

On The Numerical Solution of Two Dimensional Model of an Alloy Solidification Problem

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Abstract

In this paper, a linearized three level difference scheme is derived for two-dimensional model of an alloy solidification problem called Sivashinsky equation. Further, it is proved that the scheme is uniquely solvable and convergent with convergence rate of order two in a discrete L^∞ -norm. At last, numerical experiments are carried out to support the theoretical claims.

Keywords

Solidification Problem, Sivashinsky Equation, Linearized Difference Scheme, Solvability, Convergence

1. Introduction

In the solidification of a dilute binary alloy, a planer solid-liquid interface is often to be instable, spontaneously assuming a cellular structure. This situation enables one to derive an asymptotic nonlinear equation which directly describes the dynamic of the onset and stabilization of cellular structure

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial}{\partial x} \left[(2-u) \frac{\partial u}{\partial x} \right] + \alpha u = 0, \quad (1.1)$$

where α is a positive constant, (see [1] [2]). Equation (1.1) is referred as the Sivashinsky equation.

In this article, we introduce the mathematical model for a finite difference discretization to the solution of the periodical boundary of two-dimensional Sivashinsky equation:

$$u_t + \Delta^2 u + \alpha u = \Delta f(u), \quad (x, y) \in \mathbb{R}^2, \quad 0 < t \leq T, \quad (1.2)$$

with the initial condition

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \mathbb{R}^2, \quad (1.3)$$

subject to the (L_1, L_2) -periodic boundary conditions

$$u(x + L_1, y, t) = u(x, y, t), \quad u(x, y + L_2, t) = u(x, y, t), \quad 0 < t \leq T, \quad (1.4)$$

where $f(u) = \frac{1}{2}u^2 - 2u$, $\alpha > 0$, $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is the Laplacian operator, and $u_0(x, y)$ is a given (L_1, L_2) -periodic smooth function.

Several numerical methods have been proposed in the literature for discretizing Sivashinsky equation. A semi-implicit finite difference scheme and a linearized finite difference method for the Sivashinsky equation in one-dimensional have been proposed respectively in [3] [4]. A semidiscrete approximation of the two dimensional Sivashinsky equation with lumped-mass method and optimal order error bounds for the piecewise linear approximation are derived in [5]. There are many papers that have already been published to study the finite difference method for fourth-order nonlinear equation, for example [5]-[14] and so on.

In this work, we investigate a linearized three level difference scheme for two-dimensional Sivashinsky equations. The remainder of this paper is organized as follows. In Section 2, a linearized difference scheme for (1.2) is derived. The unique solvability of the approximate solutions is shown in Section 3. A second order convergent linearized difference scheme is proved in Section 4. At last section, some numerical examples are presented to improve the theoretical results.

2. Linearized Difference Scheme

To solve the periodic initial-value problems (1.2)-(1.4), one can restrict it on a bounded domain

$\Omega = (0, L_1) \times (0, L_2)$. For a positive integer N , let time-step $\tau = \frac{T}{N}$, $t_n = n\tau$, $0 \leq n \leq N$, and

$t_{n+\frac{1}{2}} = \frac{1}{2}(t_n + t_{n+1})$, $0 \leq n \leq N-1$. We define a partition of $[0, L_1] \times [0, L_2]$ by the rectangles

$[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ with $x_i = ih_1$, $y_j = jh_2$, $i = 0, 1, 2, \dots, M_1 := \left\lceil \frac{L_1}{h} \right\rceil$, $j = 0, 1, 2, \dots, M_2 := \left\lceil \frac{L_2}{h} \right\rceil$, such that

$h_1 = \gamma_1 h$, $h_2 = \gamma_2 h$, $\tau = \gamma_3 h^{\varepsilon + \frac{1}{2}}$, where $\gamma_1, \gamma_2, \gamma_3$ and ε are positive constants. The optimal choice for ε is $\frac{1}{2}$. Denote

$$\Omega_h = \left\{ (x_i, y_j) / 1 \leq i \leq M_1, 1 \leq j \leq M_2 \right\}, \quad \Omega_\tau = \{t_n / 0 \leq n \leq N\}.$$

We define the space of periodic grid functions on Ω_h as:

$$\mathcal{V}_h = \left\{ V = (V_{i,j})_{i,j \in \mathbb{Z}} / V_{i,j} \in \mathbb{R}, V_{i+M_1,j} = V_{i,j}, V_{i,j+M_2} = V_{i,j}, i, j \in \mathbb{Z} \right\}.$$

For $V \in \mathcal{V}_h$, denote

$$\delta_{+x} V_{i,j} = \frac{V_{i+1,j} - V_{i,j}}{h}, \quad \delta_{+y} V_{i,j} = \frac{V_{i,j+1} - V_{i,j}}{h},$$

$$\delta_{-x} V_{i,j} = \frac{V_{i,j} - V_{i-1,j}}{h}, \quad \delta_{-y} V_{i,j} = \frac{V_{i,j} - V_{i,j-1}}{h},$$

$$\delta_x^2 V_{i,j} = \delta_{+x} \delta_{-x} V_{i,j}, \quad \delta_y^2 V_{i,j} = \delta_{+y} \delta_{-y} V_{i,j},$$

$$\Delta_h V_{i,j} = (\delta_x^2 + \delta_y^2) V_{i,j}, \quad \Delta_h^2 V_{i,j} = \Delta(\Delta_h V_{i,j}).$$

Further, define operators $V^{n+\frac{1}{2}}$, $V^{\hat{n}}$ and $\partial_t V^n$, respectively, as

$$V^{n+\frac{1}{2}} = \frac{V^{n+1} + V^n}{2}, \quad V^{\hat{n}} = \frac{3}{2}V^n - \frac{1}{2}V^{n-1}, \quad \partial_t V^n = \frac{V^{n+1} - V^n}{\tau}.$$

For $U \in \mathcal{V}_h$ and $V \in \mathcal{V}_h$ define the inner product

$$(U, V)_h = h^2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} U_{i,j} \cdot V_{i,j},$$

and Sobolev norms (or seminorms)

$$\|V\|_h = (V, V)_h^{\frac{1}{2}}, \quad \|V\|_{\infty, h} = \max_{1 \leq i \leq M_1, 1 \leq j \leq M_2} |V_{i,j}|,$$

$$|V|_{1,h} = \left[h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \left(|\delta_{+y} V_{i,j}|^2 + |\delta_{+x} V_{i,j}|^2 \right) \right]^{\frac{1}{2}}, \quad \|V\|_{2,h} = \left[h_1 h_2 \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} |\Delta_h V_{i,j}|^2 \right]^{\frac{1}{2}}.$$

Define $C_{x,y,t}^{6,6,3}$ as the space of functions $u(x, y, t)$ which are of class C^6 with respect to x, y and class C^3 with respect to t .

It follows from summation by parts that the following Lemma holds [5] [6].

Lemma 1. For $U, V \in \mathcal{V}_h$, we have

$$(\Delta_h V, U)_h = (V, \Delta_h U)_h \quad (2.1)$$

$$-(\Delta_h V, V)_h = |V|_{1,h}^2 \quad (2.2)$$

$$(\Delta_h^2 V, V)_h = |V|_{2,h}^2. \quad (2.3)$$

We discretize problems (1.2)-(1.4) by the following finite difference scheme: we approximate $u^n \in \mathcal{V}_h$, $u_{i,j}^n = u(x_i, y_j, t^n)$, by $U^n \in \mathcal{V}_h$

$$\partial_t U_{i,j}^n + \Delta_h^2 U_{i,j}^{n+\frac{1}{2}} + \alpha U_{i,j}^{n+\frac{1}{2}} = \Delta_h f(U_{i,j}^{\hat{n}}), \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2, 1 \leq n \leq N-1. \quad (2.4)$$

$$U_{i,j}^0 = u_0(x_i, y_j), \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2, \quad (2.5)$$

$$U_{i,j}^1 = u_0(x_i, y_j) + \tau \left[-\Delta^2 u_0(x_i, y_j) - \alpha u_0(x_i, y_j) + \Delta f(u_0(x_i, y_j)) \right], \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2. \quad (2.6)$$

3. Solvability of the Difference Scheme

Next, we will discuss the unique solvability of the difference schemes (2.4)-(2.6).

Theorem 1. *Difference schemes (2.4)-(2.6) have a unique solution.*

Proof. It is obvious that U^0 and U^1 are uniquely determined by the initial conditions (2.5) and (2.6). Now, we suppose that U^0, U^1, \dots, U^n ($0 \leq n \leq N-1$) can be solved uniquely. Consider the homogeneous equation of (2.4) for U^{n+1} :

$$\frac{1}{\tau} U_{i,j}^{n+1} + \frac{1}{2} \Delta_h^2 U_{i,j}^{n+1} + \frac{\alpha}{2} U_{i,j}^{n+1} = 0, \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2. \quad (3.1)$$

Taking the inner product of (3.1) with $U_{i,j}^{n+1}$, it follows from Lemma 1 that

$$\frac{1}{\tau} \|U^{n+1}\|_h^2 + \frac{1}{2} |U^{n+1}|_{2,h}^2 + \frac{\alpha}{2} \|U^{n+1}\|_h^2 = 0.$$

This implies,

$$\|U^{n+1}\|_h^2 = 0.$$

That is, (3.1) has only a trivial solution. Thus, by the induction principle, (2.4) determines U^{n+1} uniquely. This completes the proof.

4. Convergence of the Difference Scheme

For a smooth function u , we have

$$u^{\hat{n}} = \frac{3}{2}u^n - \frac{1}{2}u^{n-1} = u^{n+\frac{1}{2}} + O(\tau^2) \quad \text{as } \tau \rightarrow 0.$$

Therefore, the extrapolation just proposed will give second-order accuracy. To show the convergence of the difference scheme, we need the following Lemmas.

Lemma 2. [15] [16]. *Let b_1, b_2 and $a_i, i=1,2,3,\dots$, be positive and satisfy*

$$a_{i+1} \leq (1+b_1\tau)a_i + b_2\tau, \quad i=1,2,\dots;$$

then

$$a_{i+1} \leq \exp(b_1i\tau) \left(a_1 + \frac{b_2}{b_1} \right).$$

Lemma 3. [17]. *For any grid function v on $\Omega_h = \{(x_i, y_j) / 1 \leq i \leq M_1, 1 \leq j \leq M_2\}$ there is a positive constant c independent h such that*

$$\|v\|_{\infty, h} \leq c \|v\|_h^{\frac{1}{2}} \left(|v|_{2, h} + |v|_h \right)^{\frac{1}{2}}.$$

The main result of this article is the following Theorem.

Theorem 2. *Assume the solution of $u(x, y, t)$ of (1.2)-(1.4) belong to $C_{x,y,t}^{6,6,3}([0, L_1] \times [0, L_2] \times [0, T])$. Then, the solution of difference schemes (2.4)-(2.6) converges to the solution of the problems (1.2)-(1.4) with the convergence order of $O(h_1^2 + h_2^2 + \tau^2)$ in the discrete L^∞ -norm.*

Proof. Define the net function $u_{i,j}^n = u(x_i, y_j, t^n)$, $1 \leq i \leq M_1, 1 \leq j \leq M_2, 0 \leq n \leq N$.

Therefore, From Taylor expansion, we have for $1 \leq i \leq M_1, 1 \leq j \leq M_2$,

$$\partial_t u_{i,j}^n + \Delta_h^2 u_{i,j}^{n+\frac{1}{2}} + \alpha u_{i,j}^{n+\frac{1}{2}} = \Delta_h f(u_{i,j}^{\hat{n}}) + F_{i,j}^n, \quad 1 \leq n \leq N-1. \quad (4.1)$$

$$u_{i,j}^0 = u_0(x_i, y_j), \quad (4.2)$$

$$u_{i,j}^1 = u_0(x_i, y_j) + \tau \left[-\Delta^2 u_0(x_i, y_j) - \alpha u_0(x_i, y_j) + \Delta f(u_0(x_i, y_j)) \right] + G_{i,j}, \quad (4.3)$$

where $F_{i,j}^n$ and $G_{i,j}$ are truncation errors of difference schemes (2.4)-(2.6) and there exists a constant c_1 such that

$$|F_{i,j}^n| \leq c_1 (h_1^2 + h_2^2 + \tau^2), \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2, 1 \leq n \leq N-1. \quad (4.4)$$

$$|G_{i,j}| = \tau^2 \left| \int_0^1 u_{ii}(x_i, y_j, s\tau)(1-s) ds \right| \leq c_1 \tau^2, \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2. \quad (4.5)$$

Let $E_{i,j}^n = u_{i,j}^n - U_{i,j}^n$ and subtracting (2.4)-(2.6) from (4.1)-(4.3), we obtain

$$\partial_t E_{i,j}^n + \Delta_h^2 E_{i,j}^{n+\frac{1}{2}} + \alpha E_{i,j}^{n+\frac{1}{2}} = \Delta_h f(u_{i,j}^{\hat{n}}) - \Delta_h f(U_{i,j}^{\hat{n}}) + F_{i,j}^n, \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2, 1 \leq n \leq N-1 \quad (4.6)$$

$$E_{i,j}^0 = 0, \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2, \quad (4.7)$$

$$E_{i,j}^1 = G_{i,j}, \quad 1 \leq i \leq M_1, 1 \leq j \leq M_2. \quad (4.8)$$

We prove by inductive method that

$$\|E^n\|_h \leq c_2 (h_1^2 + h_2^2 + \tau^2), \quad 0 \leq n \leq N. \quad (4.9)$$

From (4.5) and (4.7)-(4.8), we have

$$\|E^0\|_h = 0, \quad \|E^1\|_h \leq c_1 (h_1^2 + h_2^2 + \tau^2). \quad (4.10)$$

It follows from (4.10) that (4.9) is valid for $n=0$ and $n=1$. Now suppose that (4.9) is true for n from 0 to l ($1 \leq l \leq N-1$). Therefore, for h sufficiently small

$$|E_{i,j}^n| \leq c_2 (h_1^2 + h_2^2 + \tau^2) / \sqrt{h_1 h_2} \leq 1, \quad 1 \leq i \leq M_1, \quad 1 \leq j \leq M_2, \quad 1 \leq n \leq l. \quad (4.11)$$

Thus,

$$|U_{i,j}^n| = |u_{i,j}^n - E_{i,j}^n| \leq |u_{i,j}^n| + |E_{i,j}^n| \leq s+1, \quad 1 \leq i \leq M_1, \quad 1 \leq j \leq M_2, \quad 1 \leq n \leq l, \quad (4.12)$$

where

$$s = \max_{0 \leq x \leq L_1, 0 \leq y \leq L_2, 0 \leq t \leq T} |u(x, y, t)|.$$

For $1 \leq n \leq l$, taking in (4.6) the inner product with $E_{i,j}^{n+\frac{1}{2}}$

$$\left(\partial_t E^n, E^{n+\frac{1}{2}} \right)_h + \left| E^{n+\frac{1}{2}} \right|_{2,h}^2 + \alpha \left| E^{n+\frac{1}{2}} \right|_h^2 = \left(f(u^{\hat{n}}) - f(U^{\hat{n}}), \Delta_h E^{n+\frac{1}{2}} \right)_h + \left(F^n, E^{n+\frac{1}{2}} \right)_h. \quad (4.13)$$

Noting that from the Lipschitz condition of f

$$\left| f(u_{i,j}^{\hat{n}}) - f(U_{i,j}^{\hat{n}}) \right| \leq c_3 |E_{i,j}^{\hat{n}}|, \quad 1 \leq i \leq M_1, \quad 1 \leq j \leq M_2, \quad (4.14)$$

where

$$c_3 = \max_{-(s+1) \leq z \leq s+1} \left| \frac{df}{dz}(z) \right|.$$

For $\alpha > 0$, it follows from (4.13) and (4.14) that

$$\frac{1}{2k} \left(\|E^{n+1}\|_h^2 - \|E^n\|_h^2 \right) + \left| E^{n+\frac{1}{2}} \right|_{2,h}^2 \leq \frac{c_3^2}{4} \|E^{\hat{n}}\|_h^2 + \left| E^{n+\frac{1}{2}} \right|_{2,h}^2 + \frac{1}{2} \|F^n\|_h^2 + \frac{1}{2} \left\| E^{n+\frac{1}{2}} \right\|_h^2.$$

Using (4.4), we get

$$\frac{1}{2\tau} \left(\|E^{n+1}\|_h^2 - \|E^n\|_h^2 \right) \leq c_4 \left[\|E^{n-1}\|_h^2 + \|E^n\|_h^2 + \|E^{n+1}\|_h^2 \right] + c_1^2 (h_1^2 + h_2^2 + \tau^2)^2.$$

This yields

$$(1 - 2c_4\tau) \|E^{n+1}\|_h^2 \leq (1 + 2c_4\tau) \|E^n\|_h^2 + 2c_4\tau \|E^{n-1}\|_h^2 + 2c_1^2\tau (h_1^2 + h_2^2 + \tau^2)^2.$$

Therefore, when $\tau \leq \frac{1}{6c_4}$

$$\|E^{n+1}\|_h^2 \leq (1 + 6c_4\tau) \|E^n\|_h^2 + 3c_4\tau \|E^{n-1}\|_h^2 + 3c_1^2\tau (h_1^2 + h_2^2 + \tau^2)^2.$$

It follows easily from this inequality that

$$\max \left(\|E^{n+1}\|_h^2, \|E^n\|_h^2 \right) \leq (1 + 9c_4\tau) \max \left(\|E^n\|_h^2, \|E^{n-1}\|_h^2 \right) + 3c_1^2\tau (h_1^2 + h_2^2 + \tau^2)^2.$$

Applying Lemma 2, we obtain

$$\max \left(\|E^{l+1}\|_h^2, \|E^l\|_h^2 \right) \leq \exp(9c_4\tau l) \left[\max \left(\|E^l\|_h^2, \|E^0\|_h^2 \right) + \frac{c_1^2}{3c_4} (h_1^2 + h_2^2 + \tau^2)^2 \right].$$

Using (4.10), we get

$$\max\left(\|E^{l+1}\|_h^2, \|E^l\|_h^2\right) \leq \exp(9c_4 T) \left(1 + \frac{1}{3c_4}\right) c_1^2 (h_1^2 + h_2^2 + \tau^2)^2,$$

and hence,

$$\|E^{l+1}\|_h \leq c_5 (h_1^2 + h_2^2 + \tau^2),$$

where c_5 is constant dependent on T, c_1 and c_4 . That means, by the induction principle (4.9) is true.

Second, we will prove that

$$|E^n|_{2,h} \leq c_6 (h_1^2 + h_2^2 + \tau^2), \quad 0 \leq n \leq N. \quad (4.15)$$

From (4.7), we find

$$|E^0|_{2,h} = 0. \quad (4.16)$$

Using (4.5), we obtain

$$\Delta_{xh} G_{i,j} = \tau^2 \int_0^1 \frac{u_{tt}(x_{i+1}, y_j, s\tau) - 2u_{tt}(x_i, y_j, s\tau) + u_{tt}(x_{i-1}, y_j, s\tau)}{h_1^2} (1-s) ds.$$

This implies that

$$\Delta_{xh} G_{i,j} = \tau^2 \int_0^1 u_{xxx}(\chi_i, y_j, s\tau) (1-s) ds,$$

where $\chi \in (x_i, x_{i+1})$. Thus, we get $\|\Delta_{xh} G\|_h \leq C\tau^2$. Similarly we find, $\|\Delta_{yh} G\|_h \leq C\tau^2$. Then

$$|E^1|_{2,h} \leq c_7 (h_1^2 + h_2^2 + \tau^2). \quad (4.17)$$

Taking now in (4.6) the inner product with $\partial_t E^n$, we obtain for $n = 1, 2, \dots, N$

$$\|\partial_t E^n\|_h^2 + \frac{1}{2} \partial_t (|E^n|_{2,h}^2) = -\alpha \left(E^{n+\frac{1}{2}}, \partial_t E^n \right)_h + \left(\Delta_h f(u^{\hat{n}}) - \Delta_h f(U^{\hat{n}}), \partial_t E^n \right)_h + (F^n, \partial_t E^n)_h.$$

Using the differentiability of f and the Cauchy Schwartz inequality, we obtain

$$\|\partial_t E^n\|_h^2 + \frac{1}{2} \partial_t (|E^n|_{2,h}^2) \leq \frac{\alpha^2}{2} \|E^{n+\frac{1}{2}}\|_h^2 + \frac{1}{2} \|\partial_t E^n\|_h^2 + c_3^2 \|E^{\hat{n}}\|_h^2 + \frac{1}{4} \|\partial_t E^n\|_h^2 + \|F^n\|_h^2 + \frac{1}{4} \|\partial_t E^n\|_h^2.$$

This yields by (4.4)

$$\partial_t (|E^n|_{2,h}^2) \leq c_8 \left[\|E^{n-1}\|_h^2 + \|E^n\|_h^2 + \|E^{n+1}\|_h^2 + (h_1^2 + h_2^2 + \tau^2)^2 \right].$$

It follows from (4.9) that

$$\frac{1}{\tau} (|E^{n+1}|_{2,h}^2 - |E^n|_{2,h}^2) \leq c_9 (h_1^2 + h_2^2 + \tau^2)^2.$$

Here, by above,

$$|E^n|_{2,h}^2 \leq c_{10} n \tau (h_1^2 + h_2^2 + \tau^2)^2,$$

and hence,

$$|E^n|_{2,h} \leq C(u, T) (h_1^2 + h_2^2 + \tau^2), \quad 1 \leq n \leq N. \quad (4.18)$$

Applying Lemma 3, (4.9) and (4.16)-(4.18), we obtain

$$|E^n|_{\infty,h} \leq C(h_1^2 + h_2^2 + \tau^2).$$

This completes the proof.

5. Numerical Experiments

In this section, we give some numerical experiments to verify our theoretical results that are given in the previous sections. For that purpose, we consider the following periodic inhomogeneous Sivashinsky equation

$$u_t + \Delta^2 u + 2u + \Delta \left(2u - \frac{1}{2} u^2 \right) = g(x, y, t), \quad (x, y) \in (0, 2\pi) \times (0, 2\pi), \quad t \in [0, 1], \quad (5.1)$$

with the initial condition

$$u(x, y, 0) = \cos(x + y), \quad (x, y) \in [0, 2\pi] \times [0, 2\pi], \quad (5.2)$$

where

$$g(x, y, t) = 4 \cos^2(x + y + t) - \sin(x + y + t) - 2 \sin^2(x + y + t).$$

For which the exact solution is $u(x, y, t) = \cos(x + y + t)$.

In the runs, we use the same spacing h in each direction, $h_1 = h_2 = h$, and compute the maximum norm errors of the numerical solution

$$e_\infty(h, \tau) = \max_{0 \leq n \leq N} \|u^n - U^n\|_\infty.$$

The convergence order in spatial direction is defined as

$$rate_1 = \log_2 \left(\frac{e_\infty(2h, \tau)}{e_\infty(h, \tau)} \right),$$

when τ is sufficiently small. The convergence order in temporal direction is defined as

$$rate_2 = \log_2 \left(\frac{e_\infty(h, 2\tau)}{e_\infty(h, \tau)} \right),$$

when h is sufficiently small. We also define the rate of convergence

$$rate_3 = \log_2 \left(\frac{e_\infty(2h, 2\tau)}{e_\infty(h, \tau)} \right),$$

when both h and τ are sufficiently small.

By computing the problems (5.1)-(5.2) with the difference schemes (2.4)-(2.6), we carry out the spatial and temporal convergence in the sense of the maximum norm. **Table 1** and **Table 2** give the errors between numerical solutions and exact solutions for spatial and temporal convergence, respectively. Once again, we conclude from **Tables 1-3**, that the difference schemes (2.4)-(2.6) are convergent with the convergence order of two both in space and in time. This is in accordance with Theorem 2.

6. Conclusion

In this paper, we use the discrete energy method to study the convergence of a linearized difference scheme for solving the two-dimensional Sivashinsky equation. The convergence is proved to be second order in the maxi-

Table 1. The spatial convergence orders in maximum norm for difference schemes (2.1)-(2.3) to the inhomogeneous Sivashinsky Equations (5.1) and (5.2), with $\tau = 0.0025$.

h	$e_\infty(h, \tau)$	$rate_1$
0.1	5.832E-3	*
0.05	1.454E-3	2.003
0.025	3.462E-4	1.989
0.0125	8.445E-5	1.997

Table 2. The temporal convergence orders in maximum norm for difference schemes (2.1)-(2.3) to the inhomogeneous Sivashinsky Equations (5.1) and (5.2), with $h = 0.0025$.

h	$e_{\infty}(h, \tau)$	rate ₂
0.1	8.745E-4	*
0.05	1.913E-4	2.187
0.025	4.115E-5	2.274
0.0125	9.570E-6	2.042

Table 3. The maximum norm errors and convergence orders for difference schemes (2.1)-(2.3) to the inhomogeneous Sivashinsky Equations (5.1) and (5.2).

h	τ	$e_{\infty}(h, \tau)$	rate ₃
0.1	0.1	7.452E-3	*
0.05	0.05	1.733E-3	2.216
0.025	0.025	4.266E-4	2.012
0.0125	0.0125	1.043E-4	2.003

imum norm, which extends the result in [3] [4] where they only prove the second order convergence of the difference scheme for one-dimensional Sivashinsky equation in the discrete L^2 -norm. For obtaining the approximate solution for the two dimensional Sivashinsky equation by finite element Galerkin method, one must need polynomials of the degree ≥ 3 . It means that they have to construct minimum 10 node triangle for approximating the solution. Computationally, it is very expensive and difficult to impose inter-element C^1 continuity condition. If the boundary is curved, imposition of boundary conditions causes some more difficulties. Therefore, based on the linearized difference schemes (2.4)-(2.6), this article proposes a recipe to eradicate such numerical difficulties.

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