Approximate Analysis of an M/M/1 Markovian Queue Using Unit Step Function

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Abstract

This study analyzes a single server queueing model with a time-dependent arrival rate and service rate is constant. In this model, the incoming arrivals are Poisson stream; service time is exponentially distributed and the first-come first-served queueing discipline. We obtain an explicit expression for the state probability distribution with time-dependent arrival rate using unit step function.

Keywords

Transient Analysis, Unit Step Function, Queueing System

Subject Areas: **Applied Statistical Mathematics, Numerical Mathematics**

1. Introduction

The vast majority of the queueing literature considers only models where the arrivals and service processes are time-homogeneous. Yet it is clear that in many situations it is quite natural to have the arrival rate varying with time. In the past few years a number of interesting literatures that discuss the transient behaviour of the M/M/1 queue, an aspect which is very important for practical applications have been focused by cf. Abate and Whitt [1]-[4]. Transient solutions of queues have been obtained by different methods: generating function method by Bailey [5], spectral method by Lederman and Reuter [6], difference equation method by Conolly [7], alternative approach of generating functions by Parthasarathy [8]. Specific case studies and applications of time-dependent (*i.e.* non-stationary) queues are given in [9] [10]. The relatively small body of literature on time-dependent queues is probably due to the fact that standard methods for treating time-independent models, such as generating functions and Laplace transforms, do not easily generalize to non-stationary models. In most of the literatures it is assumed that the parameters of the system are constant, *i.e.* do not vary over time. But in our model one parameter is time-dependent and the other is time-independent. Our primary objective in this paper is to approximate the analysis of a single server queue, arrival rate varying with time and service rate is constant using

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step function. In this paper, the incoming arrivals are Poisson stream; service time is negative exponentially distributed and the first-come first-served queueing discipline. For a special form of the traffic intensity, we obtain an explicit analytic expression for the probability distribution function with time-dependent arrival rate using unit step function. The rest of the paper is organized as follows: in Section 2, the mathematical description of the considered queueing model is derived; in Section 3, the transient solution to the model is derived.

2. Model Description

We consider a single server queue at which customers arrive to a Poisson stream of rate. The service time is exponentially distributed time-independent rate. Let the system take initially non-empty with FCFS discipline *i.e.*, "*j*" is in the system at $t = 0$. In view of the arbitrary rate functions, there is no loss of generality in assuming that we start the process at $t = 0$, by introduce a new function *i.e.* unit step function define as

$$1-u(t)=\begin{cases}1 & \text{if } t<T\\ 0 & \text{if } t>T\end{cases} \tag{1}$$

This function if useful in expressing boundary conditions which is depends upon time. The basis of the approximate differential equation derived from the time dependent arrival rate and time independent service rate. Let $P_n(t)$ = the probability that there are n customers in the system at time t given that there were j customers in the system at time 0. Therefore,

Probability that [one arrival in the time interval $(t,t+\Delta t)$] = $\lambda\left(1+tT^{-1}\right)\left(1-u_T(t)\right)\Delta t+0(\Delta t)$;

Probability that [one departure in the time interval $(t,t+\Delta t)$] = $\mu\Delta t+0(\Delta t)$.

Transient state equations for the finite source M/M/1 queueing system are given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}p_n(t)&=-\left[\lambda\left(1+tT^{-1}\right)\left(1-u_T(t)\right)+\mu\right]p_n(t)\\&\quad+\left[\lambda\left(1+tT^{-1}\right)\left(1-u_T(t)\right)\right]p_{n-1}(t)+\mu p_{n+1}(t):n>0\end{aligned} \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}p_0(t)=-\left[\lambda\left(1+tT^{-1}\right)\left(1-u_T(t)\right)+\mu\right]p_0(t)+\mu p_1(t):n=0 \tag{3}$$

3. The Model Solution

The solution of our problem of this queueing system of differential-difference equations under the initial conditions

$$P_n(t=0)=\psi_{jn},\ \text{where } \psi \text{ is the Kronecker delta}\left(\psi_{jn}=\begin{cases}1 & \text{if } n=j\\ 0 & \text{if } n\neq j\end{cases}\right) \tag{4}$$

We define,

$$\frac{\mathrm{d}}{\mathrm{d}t}p_n(t)=\mu\frac{\mathrm{d}}{\mathrm{d}\pi}p_n(\pi) \tag{5}$$

$$Q_n(\pi,T)=\mathrm{Exp}\cdot\left\{\pi\left[1+R(\pi,T)\right]\right\}p_n(\pi) \tag{6}$$

$$\rho(\pi,T)=\left[\frac{\lambda\left(1+tT^{-1}\right)\left(1-u_T(t)\right)}{\mu}\right] \tag{7}$$

$$R(\pi,T)=\left\{\frac{\int_0^t\left[\lambda\left(1+tT^{-1}\right)\left(1-u_T(t)\right)\right]\mathrm{d}t}{\mu}\right\} \tag{8}$$

Solving the Equations (2), (3) recursively, using (6)-(8), we obtain

$$p_n'(\pi) = -[1+\rho(\pi,T)]p_n(\pi) + \rho(\pi,T)p_{n-1}(\pi) + p_{n+1}(\pi) \quad (9)$$

$$p_0'(\pi) = -\rho(\pi,T)p_0(\pi) + p_1(\pi) \quad (10)$$

Rewritten and differentiating these equations, we have,

$$\frac{\mathrm{d}}{\mathrm{d}\pi}Q_0(\pi,T) = -[R(\pi,T)] + \rho(\pi,T)\mathrm{Exp}\cdot\{\pi[1+R(\pi,T)]\} \quad (11)$$

$$\frac{\mathrm{d}}{\mathrm{d}\pi}Q_n(\pi,T) = \left[p_n'(\pi,T) + \{1+R(\pi,T)+\pi(-[R(\pi,T)]+\rho(\pi,T))\}p_n(\pi)\right]\mathrm{Exp}\cdot\{\pi[1+R(\pi,T)]\} \quad (12)$$

The initial conditions $[Q_n(t)]_{t=0} = \psi_{jn} : j,n = 0,1,2,\cdots$ exist and satisfy the system of linear equations. In order to reduce this differential-difference Equation (11) (12), we introduce the generating function:

$$Q(z,\pi,T) = \sum_{n=0}^{\infty} Q_n(\pi,T)\frac{(z-\pi)^n}{n!} \quad (13)$$

This function will be analytic in z and continuously differentiable in π. Differentiation with respect to z and π gives, using (11) (12). Restate the differential difference equations in term of $\frac{\mathrm{d}}{\mathrm{d}\pi}Q_n(\pi,T)$, taking summations on both sides with limits $n = 0$ to $n = \infty$ as a partial differential equation-w.r.t. π, we get

$$\frac{\partial}{\partial\pi}Q(z,\pi,T) = -\sum_{n=0}^{\infty} Q_n(\pi,T)\frac{(z-\pi)^{n-1}}{(n-1)!} + \sum_{n=0}^{\infty}\frac{(z-\pi)^n}{n!}\frac{\partial}{\partial\pi}Q_n(\pi,T) \quad (14)$$

$$\frac{\partial}{\partial\pi}Q(z,\pi,T) = Q_0(\pi,T) + \rho(\pi,T)\sum_{n=0}^{\infty} Q_n(\pi,T)\frac{(z-\pi)^{n+1}}{(n+1)!} \quad (15)$$

Again partial differential equation for $Q(z,\pi,T)$ by using (11) (12) w.r.t. z:

$$\frac{\partial^2}{\partial z\partial\pi}Q(z,\pi,T) = \rho(\pi,T)Q(z,\pi,T) \quad (16)$$

These are given by (11) (12) and (14) (15) which transform in the form of

$$\left[\frac{\partial}{\partial\pi}Q(z,\pi,T)\right]_{z=\pi} = Q_0(\pi,T) \quad (17)$$

$$[Q(z,\pi,T)]_{z=\pi} = Q_0(\pi,T) + \left[\sum_{n=1}^{\infty} Q_n(\pi,T)\frac{(z-\pi)^n}{n!}\right]_{z=\pi} \quad (18)$$

Also

$$Q_j(z,0,T) = \frac{z^j}{j!} \quad (19)$$

The method of the solution to be used here, we first solve for $Q(z,\pi,T)$ in terms of the function, we define:

$$f_j(\pi,T) = \frac{\partial}{\partial\pi}Q(0,\pi,T) \quad (20)$$

Taking Laplace transform of (12) and (16) using (20) w.r.t. z and we obtain

$$\frac{\partial}{\partial\pi}Q^*(s,\pi,T) - \frac{\rho(\pi,T)Q^*(s,\pi,T)}{s} = \frac{1}{s}f_j(\pi,T) \quad (21)$$

$$Q^*(s,\pi,T) = \frac{1}{\mathrm{e}^{-\frac{1}{s}[\pi R(\pi,T)]}}\int_0^{\pi}\frac{1}{s}f_j(\vartheta,T)\mathrm{e}^{-\frac{1}{s}[\vartheta R(\vartheta,T)]}\mathrm{d}\vartheta + c \quad (22)$$

$$Q^*(s,\pi,T)=\sum_{n=0}^{\infty}\frac{Q_n(\pi,T)}{n!}\left[c_0^n(-1)^0(\pi)^0\frac{n!}{s^{(n-0+1)}}+\sum_{k=1}^{n}(-1)^k c_k^n(\pi)^k\frac{(n-k)!}{s^{(n-k+1)}}\right] \tag{23}$$

At $t = 0$, then system is in initial state. Therefore

$$Q^*(s,0,T)=\sum_{n=0}^{\infty}\frac{Q_n(0,T)}{n!}\frac{n!}{s^{(n+1)}} \tag{24}$$

The solution of (21)-(23) is

$$Q^*(s,\pi,T)=e^{\frac{1}{s}[\pi R(\pi,T)]}s^{-(j+1)}+\int_0^{\pi}\frac{1}{s}f_j(\vartheta,T)e^{\left[\frac{1}{s}[\pi R(\pi,T)]-\frac{1}{s}[\vartheta R(\vartheta,T)]\right]}\mathrm{d}\vartheta \tag{25}$$

Taking Laplace inverse (25) and using the modified Bessel function, with boundary conditions. Then, we get

$$\begin{aligned}Q_j(z,\pi,T)&=[\pi R(\pi,T)]^{-j/2}z^{j/2}I_j\left[2\{\pi zR(\pi,T)\}^{1/2}\right]\\&+\int_0^{\pi}I_0\left[2\{(\pi R(\pi,T)-\vartheta R(\vartheta,T)z)\}^{1/2}\right]f_j(\vartheta,T)\mathrm{d}\vartheta\end{aligned} \tag{26}$$

$$Q_j(z,\pi,T)=\Upsilon_j(0,z,\pi,T)+\int_0^{\pi}\Upsilon_0(\vartheta,z,\pi,T)f_j(\vartheta,T)\mathrm{d}\vartheta \tag{27}$$

$$\Upsilon_n(\vartheta,z,\pi,T)=[\pi R(\pi,T)-\vartheta R(\vartheta,T)]^{-n/2}z^{n/2}I_n\left[2\{(\pi R(\pi,T)-\vartheta R(\vartheta,T))z\}^{1/2}\right]$$

$$\frac{\partial}{\partial z}\Upsilon_n(\vartheta,z,\pi,T)=\Upsilon_{n-1}(\vartheta,z,\pi,T) \tag{28}$$

Noting that

$$\frac{\partial}{\partial\pi}\Upsilon_n(\vartheta,z,\pi,T)=\frac{\partial}{\partial\pi}\left[[\pi R(\pi,T)-\vartheta R(\vartheta,T)]^{-n/2}z^{n/2}I_n\left[2\{(\pi R(\pi,T)-\vartheta R(\vartheta,T))z\}^{1/2}\right]\right] \tag{29}$$

$$\frac{\partial}{\partial\pi}\Upsilon_n(\vartheta,z,\pi,T)=\rho(\pi,T)\Upsilon_{n+1}(\vartheta,z,\pi,T) \tag{30}$$

Define

$$\Upsilon_n(0,0,0,T)=\psi_{0n} \tag{31}$$

$$\Upsilon_{-n}(\pi,\pi,\pi,T)=0:n>0 \tag{32}$$

$$\Upsilon_0(\pi,\pi,\pi,T)=1 \tag{33}$$

Partial differentiation (27) w.r.t. π and using (18) (29) (30), we get

$$\begin{aligned}\left[\frac{\partial}{\partial\pi}Q_j(z,\pi,T)\right]_{z=\pi}&=\rho(\pi,T)\Upsilon_{j+1}(0,\pi,\pi,T)+\Upsilon_0(\pi,\pi,\pi,T)f_j(\pi,T)\\&+\int_0^{\pi}\rho(\pi,T)\Upsilon_1(\vartheta,\pi,\pi,T)f_j(\vartheta,T)\mathrm{d}\vartheta\end{aligned} \tag{34-35}$$

$$f_j(\pi,T)=Q(\pi,\pi,T)-\rho(\pi,T)\Upsilon_{j+1}(0,\pi,\pi,T)-\int_0^{\pi}\rho(\pi,T)\Upsilon_1(\vartheta,\pi,\pi,T)f_j(\vartheta,T)\mathrm{d}\vartheta \tag{36}$$

Then,

$$p_n(\pi)=e^{-\pi[1+R(\pi,T)]}\left[\Upsilon_{j-n}(0,\pi,\pi,T)+\int_0^{\pi}\Upsilon_{-n}(\vartheta,\pi,\pi,T)f_j(\vartheta,T)\mathrm{d}\vartheta\right] \tag{37}$$

$$f_j(\pi,T)=\varpi_j(0,\pi,\pi,T)+\int_0^{\pi}\varpi_0(\vartheta,\pi,\pi,T)f_j(\vartheta,T)\mathrm{d}\vartheta \tag{38}$$

Here,

$$\varpi_j(\vartheta,\pi,\pi,T)=\Upsilon_j(\vartheta,\pi,\pi,T)-\rho(\pi,T)\Upsilon_{j+1}(\vartheta,\pi,\pi,T)$$

Consequently, Equation (37) gives the approximate solution to the single server queue which arrival rate is time dependent and service rate is constant with step function, provided a solution (38) can be found.

4. Conclusion

In this paper we have obtained the analytic transient solution of M/M/1/N queuing system with a time-dependent arrival rate $\lambda(1+tT^{-1})$ using unit step function and service rate μ is constant. This model finds its application in computer-networks, telecommunications, traffic problems and production systems.

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