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Optimal Investment Problem with Multiple Risky Assets under the Constant Elasticity of Variance (CEV) Model

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ABSTRACT

This paper studies the optimal investment problem for utility maximization with multiple risky assets under the constant elasticity of variance (CEV) model. By applying stochastic optimal control approach and variable change technique, we derive explicit optimal strategy for an investor with logarithmic utility function. Finally, we analyze the properties of the optimal strategy and present a numerical example.

Keywords: Constant Elasticity of Variance Model; Stochastic Optimal Control; Hamilton-Jacobi-Bellman Equation; Portfolio Selection; Multiple Risky Assets; Stochastic Volatility

1. Introduction

Optimal investment problem of utility maximization is a fundamental problem in mathematical finance and has been studied in many articles. This problem is usually studied via two approaches in literatures. One is stochastic control approach used by Merton [1,2] for the first time. By this approach, Browne [3] found the optimal investment strategy to maximize the expected exponential utility of terminal wealth for an insurance company. Yang and Zhang [4] studied a similar problem for an insurer with exponential utility via stochastic control approach. Another method is the martingale approach which was adapted to the problem of utility maximization by Pliska [5], Karatzas, Lehoczky and Shreve [6] and Cox and Huang [7]. Much of this development appeared in [8,9]. Applying the martingale approach, Karatzas *et al*. [10] investigated the utility maximization problem in an incomplete market and Zhang [11] considered a similar problem. In [12], closed-form strategies were obtained for different utilities maximization of an insurer through martingale approach. Zhou [13] applied the martingale approach to study the exponential utility maximization for an insurer in the Lévy market.

The above mentioned researches using the martingale method have provided results for general risky assets' prices, but most found specific solutions for geometric Brownian motion (GBM) model or a similar one merely. Meanwhile the works applying stochastic control theory generally supposed the risky assets' prices satisfy geometric Brownian motions. However, numerous studies (see e.g., [14] and the references therein) have shown that empirical evidence does not support the assumptions of GBM model and a model with stochastic volatility will be more practical.

The constant elasticity of variance (CEV) model with stochastic volatility is a natural extension of geometric Brownian motion and can explain the empirical bias exhibited by the GBM model, such as volatility smile. The CEV model allows the volatility to change with the underlying price and was first proposed by Cox and Ross [15]. In comparison with other stochastic volatility models, the CEV model is easier to deal with analytically and the GBM model can be seen as its special case. The CEV model was usually applied for option pricing and sensitivity analysis of options in most literatures, see [16-19] for example. Recently, the CEV model has been applied in the research of optimal investment, as was done by Xiao, Zhai and Qin [20]. Gao [21,22] investigated the utility maximization problem for a participant in a defined-contribution pension plan under the CEV model. Gu, Yang, Li and Zhang [23] used the CEV model for studying the optimal investment and reinsurance problems.

However, the above researches of optimization problem under the CEV model concerned only one risky asset and a risk-free asset. But actually, an investor needs to invest in multiple risky assets to disperse risk and increase his/her profit. Thus, to make the optimization problem even more realistic, we deal with the investment problem with a risk-free asset and multiple risky assets under the CEV model. Although Zhao and Rong [24] have studied portfolio selection problem with multiple risky assets under the CEV model, they obtained closed-form solutions only for special model parameters. Whereas in this paper, considering to maximize the expected logarithmic utility of an investor's terminal wealth, we derive optimal strategy explicitly for all values of the elasticity coefficient. By applying the methods of stochastic optimal control, we derive a complicated nonlinear partial differential equation (PDE). However, there are terms that contain variables concerning different assets' prices, which makes it difficult to characterize the solution structure. Therefore, we conjecture a corresponding solution to this PDE via separating variables partially and simplify it into several PDEs. The coefficient variables of these simplified PDEs are closely correlated and therefore we use a power transformation and a variable change technique to solve them.

It is noteworthy that the introduction of multiple risky assets does give rise to difficulties and this research is not a routine extension of the case of one risky asset. For portfolio selection problems concerning risky assets with the CEV price processes, the characterization of solution under n dimensional case is quite different from one dimensional case. Owing to the consideration of multiple risky assets, we conjecture the solution through separating variables represented different assets' prices and combining each price variable with time variable respectively.

Furthermore, we compare our result with that under the GBM model and that of one dimensional case. Firstly, the optimal policy for an investor with logarithmic utility under the CEV model is similar to that under the GBM model in form except for a stochastic volatility. Secondly, our solution is just the result of [20] when there is only one risky asset. Moreover, we present a numerical simulation to analyze the properties of the optimal strategy under the CEV model.

This paper proceeds as follows. Section 2 proposes the utility maximization problem with multiple risky assets whose prices are driven by the CEV models and provides the general framework to solve the optimization problem. In Section 3, we derive the optimal investment strategy for logarithmic utility function and compare our result with the previous works. Section 4 provides a numerical analysis to illustrate our results. Section 5 concludes the paper.

2. Problem Formulation

We consider a financial market consisting of a risk-free asset (hereinafter called "bond") and n risky assets (hereinafter called "stocks"). The price process $\{S_0(t), t \geq 0\}$ of the bond follows

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t, \quad S_0(0) = 1, \tag{1}$$

where r is the interest rate. The price processes $\{S_1(t), t \geq 0\}, \cdots, \{S_n(t), t \geq 0\}$ of the n stocks are described by the CEV model

$$\mathrm{d}S_i(t) = S_i(t)\left(\mu_i \mathrm{d}t + \sum_{j=1}^{d} \sigma_{ij}\left(S_i(t)\right)^{\beta} \mathrm{d}W_j(t)\right), \quad i = 1, 2, \cdots, n, \tag{2}$$

where $W := (W_1, \cdots, W_d)^{\mathrm{T}}$ is a d-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathrm{P})$ and $d \geq n$. $(\mathcal{F}_t)$ is an augmented filtration generated by the Brownian motion with $\mathcal{F} = \mathcal{F}_T$, where T is a fixed and finite time horizon. μ_i is the appreciation rate of the i th stock and $\mu := (\mu_1, \cdots, \mu_n)^{\mathrm{T}}$. Define $\sigma = \{\sigma_{ij}\}_{n \times d}$ and

$$S^{\beta}(t) = \begin{pmatrix} (S_1(t))^{\beta} & 0 & \cdots & 0 \\ 0 & (S_2(t))^{\beta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (S_n(t))^{\beta} \end{pmatrix},$$

then $S^{\beta}(t) \cdot \sigma$ is the instantaneous volatility matrix. The elasticity parameter β satisfies $\beta \leq 0$. If $\beta = 0$, the volatility matrix is constant with respect to the stock prices and Equation (2) reduces to the standard Black-Scholes model. In addition, we assume that $\sigma\sigma^{\mathrm{T}}$ is positive definite throughout this paper.

The investor is allowed to invest in those n stocks as well as in the bond. Let $\pi_i(t)$ be the money amount invested in the i th stock at time t for $i = 1, 2, \cdots, n$. Denote by $\pi(t) := (\pi_1(t), \cdots, \pi_n(t))^{\mathrm{T}}$ and each $(\pi_i(t))$ is an $(\mathcal{F}_t)$ -predictable process for $i = 1, 2, \cdots, n$. Corresponding to a trading strategy $\{\pi(t), t \geq 0\}$ and an initial capital V, the investor's wealth process $\{X(t), t \geq 0\}$ follows the dynamics

$$\begin{aligned} \mathrm{d}X(t) &= \left[rX(t) + \pi^{\mathrm{T}}(t)(\mu - r\mathbf{1}_n)\right]\mathrm{d}t \\ &\quad + \pi^{\mathrm{T}}(t)S^{\beta}(t)\sigma \mathrm{d}W(t) \\ X(0) &= V, \end{aligned} \tag{3}$$

where $\mathbf{1}_n = (1, \cdots, 1)^{\mathrm{T}}$ is an $n \times 1$ vector.

Suppose that the investor has a utility function U which is strictly concave and continuously differentiable on $(-\infty, \infty)$. Then the investor aims to

$$\max_{\{\pi(t)\}} \mathrm{E}\left[U(X(T))\right]. \tag{4}$$

By applying the classical tools of stochastic optimal control, we define the value function as

$$\begin{aligned} &H(t, s_1, s_2, \cdots, s_n, x) \\ &= \sup_{\{\pi(t)\}} \mathrm{E}\left\{U(X(T)) \middle| S_1(t) = s_1\right. \\ &\left. S_2(t) = s_2, \cdots, S_n(t) = s_n, X(t) = x\right\}, \quad 0 < t < T \end{aligned} \tag{5}$$

with $H(T,s_1,s_2,\cdots,s_n,x)=U(x)$.

The Hamilton-Jacobi-Bellman (HJB) equation associated with the portfolio selection problem under the CEV model is

$$\begin{aligned}&H_t+\mu^{\mathrm{T}}SH_s+rxH_x\\&+\frac{1}{2}\sum_{i=1}^{n}I_i^{\mathrm{T}}\left[S^{(\beta+1)}\sigma\sigma^{\mathrm{T}}S^{(\beta+1)}H_{ss}\right]I_i\\&+\sup_{\pi}\left\{\pi^{\mathrm{T}}(\mu-r\mathbf{1}_n)H_x+\pi^{\mathrm{T}}S^{\beta}\sigma\sigma^{\mathrm{T}}S^{(\beta+1)}H_{xs}\right.\\&\left.+\frac{1}{2}\pi^{\mathrm{T}}S^{\beta}\sigma\sigma^{\mathrm{T}}S^{\beta}\pi H_{xx}\right\}=0,\end{aligned}\tag{6}$$

where

$$S:=\begin{pmatrix}s_1&0&\cdots&0\\0&s_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&s_n\end{pmatrix},$$

$$S^{\beta}:=\begin{pmatrix}s_1^{\beta}&0&\cdots&0\\0&s_2^{\beta}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&s_n^{\beta}\end{pmatrix},$$

$$H_s:=\left(H_{s_1},\cdots,H_{s_n}\right)^{\mathrm{T}},\ H_{xs}:=\left(H_{xs_1},\cdots,H_{xs_n}\right)^{\mathrm{T}}$$

and

$$H_{ss}:=\begin{pmatrix}H_{s_1s_1}&\cdots&H_{s_ns_1}\\\vdots&\ddots&\vdots\\H_{s_1s_n}&\cdots&H_{s_ns_n}\end{pmatrix}.$$

Besides, we define $I_i:=(0,\cdots,1,\cdots,0)^{\mathrm{T}},i=1,\cdots,n$, whose i th component is 1. Differentiating with respect to π in Equation (6) gives the optimal policy

$$\pi^*=-\left(S^{\beta}\sigma\sigma^{\mathrm{T}}S^{\beta}\right)^{-1}(\mu-r\mathbf{1}_n)\cdot\frac{H_x}{H_{xx}}-SH_{xs}\cdot\frac{1}{H_{xx}}.\tag{7}$$

Putting Equation (7) into HJB Equation (6), after simplification, we have

$$\begin{aligned}&H_t+\mu^{\mathrm{T}}SH_s+rxH_x\\&+\frac{1}{2}\sum_{i=1}^{n}I_i^{\mathrm{T}}\left[S^{(\beta+1)}\sigma\sigma^{\mathrm{T}}S^{(\beta+1)}H_{ss}\right]I_i\\&-(\mu-r\mathbf{1}_n)^{\mathrm{T}}SH_{xs}\cdot\frac{H_x}{H_{xx}}\\&-\frac{1}{2}(\mu-r\mathbf{1}_n)^{\mathrm{T}}\left(S^{\beta}\sigma\sigma^{\mathrm{T}}S^{\beta}\right)^{-1}(\mu-r\mathbf{1}_n)\cdot\frac{H_x^2}{H_{xx}}\\&-\frac{1}{2}H_{xs}^{\mathrm{T}}S^{(\beta+1)}\sigma\sigma^{\mathrm{T}}S^{(\beta+1)}H_{xs}\cdot\frac{1}{H_{xx}}=0.\end{aligned}\tag{8}$$

The problem now is to solve the nonlinear partial differential equation (PDE) (8) for H and recover π^* from derivatives of H.

3. Optimal Strategy for the Logarithmic Utility

In this paper, we consider the investment problem for logarithmic utility function

$$U(x)=\ln x.\tag{9}$$

A solution to Equation (8) is conjectured in the following form:

$$\begin{aligned}&H(t,s_1,\cdots,s_n,x)\\&=\ln x\sum_{i=1}^{n}g^{(i)}(t,s_i)+\sum_{i=1}^{n}d^{(i)}(t,s_i)\end{aligned}\tag{10}$$

with the boundary conditions given by

$$\sum_{i=1}^{n}d^{(i)}(T,s_i)=0,\sum_{i=1}^{n}g^{(i)}(T,s_i)=1.$$

Then

$$\begin{aligned}&H_t=\ln x\sum_{i=1}^{n}g_t^{(i)}+\sum_{i=1}^{n}d_t^{(i)},H_{s_i}=g_{s_i}^{(i)}\ln x+d_{s_i}^{(i)},\\&H_{s_is_i}=g_{s_is_i}^{(i)}\ln x+d_{s_is_i}^{(i)},H_{s_is_j}=0,\\&H_x=\frac{1}{x}\sum_{i=1}^{n}g^{(i)},\\&H_{xx}=-\frac{1}{x^2}\sum_{i=1}^{n}g^{(i)},H_{xs_i}=\frac{g_{s_i}^{(i)}}{x},\end{aligned}$$

where $i\neq j$ and $i,j=1,2,\cdots,n$. Plugging these derivatives into Equation (8) gives

$$\begin{aligned}&\left(\sum_{i=1}^{n}g_t^{(i)}+\sum_{i=1}^{n}\mu_is_ig_{s_i}^{(i)}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{d}\sigma_{ij}^2s_i^{2\beta+2}g_{s_is_i}^{(i)}\right)\ln x\\&+\sum_{i=1}^{n}d_t^{(i)}+\sum_{i=1}^{n}\mu_is_id_{s_i}^{(i)}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{d}\sigma_{ij}^2s_i^{2\beta+2}d_{s_is_i}^{(i)}\\&+r\sum_{i=1}^{n}g^{(i)}+\sum_{i=1}^{n}(\mu_i-r)s_ig_{s_i}^{(i)}\\&+\frac{1}{2|\sigma\sigma^{\mathrm{T}}|}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mu_i-r)(\mu_j-r)\Psi_{ij}s_i^{-\beta}s_j^{-\beta}\sum_{k=1}^{n}g^{(k)}\\&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{d}\sigma_{ik}\sigma_{jk}s_i^{\beta+1}s_j^{\beta+1}\frac{g_{s_i}^{(i)}g_{s_j}^{(j)}}{\sum_{i=1}^{n}g^{(i)}}=0,\end{aligned}\tag{11}$$

here Ψ denotes the adjoint matrix of $\sigma\sigma^{\mathrm{T}}$, Ψ_{ij} is the element of Ψ in the i th row and j th column and $\left|\sigma\sigma^{\mathrm{T}}\right|$ represents the determinant of the matrix $\sigma\sigma^{\mathrm{T}}$, namely, $\left(\sigma\sigma^{\mathrm{T}}\right)^{-1}=\frac{1}{\left|\sigma\sigma^{\mathrm{T}}\right|}\cdot\Psi$.

In order to eliminate the dependence on x, we can split Equation (11) into two equations:

$$\sum_{i=1}^{n} g_t^{(i)} + \sum_{i=1}^{n} \mu_i s_i g_{s_i}^{(i)} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{d} \sigma_{ij}^2 s_i^{2\beta+2} g_{s_i s_i}^{(i)} = 0, \tag{12}$$

$$\begin{aligned} &\sum_{i=1}^{n} d_t^{(i)} + \sum_{i=1}^{n} \mu_i s_i d_{s_i}^{(i)} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{d} \sigma_{ij}^2 s_i^{2\beta+2} d_{s_i s_i}^{(i)} \\ &+ r \sum_{i=1}^{n} g^{(i)} + \sum_{i=1}^{n} (\mu_i - r) s_i g_{s_i}^{(i)} \\ &+ \frac{1}{2|\sigma\sigma^{\mathrm{T}}|} \sum_{i=1}^{n} \sum_{j=1}^{n} (\mu_i - r) \\ &\cdot (\mu_j - r) \Psi_{ij} s_i^{-\beta} s_j^{-\beta} \sum_{k=1}^{n} g^{(k)} \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{d} \sigma_{ik} \sigma_{jk} s_i^{\beta+1} s_j^{\beta+1} \frac{g_{s_i}^{(i)} g_{s_j}^{(j)}}{\sum_{i=1}^{n} g^{(i)}} = 0. \end{aligned} \tag{13}$$

For Equation (12), we use a power transform and a variable change technique proposed by Cox [16] to solve it. Let

$$g^{(i)}(t, s_i) = m^{(i)}(t, y_i) \quad \text{and} \quad y_i = s_i^{-2\beta} \tag{14}$$

with the boundary condition $\sum_{i=1}^{n} m^{(i)}(T, y_i) = 1$.

Hence,

$$g_t^{(i)} = m_t^{(i)}, g_{s_i}^{(i)} = -2\beta s_i^{-2\beta-1} m_{y_i}^{(i)},$$

$$g_{s_i s_i}^{(i)} = 2\beta(2\beta+1) s_i^{-2\beta-2} m_{y_i}^{(i)} + 4\beta^2 s_i^{-4\beta-2} m_{y_i y_i}^{(i)}.$$

Bringing these derivatives into Equation (12), we obtain

$$\begin{aligned} &\sum_{i=1}^{n} m_t^{(i)} + \beta \sum_{i=1}^{n} \left[(2\beta+1) \sum_{j=1}^{d} \sigma_{ij}^2 - 2\mu_i y_i \right] m_{y_i}^{(i)} \\ &+ 2\beta^2 \sum_{i=1}^{n} \sum_{j=1}^{d} \sigma_{ij}^2 y_i m_{y_i y_i}^{(i)} = 0. \end{aligned} \tag{15}$$

We conjecture a solution to Equation (15) in the following form:

$$\sum_{i=1}^{n} m^{(i)}(t, y_i) = I(t) + \sum_{i=1}^{n} L^{(i)}(t) y_i \tag{16}$$

with the boundary conditions given by $I(T) = 1, L^{(i)}(T) = 0$ for $i = 1, \cdots, n$. Putting Equation (16) into Equation (15), we get:

$$\begin{aligned} &I_t + \beta(2\beta+1) \sum_{i=1}^{n} \sum_{j=1}^{d} \sigma_{ij}^2 L^{(i)}(t) \\ &+ \sum_{i=1}^{n} \left[L_t^{(i)} - 2\beta\mu_i L^{(i)}(t) \right] y_i = 0. \end{aligned} \tag{17}$$

Again to eliminate the dependence on $y_i, i = 1, \cdots, n$, we can split Equation (17) into $n+1$ conditions:

$$I_t + \beta(2\beta+1) \sum_{i=1}^{n} \sum_{j=1}^{d} \sigma_{ij}^2 L^{(i)}(t) = 0. \tag{18}$$

$$L_t^{(i)} - 2\beta\mu_i L^{(i)}(t) = 0, \quad i = 1, \cdots, n. \tag{19}$$

Taking into account the boundary conditions, the solutions to Equations (18) and (19) are

$$I(t) = 1, L^{(i)}(t) = 0, i = 1, \cdots, n. \tag{20}$$

Subsequently, we have the following theorem.

Theorem 1. *The optimal strategy for the logarithmic utility maximization with multiple stocks under the CEV model is given by*

$$\pi^*(t) = \left(\left(S^{\beta}(t) \sigma \right) \left(S^{\beta}(t) \sigma \right)^{\mathrm{T}} \right)^{-1} (\mu - r\mathbf{1}_n) X(t), \tag{21}$$

and the value function is given by

$$H(t, s_1, \cdots, s_n, x) = \ln x + \sum_{i=1}^{n} d^{(i)}(t, s_i),$$

where $d^{(i)}(t, s_i), i = 1, \cdots, n$ satisfy

$$\begin{aligned} &\sum_{i=1}^{n} d_t^{(i)} + \sum_{i=1}^{n} \mu_i s_i d_{s_i}^{(i)} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{d} \sigma_{ij}^2 s_i^{2\beta+2} d_{s_i s_i}^{(i)} + r \\ &+ \frac{1}{2|\sigma\sigma^{\mathrm{T}}|} \sum_{i=1}^{n} \sum_{j=1}^{n} (\mu_i - r)(\mu_j - r) \Psi_{ij} s_i^{-\beta} s_j^{-\beta} = 0. \end{aligned}$$

Proof. Equations (7) and (10) leads to

$$\pi^* = \left(S^{\beta} \sigma\sigma^{\mathrm{T}} S^{\beta} \right)^{-1} (\mu - r\mathbf{1}_n) x + \frac{x}{\sum_{i=1}^{n} g^{(i)}} S g_s,$$

where $g_s := \left(g_{s_1}^{(1)}, \cdots, g_{s_n}^{(n)} \right)^{\mathrm{T}}$. Due to Equations (14), (16) and (20), we have for each $1 \le i \le n$,

$$g_{s_i}^{(i)} = -2\beta s_i^{-2\beta-1} m_{y_i}^{(i)} = 0,$$

and then

$$\pi^*(t) = \left(\left(S^{\beta}(t) \sigma \right) \left(S^{\beta}(t) \sigma \right)^{\mathrm{T}} \right)^{-1} (\mu - r\mathbf{1}_n) X(t).$$

According to Equations (14), (16) and (20), we obtain $\sum_{i=1}^{n} g^{(i)} = 1$. This together with Equations (10) and (13) immediately completes the proof.

Remark 2. From Equation (21), we find that the optimal investment proportion $\pi^*(t)/X(t)$ is independent of the wealth. This can be explained by the relative risk tolerance $-U'(x)/(xU''(x))$, which is a constant for logarithmic utility. Thus, the wealth has no influence on the optimal proportion invested in stocks.

Remark 3. For a logarithmic utility function, the optimal policy under the CEV model is similar to that under a geometric Brownian motion (GBM) model. However, the volatility matrix of the stocks $\left(S(t)\right)^{\beta}\sigma$ is not constant but related to the prices of stocks. This result implies that the CEV model fully considers the role of the stochastic price of the stock market.

Remark 4. In the case where there is only one stock and a bond, *i.e.* $n=1$, $\mu,\sigma,S(t)$ are unidimensional and we are back to the settings of [20]. Equation (21) reduces to

$$\pi^{*}(t)=\frac{\mu-r}{\sigma^{2}\left(S(t)\right)^{2\beta}}X(t), \tag{22}$$

which is the same as the optimal policy derived by Xiao, Zhai and Qin [20]. From Equation (22), we find that $\pi^{*}(t)$ decreases with β. A bigger β means a larger volatility, which increases risks for investors. Thus, investors would reduce the amount invested in the stock to avoid risks.

4. Numerical Analysis

In this section, we provide some numerical simulations to analyze the properties of the optimal strategy and illustrate the dynamic behavior of the optimal strategy.

We assume that there are two stocks and one bond in the market during the time horizon $T=10$ (years), *i.e.*, $n=2$. Throughout the numerical analysis, we use the optimal proportion invested in stocks at time t, *i.e.*, $\pi^{*}(t)/X(t)$ to denote the optimal strategy.

Let

$$r=0.03,\mu=(0.12,0.1),\sigma=\begin{pmatrix}18.16 & 12.15\\ 12.03 & 13.10\end{pmatrix},$$
$$\beta=-1,S_1(0)=13.5,S_2(0)=12.5.$$

Figure 1 shows the effects of the appreciation rate μ_1 on the optimal strategies. As expected, the optimal proportion invested in stock 1 increases with respect to its appreciation rate μ_1. Since multiple stocks are considered, we can analyze the impact of one stock on the other stock. From **Figure 1**, we find that there is an inverse relationship between μ_1 and the optimal strategy of stock 2. This is consistent with intuition. When the appreciation rate of one stock increases constantly, it is optimal to increase the proportion of wealth in this stock and reduce investment in the other stock. Furthermore, **Figure 1** also shows that the total proportion invested in two stocks changes moderately with μ_1. This implies that the influence of one stock's appreciation rate on the total investment is not obvious.

In **Figure 2**, the parameters are given by:

$$r=0.03,\mu=(0.12,0.1),\sigma=\begin{pmatrix}1 & 2\\ 1.5 & 1.2\end{pmatrix},$$
$$\beta=-0.4,S_1(0)=8,S_2(0)=6.$$

Figures 2(a) and **(b)** plot the evolution of the stocks' prices and the optimal strategy over time under the CEV model, respectively. Unlike the GBM model, the optimal proportion invested in each stock under the CEV model

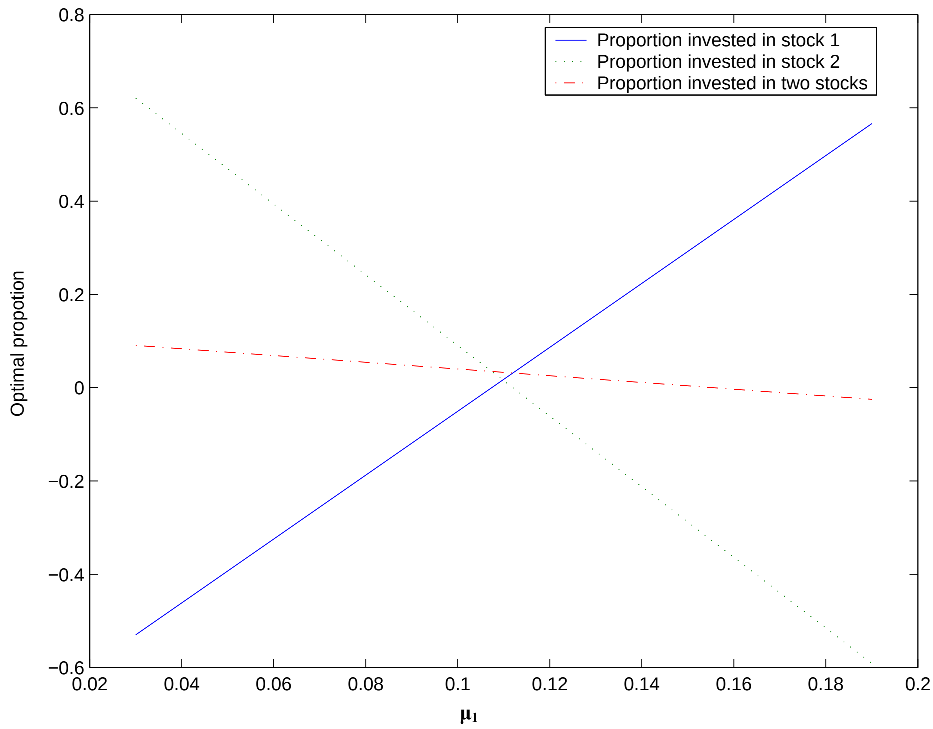
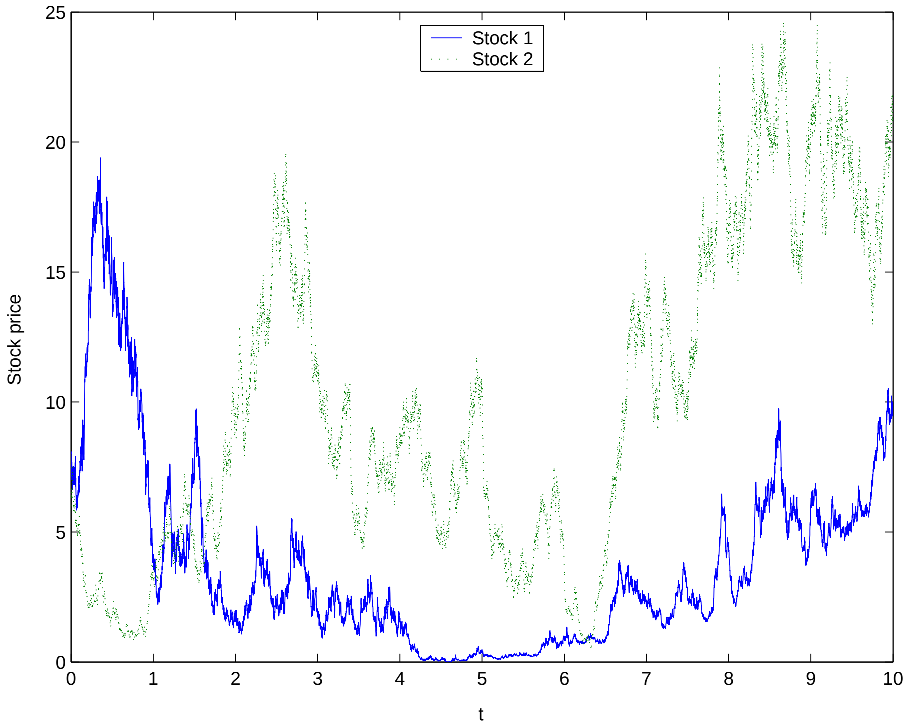
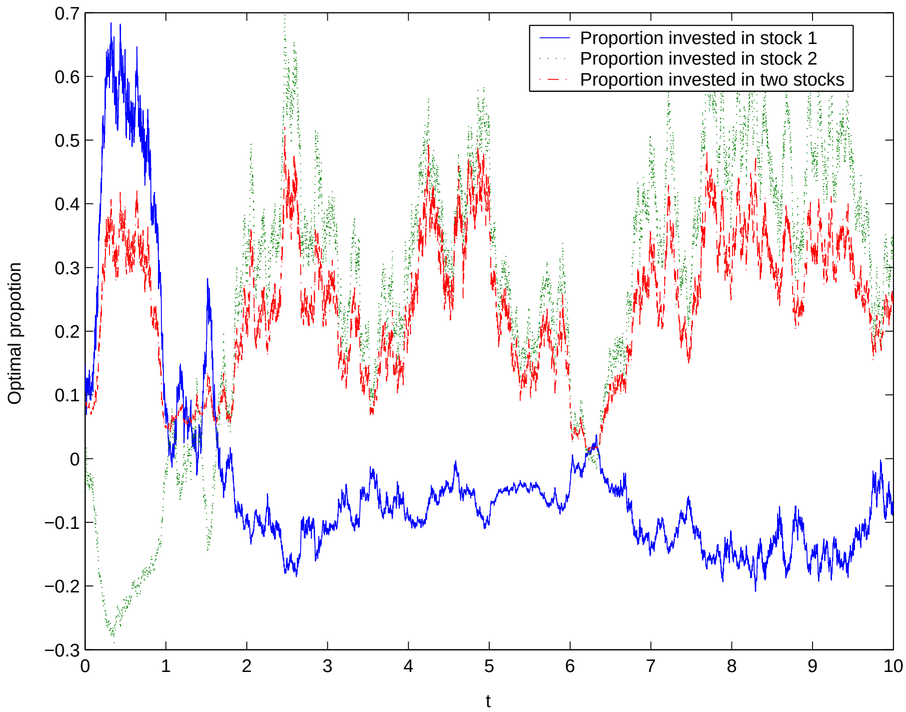


Figure 1. The effect of μ_1 on the optimal strategy.



(a)



(b)

Figure 2. (a) Evolution of stocks' prices over time; (b) Evolution of optimal strategy over time.

fluctuates with stocks' prices. As shown in **Figure 2(b)**, the overall tendency of optimal proportion invested in stock 1 decreases with respect to time. This is because that the actual price of stock 1 has a decreasing trend over time (see **Figure 2(a)**). Moreover, **Figure 2(b)** also indicates that sometimes it is optimal to sell short stock 1. On the contrary, the optimal strategy of stock 2 increases in general due to the rising tendency of its price. Consequently, the total proportion invested in stocks is relatively steady over time.

5. Conclusion

By considering multiple risky assets and a risk-free asset in a financial market, this paper extends the portfolio selection problem under the constant elasticity of variance (CEV) model. We propose the framework of portfolio selection problem with multiple risky assets under the CEV model. Explicit solution for the logarithmic utility maximization has been derived via stochastic control approach. It is shown that for portfolio selection problems concerning risky assets with the CEV price processes, there are differences in solution characterization and calculations between one dimensional case and n dimensional case. The optimal policy under the CEV model is the same as that under the GBM model in form except for a stochastic volatility matrix. Finally, numerical results demonstrate the properties of the multidimensional optimal strategy under the CEV model.

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REFERENCES

[1] R. C. Merton, "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *The Review of Economics and Statistics*, Vol. 51, No. 3, 1969, pp. 247-257. doi:10.2307/1926560

[2] R. C. Merton, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, Vol. 3, No. 4, 1971, pp. 373-413. doi:10.1016/0022-0531(71)90038-X

[3] S. Browne, "Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin," *Mathematics of Operations Research*, Vol. 20, No. 4, 1995, pp. 937-958. doi:10.1287/moor.20.4.937

[4] H. Yang and L. Zhang, "Optimal Investment for Insurer with Jump-Diffusion Risk Process," *Insurance*: *Mathematics and Economics*, Vol. 37, No. 3, 2005, pp. 615-634. doi:10.1016/j.insmatheco.2005.06.009

[5] S. Pliska, "A Stochastic Calculus Model of Continuous Trading: Optimal Portfolios," *Mathematics of Operations Research*, Vol. 11, No. 2, 1986, pp. 371-382. doi:10.1287/moor.11.2.371

[6] I. Karatzas, J. P. Lehoczky and S. E. Shreve, "Optimal Portfolio and Consumption Decisions for a 'Small Investor' on a Finite Horizon," *SIAM Journal on Control and Optimization*, Vol. 25, No. 6, 1987, pp. 1557-1586. doi:10.1137/0325086

[7] J. C. Cox and C. F. Huang, "Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory*, Vol. 49, No. 1, 1989, pp. 33-83. doi:10.1016/0022-0531(89)90067-7

[8] I. Karatzas, "Optimization Problems in the Theory of Continuous Trading," *SIAM Journal on Control and Optimization*, Vol. 27, No. 6, 1989, pp. 1221-1259. doi:10.1137/0327063

[9] I. Karatzas and S. E. Shreve, "Brownian Motion and Stochastic Calculus," Springer Verlag, New York, 1991. doi:10.1007/978-1-4612-0949-2

[10] I. Karatzas, J. P. Lehoczky, S. E. Shreve and G. L. Xu, "Martingale and Duality Methods for Utility Maximization in an Incomplete Market," *SIAM Journal on Control and Optimization*, Vol. 29, No. 3, 1991, pp. 702-730. doi:10.1137/0329039

[11] A. Zhang, "A Secret to Create a Complete Market from an Incomplete Market," *Applied Mathematics and Computation*, Vol. 191, No. 1, 2007, pp. 253-262. doi:10.1016/j.amc.2007.02.086

[12] Z. Wang, J. Xia and L. Zhang, "Optimal Investment for an Insurer: The Martingale Approach," *Insurance*: *Mathematics and Economics*, Vol. 40, No. 2, 2007, pp. 322-334. doi:10.1016/j.insmatheco.2006.05.003

[13] Q. Zhou, "Optimal Investment for an Insurer in the Lévy Market: The Martingale Approach," *Statistics & Probability Letters*, Vol. 79, No. 14, 2009, pp. 1602-1607. doi:10.1016/j.spl.2009.03.027

[14] D. G. Hobson and L. C. G. Rogers, "Complete Models with Stochastic Volatility," *Mathematical Finance*, Vol. 8, No. 1, 1998, pp. 27-48. doi:10.1111/1467-9965.00043

[15] J. C. Cox and S. A. Ross, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics*, Vol. 3, No. 1-2, 1976, pp. 145-166. doi:10.1016/0304-405X(76)90023-4

[16] J. C. Cox, "The Constant Elasticity of Variance Option Pricing Model," *The Journal of Portfolio Management*, Vol. 22, 1996, pp. 15-17.

[17] C. F. Lo, P. H. Yuen and C. H. Hui, "Constant Elasticity of Variance Option Pricing Model with Time-Dependent Parameters," *International Journal of Theoretical and Applied Finance*, Vol. 3, No. 4, 2000, pp. 661-674. doi:10.1142/S0219024900000814

[18] D. Davydov and V. Linetsky, "Pricing and Hedging Path-Dependent Options under the CEV Process," *Management Science*, Vol. 47, No. 7, 2001, pp. 949-965. doi:10.1287/mnsc.47.7.949.9804

[19] Y. L. Hsu, T. I. Lin and C. F. Lee, "Constant Elasticity of Variance (CEV) Option Pricing Model: Integration and Detailed Derivation," *Mathematics and Computers in Simulation*, Vol. 79, No. 1, 2008, pp. 60-71. doi:10.1016/j.matcom.2007.09.012

[20] J. Xiao, H. Zhai and C. Qin, "The Constant Elasticity of Variance (CEV) Model and the Legendre Transform-Dual Solution for Annuity Contracts," *Insurance*: *Mathematics and Economics*, Vol. 40, No. 2, 2007, pp. 302-310. doi:10.1016/j.insmatheco.2006.04.007

[21] J. Gao, "Optimal Portfolios for DC Pension Plans under a CEV Model," *Insurance*: *Mathematics and Economics*, Vol. 44, No. 3, 2009, pp. 479-490. doi:10.1016/j.insmatheco.2009.01.005

[22] J. Gao, "Optimal Investment Strategy for Annuity Contracts under the Constant Elasticity of Variance (CEV) Model," *Insurance*: *Mathematics and Economics*, Vol. 45, No. 1, 2009, pp. 9-18. doi:10.1016/j.insmatheco.2009.02.006

[23] M. Gu, Y. Yang, S. Li and J. Zhang, "Constant Elasticity of Variance Model for Proportional Reinsurance and Investment Strategies," *Insurance*: *Mathematics and Economics*, Vol. 46, No. 3, 2010, pp. 580-587. doi:10.1016/j.insmatheco.2010.03.001

[24] H. Zhao and X. Rong, "Portfolio Selection Problem with Multiple Risky Assets under the Constant Elasticity of Variance Model," *Insurance*: *Mathematics and Economics*, Vol. 50, No. 1, 2012, pp. 179-190. doi:10.1016/j.insmatheco.2011.10.013

