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# One Dimensional Solute Transport Originating from a Exponentially Decay Type Point Source along Unsteady Flow through Heterogeneous Medium

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## Abstract

One dimensional advection dispersion equation is analytically solved initially in solute free domain by considering uniform exponential decay input condition at origin. Heterogeneous medium of semi infinite extent is considered. Due to heterogeneity velocity and dispersivity coefficient of the advection dispersion equation are considered functions of space variable and time variable. Analytical solution is obtained using Laplace transform technique when dispersivity depended on velocity. The effects of first order decay term and adsorption are studied. The graphical representations are made using MATLAB.

## 1. Introduction

Managing the groundwater resources and rehabilitation of polluted aquifers, mathematical modeling is a powerful tool. The contaminant concentration distribution behaviour along/against unsteady groundwater flow in aquifer is studied through mathematical modeling as it is an important approach to formulate the geo-environmental problems and provides the best possible solution for reducing its impact on the environment. The pollutant's solute transport from a source through a medium of air or water is described by a partial differential equation of parabolic type derived on the principle of conservation of mass, and is known as advection–diffusion equation (ADE). In one-dimension it contains two coefficients, one represents the diffusion parameter and the second represents the velocity of the advection of the medium like air or water. In case of porous medium, like aquifer, velocity satisfies the Darcy law and in non-porous medium, like air it satisfies the laminar conditions. The dispersive property differs from pollutant to pollutant.

The literature contains analytical solutions for solute transport in homogenous and heterogeneous porous media. Analytical solutions in one-, two-, and three-dimensional advection-dispersion transport equations with constant coefficients in homogeneous medium which have been collected in various compendiums [1-4]. Some

more works in homogeneous medium has been compiled [5-10]. Using the theory [11] that relates dispersion directly to velocity, analytical solutions were obtained for solute transport along unsteady flow through homogeneous medium [12-15]. According to the dispersion theory [16] the dispersion parameter is proportional to square of velocity. Though much analytical solutions are not available based on this theory but some works [17,18] do occur. Some large sub-surface formations exhibit variable dispersivity properties either as a function of time or function of distance observed [19]. So the advection-dispersion equation with constant coefficients may not be appropriate for solute transport in heterogeneous media. Analytical solutions are available for space and/or time dependent coefficients manly in finite domain are very less in number. Analytical solutions for heterogeneous porous media for transport equation with time dependent coefficients [20-23]. Distance dependent analytical solution for one dimensional transport in porous media with an exponential dispersion function were solved [24,25] for uniform input condition and [26] for periodic input condition which describe the solute transport due to spatially dependent dispersion along uniform flow through heterogeneous semi-infinite media. The limitations of analytical solutions of the ADE with coefficients being function of space variables discussed [27]. Analytical solution of the advection-diffusion transport equation using

a change-of-variable and integral transform technique obtained [28]. Further the technique of generalized integral transform to get analytical solutions of ADE in heterogeneous media with different spatially dependent dispersivity discussed [29]. A closed form analytical solution for spatially-varying initial conditions was derived for Dirichlet and Cauchy boundary conditions each with Bateman-type source terms [30]. Some work on distance dependent [31-40]. Longitudinal and transverse dispersion in two dimensional flows in aquifer-aquitard system have been investigated analytically [41]. The numerical solution of a fractional partial differential equation with Riesz-Space fractional derivative in a finite domain is discussed [42]. They considered two types of fractional partial differential equation, first one is the Riesz fractional diffusion equation and the second is the Riesz fractional advection-diffusion equation and provided three numerical methods to deal with the Riesz-Space fractional derivative. Also a finite difference approximation for two sided space fractional partial differential equation was provided [43].

In the present work one-dimensional advection diffusion equation is solved for dispersivity depended on square of velocity. The medium is of inhomogeneous nature and is of semi infinite extent. Due to inhomogeneous medium both the parameters dispersion and fluid velocity depends on space and time. Initially aquifer is considered to be solute free. The input point source is of exponentially decreasing nature at the origin and at the other end its concentration gradient is considered to be zero. The effect of first order decay of temporally dependent and adsorption is also considered in this work to get the physical insite of the problem. Laplace transform technique is used to obtain the analytical solution.

## 2. Mathematical Formulation and Its Analytical Solution

The linear Advection-Diffusion partial differential equation in one dimension in general form with absorption and decay term may be written as

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial c}{\partial x} - u(x,t)c\right) - \mu(t)c + \gamma \quad (1)$$

where $c$ is the solute concentration at a position $x$ at time $t$, $D(x,t)$ represents the solute dispersion and $u(x,t)$ is velocity of the medium transporting the solute particles, $u(t)$ is first order decay or production term $[T^{-1}]$, $\gamma$ is source/sink of dimension $[ML^{-3}T^{-1}]$, $K_1$ is empirical constant and $\eta_0$ is the porosity. Initially the medium is solute free. An exponential decay type input point source concentration is assumed at the origin of the medium of uniform nature where $q$ is the contaminant

decay rate of dimension inverse of time $[T^{-1}]$. It means that the input concentration decreases with time at the source. The second boundary condition is assumed to be of second type (flux type) of homogeneous nature. Thus the initial and two boundary conditions are as follows:

$$c(x,t) = 0; t = 0, x \geq 0 \quad (2)$$

$$c(x,t) = c_0 \exp(-qt); x = 0, t \geq 0 \quad (3)$$

and

$$\frac{\partial c}{\partial x} = 0; x \to \infty, t \geq 0 \quad (4)$$

In [44] they considered the co-efficients of Equation (1) are temporally dependent and [23] assumed spatially dependent in a constant point source and derived their analytical solutions. But in this paper due to heterogeneity velocity is considered spatially dependent of linearly interpolated nature, and also velocity is assumed temporally dependent. Due to heterogeneous medium it's not always possible that the source of contaminants is constant, so in this paper the source of contaminant at the origin is of exponentially decay type. The expressions for each coefficient velocity, dispersion and first order decay are considered in degenerate forms as follows:

$$u(x,t) = u_0 f(mt)(1+ax),$$
$$D(x,t) = D_0 f^2(mt)(1+ax)^2 \text{ and } \mu(t) = \mu_0 f(mt) \quad (5)$$

where the coefficient $a$ is the heterogeneity parameter of dimension inverse of that of space variable, and $m$ is an unsteadiness parameter of dimension inverse of that of time variable, $D_0$, $u_0$ and $\mu_0$ in above expressions referred as initial dispersion coefficient of dimension $[L^2 T^{-1}]$, initial velocity of dimension $[L^2 T^{-1}]$ and initial firs order time decay rate of dimension of inverse of time $[T^{-1}]$.

## 3. Dispersion through Heterogeneous Medium along Unsteady Flow

Using the expressions (5), the advection-diffusion Equation (1) can now be written as

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D_0 f^2(mt)(1+ax)^2\frac{\partial c}{\partial x} - u_0 f(mt)(1+ax)c\right) \quad (6)$$
$$- \mu_0 f(mt)c + \gamma$$

or

$$\frac{1}{f(mt)}\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D_0 f(mt)(1+ax)^2\frac{\partial c}{\partial x} - u_0(1+ax)c\right) \quad (7)$$
$$- \mu_0 c + \frac{\gamma}{f(mt)}$$

Let us introduce a new time variable $T^*$ defined by [45] by the transformation as

$$T^* = \int_0^t f(mt)\,dt \qquad (8)$$

The dimension of $T^*$ is same as dimension of $t$, so it is referred to as a new time variable. An expression for $f(mt)$ chosen such that for $t = 0$, we get the value of $T^* = 0$, so that the initial condition not affected in new time domain. Also a space variable transformation is introduced [23,46] as

$$X = \frac{1}{a}\log(1 + ax) \qquad (9)$$

The initial value problem together with their initial and boundary conditions in new time and space variable becomes

$$\frac{\partial c}{\partial T^*} = D_0 f(mt)\frac{\partial^2 c}{\partial X^2} - u_0 f_1(mt)\frac{\partial c}{\partial X}$$
$$- (au_0 + \mu_0)c + \frac{\gamma}{f(mt)} \qquad (10)$$

$$c(X, T^*) = 0; T^* = 0, X \geq 0 \qquad (11)$$

$$c(X, T^*) = c_0(1 - qT^*); X = 0; T^* \geq 0, \qquad (12)$$

and

$$\frac{\partial c}{\partial X} = 0; X \to \infty, T^* \geq 0 \qquad (13)$$

where $f_1(mt) = 1 - \lambda f(mt)$ is another time dependent expression in non-dimensional variable $mt$ and $\lambda = (aD_0/u_0)$ is non dimensional coefficient.

To eliminate the first order decay term form the Equation (10), introducing the transformation as:

$$c = C\exp-(au_0 + \mu_0)T^* \qquad (14)$$

With the use of Equation (14), Equation (10), becomes

$$\frac{\partial C}{\partial T^*} = D_0 f(mt)\frac{\partial^2 C}{\partial X^2} - u_0 f_1(mt)\frac{\partial C}{\partial X}$$
$$+ \frac{\gamma}{f(mt)}\exp\{(au_0 + \mu_0)T^*\} \qquad (15)$$

Further using a space variable $Z$ and time variable $T$ through the transformations as:

$$Z = \frac{f_1(mt)}{f(mt)}X \qquad (16)$$

and

$$T = \int_0^t f_1^2(mt)\,dt \qquad (17)$$

The one-dimensional advection-diffusion Equation (15)

with their initial condition (11) and boundary conditions (12)-(13) may now be written as

$$\frac{\partial C}{\partial T} = D_0 \frac{\partial^2 C}{\partial Z^2} - u_0\frac{\partial C}{\partial Z} + \frac{\gamma}{f_1^2(mt)}\exp\{(au_0 + \mu_0)T^*\} \qquad (18)$$

$$C(Z, T) = 0; T = 0, Z \geq 0 \qquad (19)$$

$$C(Z, T) = c_0(1 - qT^*)\exp\{(au_0 + \mu_0)T^*\};$$
$$Z = 0; T \geq 0, Z = 0 \qquad (20)$$

$$\frac{\partial C}{\partial Z} = 0; Z \to \infty, T \geq 0 \qquad (21)$$

The time variable $T^*$ has to be expressed explicitly in terms of $T$. An expression of exponentially decreasing nature is chosen as

$$f(mt) = \exp(-mt) \qquad (22)$$

So from Equation (8), we get

$$T^* = \int_0^t \exp(-mt)\,dt = \frac{1}{m}\left[1 - \exp(-mt)\right]$$

or

$$mt = -\log(1 - mT^*)$$

Also using the transformation in Equation (17) we get

$$T = \int_0^t \{1 - \lambda f(mt)\}^2\,dt = \int_0^t \left(1 + \lambda^2 f^2(mt) - 2\lambda f(mt)\right)dt$$

or

$$T = \frac{1}{m}\left[mt + \frac{\lambda^2}{2}\{1 - \exp(-2mt)\} - 2\lambda\{1 - \exp(-mt)\}\right]$$

or

$$T = \frac{1}{m}\left[-\log(1 - mT^*) + \frac{\lambda^2}{2}\{1 - (1 - mT^*)^2\}\right.$$
$$\left. -2\lambda\{1 - (1 - mT^*)\}\right]$$

In $f(mt)$, $m$ is much smaller than one, so its second and higher degree terms in the logarithmic and binomial expansions in above equations are omitted. So we get

$$T^* = \gamma_1 T, \text{ where } \gamma_1 = (1 - \lambda)^{-2} \qquad (23)$$

Thus the initial value problem (18) and their conditions (19)-(21), becomes

$$\frac{\partial C}{\partial T} = D_0 \frac{\partial^2 C}{\partial Z^2} - u_0\frac{\partial C}{\partial Z}$$
$$+ \gamma\left[(1 + 2\lambda) - 2m\lambda\gamma_1 T\right]\exp(AT) \qquad (24)$$

$$C(Z,T) = 0; T = 0, Z \geq 0 \tag{25}$$

$$C(Z,T) = c_0(1 - q\gamma_1 T)\exp(AT); Z = 0; T \geq 0, Z = 0 \tag{26}$$

$$\frac{\partial C}{\partial Z} = 0; Z \to \infty, T \geq 0 \tag{27}$$

where $A = (au_0 + \mu_0)\gamma_1$.

Now to find the analytical solution for Equation (24), Laplace transform technique is used, but to apply it more conveniently the convective term from the Equation (24) is to be removed by the use of the transformation as

$$C(Z,T) = K(Z,T)\exp\left(\frac{u_0}{2D_0}Z - \frac{u_0^2}{4D_0}T\right) \tag{28}$$

The initial and boundary value problem from (24-27) in terms of new dependent variable $K(Z,T)$ may now be written as

$$\frac{\partial K}{\partial T} = D_0 \frac{\partial^2 K}{\partial Z^2} + \gamma\left[(1+2\lambda) - 2m\lambda\gamma_1 T\right]\exp\left(\beta^2 T - u_0 Z/2D_0\right) \tag{29}$$

$$K(Z,T) = 0; T = 0, Z \geq 0 \tag{30}$$

$$K(Z,T) = c_0(1 - q\gamma_1 T)\exp(\beta^2 T); \\ Z = 0; T \geq 0, Z = 0 \tag{31}$$

$$\frac{\partial K}{\partial Z} + \frac{u_0}{2D_0}K = 0; Z \to \infty, T \geq 0 \tag{32}$$

where $\beta^2 = A + \dfrac{u_0^2}{4D_0 R}$ and $A = (au_0 + \mu_0)\gamma_1$

Applying Laplace transformation on the above boundary value problem, the problems become in second order ordinary differential equation in the Laplacian domain $p$ as:

$$\frac{d^2\bar{K}}{dZ^2} = \frac{Rp}{D_0}\bar{K} \\ - \gamma\left[\frac{1+2\lambda}{(p - \beta^2)} - \frac{2m\lambda\gamma_1}{(p - \beta^2)^2}\right]\exp\left(-\frac{u_0}{2D_0}Z\right), \tag{33}$$

$$\bar{K}(Z,p) = c_0\left[\frac{1}{(p-\beta^2)} - (p-\beta^2)\right]Z = 0; T \geq 0, \tag{34}$$

and

$$\frac{d\bar{K}}{dZ} + \frac{u_0}{2D_0}\bar{K} = 0; Z \to \infty; T \geq 0 \tag{35}$$

After using the boundary conditions (34) and (35), its particular solution may be obtained as

$$\bar{K}(Z,p) = \left\{\frac{c_0}{(p-\beta^2)} - \frac{qc_0\gamma_1}{(p-\beta^2)^2}\right\}\exp\left(-Z\sqrt{p/D_0}\right) \\ - \frac{\gamma}{R}\left\{\frac{1+2\lambda}{(p-\alpha^2)(p-\beta^2)} - \frac{2m\lambda\gamma_1}{(p-\alpha^2)(p-\beta^2)^2}\right\} \\ \cdot\exp\left(-Z\sqrt{p/D_0}\right) \\ + \frac{\gamma}{R}\left\{\frac{1+2\lambda}{(p-\alpha^2)(p-\beta^2)} - \frac{2m\lambda\gamma_1}{(p-\alpha^2)(p-\beta^2)^2}\right\} \\ \cdot\exp\left(u_0 Z/2D_0\right) \tag{36}$$

where $\alpha^2 = u_0^2/4D_0$

Now taking the inverse Laplace transform of Equation (36), the solution in $K(Z,T)$ may be obtained. Using the transformation (28) and (14) the desired solution may be obtained as

$$c(x,t) = \frac{1}{2}\left\{c_0 + \frac{\gamma}{R}\left(\frac{1+2\lambda}{\alpha^2 - \beta^2}\right.\right. \\ \left.+ \frac{2m\lambda\gamma_1(\alpha^2\beta^2 - \alpha^4 - \beta^4)}{(\alpha^2 - \beta^2)^2}\right)(C_1 + D_1) \\ - \frac{1}{4\beta}\left(c_0 q\gamma_1 - \frac{2m\lambda\gamma\gamma_1}{(\beta^2 - \alpha^2)}\right)\left\{\left(2\beta T - Z\sqrt{\frac{1}{D_0}}\right)C_1\right. \\ \left.+ \left(2\beta T + Z\sqrt{\frac{1}{D_0}}\right)D_1\right\} \\ + \frac{\gamma}{2}\left(\frac{2m\lambda\gamma_1}{(\alpha^2 - \beta^2)^2} - \frac{1+2\lambda}{\alpha^2 - \beta^2}\right)(C_1' + D_1') \\ + \gamma\left(\frac{1+2\lambda}{\alpha^2 - \beta^2} - \frac{2m\lambda\gamma_1}{(\alpha^2 - \beta^2)^2}\right)\exp(-AT) \\ - \gamma\left(\frac{1+2\lambda}{\alpha^2 - \beta^2} + \frac{2m\lambda\gamma_1(\alpha^2\beta^2 - \alpha^4 - \beta^4)}{(\alpha^2 - \beta^2)^2} + \frac{2m\lambda\gamma_1 T}{\beta^2 - \alpha^2}\right) \tag{37}$$

where

$$C_1 = \exp\left(\frac{u_0}{2D_0}Z - \beta\sqrt{\frac{1}{D_0}}Z\right)erfc\left(\frac{Z}{2}\sqrt{\frac{1}{D_0 T}} - \beta\sqrt{T}\right);$$

$$D_1 = \exp\left(\frac{u_0}{2D_0}Z + \beta\sqrt{\frac{1}{D_0}}Z\right)erfc\left(\frac{Z}{2}\sqrt{\frac{1}{D_0 T}} + \beta\sqrt{T}\right);$$

$$C_1' = \\ \exp\left(\frac{u_0}{2D_0}Z - \alpha\sqrt{\frac{1}{D_0}}Z - AT\right)erfc\left(\frac{Z}{2}\sqrt{\frac{1}{D_0 T}} - \alpha\sqrt{T}\right);$$

$$D_1' =$$

$$\exp\left(\frac{u_0}{2D_0}Z + \alpha\sqrt{\frac{1}{D_0}}Z - AT\right)erfc\left(\frac{Z}{2}\sqrt{\frac{1}{D_0T}} + \alpha\sqrt{T}\right);$$

$$A = (au_0 + \mu_0)\gamma_1; \alpha^2 = u_0^2/4D_0 \; ; \quad \beta^2 = A + \frac{u_0^2}{4D_0} \; ;$$

$$Z = \frac{f_1(mt)}{f(mt)}X \; ; \quad X = \frac{1}{a}\log(1 + ax) \; ;$$

$$f_1(mt) = 1 - \lambda f(mt) \; ; \quad T = T^*/\gamma_1 \; ; \quad \gamma_1 = (1 - \lambda)^{-2} \; ;$$

$$\lambda = \frac{aD_0}{u_0} \; ; \quad T^* = \int_0^t f(mt)\,dt \; ;$$

The solution defined by Equation (37) describes the solute transport for exponential decay type input condition at origin in heterogeneous semi infinite domain.

## 4. Illustration and Discussion

The analytical solution of the present hydrodynamics dispersion is obtained as given in Equation (37). The concentration values $(c/C_0)$ are evaluated from the solution for the input values: reference concentration $(C_0) = 1$, initial velocity $(u_0) = 0.61$ (km/year), initial dispersivity $(D_0) = 0.71$ (km²/year), heterogeneity parameter ($a$) = 0.1 (km⁻¹), unsteady parameter ($m$) = 0.1 (km⁻¹), contaminant decay rate ($q$) = 0.1 (km⁻¹), initial first order decay ($\mu_0$) = 0.5 (year⁻¹), and the initial source/sink ($\gamma_0$) = 0.2 $(ML^{-3}T^{-1})$. Concentration attenuation with position and time is studied in the domain $0 \le x(km) \le 1$, at $t$ = 0.4, 0.7 and 1.0 (year). It is illustrated in **Figure 1**. Full line curves are drawn for decelerating flow filed represented by $f(mt) = \exp(-mt)$ and dotted curves are drawn for accelerating flow field represented by $f(mt) = \exp(mt)$. In case input concentration, *i.e.* $(c/C_0)$ at $x = 0$ decreases with time but solute transport of lower input concentration source is faster than that of source having higher input value. It is evident that, in view of the dispersion parameter being proportional to square of velocity, solute transport is much faster in case of accelerating flow field than that along decelerating flow field. The effect of heterogeneity is studied in **Figure 2**. For it concentration values $(c/C_0)$ are evaluated from solution (37) at $t = 1.0$ and $a$ = 0.1, 0.2, 0.3, for both the flow field. It may be observed that solute transport faster along accelerating flow field in a medium of higher heterogeneity (causing larger increase in velocity from origin to the end $x$) than that in a medium of lower heterogeneity. But the trend reverses in a decelerating flow field.

The effect of first order decay and zero order production are studied through **Figure 3**. It is drawn at $t = 1.0$
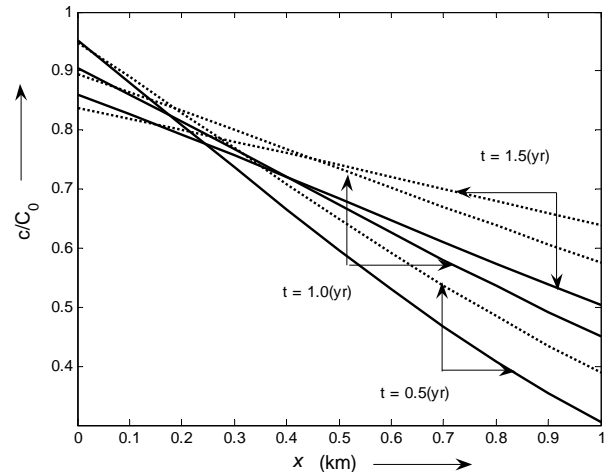


**Figure 1. Illustration of solute transport at different times when $D \propto u^2$, $a = 0.1$ (km⁻¹), $m = 0.1$ (year⁻¹), $\mu_0 = 0.5$ (year⁻¹), $\gamma_0 = 0.1$, and $q = 0.1$ (km⁻¹) described by solution (37). Solid and dotted curves are drawn for $u = u_0\exp(-mt)(1 + ax)^2$, and $u = u_0\exp(mt)(1 + ax)^2$ respectively.**
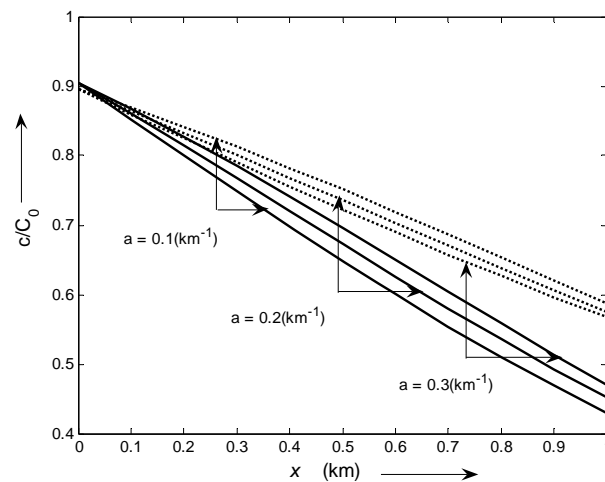


**Figure 2. Illustration of the effect of heterogeneity on the solute transport when $D \propto u^2$, described by solution (37), at $t = 0.1$ (year). Solid and dotted curves are drawn for $u = u_0\exp(-mt)(1 + ax)^2$ and $u = u_0\exp(mt)(1 + ax)^2$ respectively, $m = 0.1$ (year⁻¹).**

and $a = 0.1$. It may be observed that solute transport is fastest in the absence of both the parameters. It is slowest in the presence of first order decay but in the absence of the production term.

## 5. Conclusions

One-dimensional analytical solution of advection – diffusion equation with variable coefficients is obtained using Laplace transformation technique. The source con-
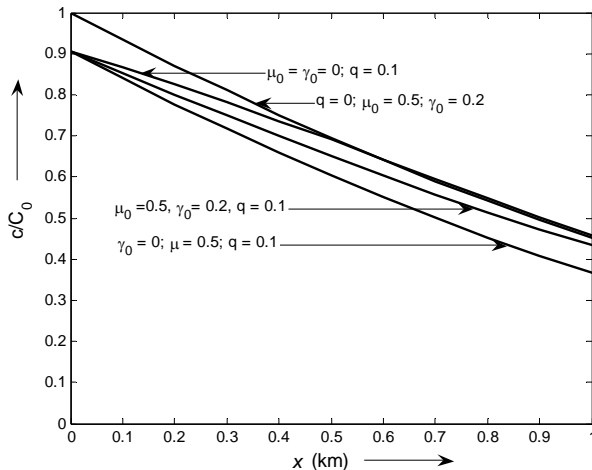
**Figure 3. Illustration of solute transport for different parameters when $D \propto u^2$, $a = 0.1$ (km-1) and $m = 0.1$ (year$^{-1}$) at time $t = 0.1$ (years).**

centration is a point uniform source of exponentially decay nature. The expressions for both the coefficients are considered in both the independent variables but in degenerate forms given by Equation (5). With the help of certain transformations the variable coefficients are reduced into constant coefficients. Such forms of the two coefficients are conceived which correspond to the different dispersion theory (Scheidegger, 1957). The change in velocity due to heterogeneity and unsteadiness may be varied by assigning appropriate values to the separate parameters of the both. It may be concluded from the present study that the concentration level in case of accelerating diffusive source along decelerating flow domain are the least. From engineering point of view this observation may be important to keep the emission of polluting solute particles from a source of accelerating nature. The effects of first order decay and adsorption are considered and their impact illustrated by graph.

# 6. Acknowledgements

# 7. References

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