,j)\cdot d(2\pi,j)\big),\\ \cos\big(\nu(\alpha,j)\big)&\equiv\cos\big(\mu(\alpha,j)\cdot d(2\pi,j)\big).\end{aligned} \tag{19}$$

To conclude the proof we estimate

$$\begin{aligned}\left|\cos(\alpha+h)-\cos(\alpha)\right|&=\left|\cos(\alpha)\cos(h)-\sin(\alpha)\sin(h)-\cos(\alpha)\right|\\ &=2\left|\sin\left(\alpha+\frac{h}{2}\right)\right|\cdot\left|\sin\left(\frac{h}{2}\right)\right|\le 2\times1\times\left|\frac{h}{2}\right|=|h|,\end{aligned} \tag{20}$$

for $h\in[-\pi/2,\pi/2]$. For such h we also have

$$\begin{aligned}\left|\sin(\alpha+h)-\sin(\alpha)\right|&=\left|\sin(\alpha)\cos(h)+\cos(\alpha)\sin(h)-\sin(\alpha)\right|\\ &=2\left|\cos\left(\alpha+\frac{h}{2}\right)\right|\cdot\left|\sin\left(\frac{h}{2}\right)\right|\le 2\times1\times\left|\frac{h}{2}\right|=|h|.\end{aligned} \tag{21}$$

The relations (18)-(21) imply the inequalities

$$\left|\cos\big(\nu(\alpha,j)\big)-\cos(\alpha)\right|\le2^{-j-1}\quad\text{and}\quad\left|\sin\big(\nu(\alpha,j)\big)-\sin(\alpha)\right|\le2^{-j-1},$$

verifying (16).

4. Conclusions

Using only elementary tools, no use of convergents of continued fraction theory, we derived two main results about the denseness:

1) For any positive irrational ζ and every $t\in[0,1)$ we constructed inductively a sequence of positive integers $N(\zeta,t,j)$ such that the appropriate sequence of fractional parts of products $\zeta N(\zeta,t,j)$ converges towards t.

2) We demonstrated constructively and quantitatively the well known fact that the ranges of cosine and sine are dense in the interval $[-1,1]$; for any real α we constructed inductively the sequence of positive integers $\nu(\alpha,j)$ such that $\max\left\{\left|\cos\big(\nu(\alpha,j)\big)-\cos(\alpha)\right|,\ \left|\sin\big(\nu(\alpha,j)\big)-\sin(\alpha)\right|\right\}\le2^{-j-1}$, for any $j\in\mathbb{N}$.

In [7] is presented very nice approach to the denseness problem which is also constructive. Essential for this paper are two lemmas.

Lemma A. [Lemma 1, p. 402] Let L be any irrational number greater than 1, and suppose that $0<x_1<x_0$, $x_0/x_1=L$ and $x_{n+2}:=x_n-\lfloor x_n/x_{n+1}\rfloor x_{n+1}$ for $n\in\mathbb{N}\cup\{0\}$. Then the sequence $(x_n)_{n\ge0}$ is well defined and

$$0<x_{n+2}<x_n/2$$

for all $n\ge0$.

Lemma B. [Lemma 2, p. 403] For each x_k defined in Lemma A we can find integers m_k and n_k such that $x_k=m_kL+n_k$, with $m_k=(-1)^k|m_k|$ and $n_k=(-1)^{k-1}|n_k|$ for $k\ge2$. As the consequence of these lemmas in [7] is constructively proved the next theorem.

Theorem. [Theorem 3, p. 404] Let $f:\mathbb{R}\to\mathbb{R}$ be continuous function with irrational period. Then, for any point α in the range of *f*, there exists a sequence $(p_k)_{k\in\mathbb{N}}$ of positive integers such that $\lim_{n\to\infty}f(p_k)=\alpha$.

This theorem could be proved and expanded also using our technique.

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