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On an *M*/*G*/1 Queueing Model with *k*-Phase Optional Services and Bernoulli Feedback

Sayeedeh Abdollahi[1], Mohammad Reza Salehi Rad[2]

Department of Statistics, Allameh Tabataba'i University, Tehran, Iran.
Email: salehirad@atu.ac.ir, rose1385@yahoo.com

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ABSTRACT

In this article an *M*/*G*/1 queueing model with single server, Poisson input, *k*-phases of heterogeneous services and Bernoulli feedback design has been considered. For this model, we derive the steady-state probability generating function (PGF) of queue size at the random epoch and at the service completion epoch. Then, we derive the Laplace-Stieltjes Transform (LST) of the distribution of response time, the means of response time, number of customers in the system and busy period.

Keywords: *M*/*G*/1 Queue; Bernoulli Feedback; Queue Size; Waiting Time and Busy Period

1. Introduction

The *M*/*G*/1 queueing model is one of the famous and applied models in which the distribution of service times is unknown. For this reason, many of real models could be considered by this model. Multi-phase service is important, because some of systems have more than one phase service, for example manufacture production lines. Feedback is also important, because in some queueing models, some customers, after completion the service, may need to go back to the end of queue to take the service again. It means that the service customer is not acceptable and must go to the end of queue. According to these conditions, we have considered an *M*/*G*/1 queueing model with *k*-Phase Optional Services and Bernoulli feedback.

For this model, first we find the steady-state probability generating function (PGF) of queue size at the random epoch and at the service completion epoch. Then, we derive the Laplace-Stieltjes Transform (LST) of the distribution of response time. The means of response time, number of customers in the system and busy period will be derived by using the PGF and LST.

In relation of this model, [1-9,11] have derived some results. The model that they have considered is two phases. But, in this article, we will consider a *k*-phase queue with optional service and Bernoulli feedback in all phases. Of course, [10] studied an *M*/*G*/1 queue with *k*-phase services and vacation, but without feedback that is different from this paper.

Following, in Section 2 we describe the model and give some definitions. In Section 3, the PGF of the system size will be derived. In Section 4, we will find some measures of effectiveness. At the final section, we provide a conclusion.

2. The Mathematical Model and Definitions

In this model, the server provides first phase of regular service to all the customers. As soon as the *i*-th (for $i = 1, \cdots, k-1$) phase of service of a customer is completed , it may leave the system or immediately go for $(i+1)$-th phase of optional service. However, after receiving each phase of unsuccessful service by a unit, then it may immediately join to the end of tail of the original queue as feedback customer to take service again. Thus, the assumptions of the model are:

1) Customers arrive at the system to a Poisson process with rate λ.

2) The service discipline of the system is FCFS[1].

3) The server provides *k*-phases of heterogeneous service for any customer. The service times for *k*-phases are independent random variable that denoted by B_i with distribution functions $B_i(x), i = 1, \cdots, k$ and LST of these distributions are $B_i^*(x), i = 1, \cdots, k$. These variables have finite moments, that is $E(B_i^l) < \infty$ for $l \geq 1$.

4) As soon as the *i*-th phase of service of a customer is completed, the customer may go to the $(i+1)$-th phase of service with probability $\theta_i, i = 1, \cdots, k$.

5) After completion of the *i*-th phase, if the customer is dissatisfied with its service for certain reason or it re-

[1]First Come First Served.

ceived unsuccessful service, in this case the customer may immediately joins the end of the original queue as a feedback customer for receiving the service again with probability p_i, for $i=1,\cdots,k$, otherwise the customer may depart the system with probability q_i, $q_i = 1-p_i$.

Definition 2.1. The modified service time or the time required by a customer to complete the service cycle is given by

$$B = \begin{cases} B_1 & \text{with probabaility } 1-\theta_1 \\ B_1+B_2 & \text{with probabaility } \theta_1(1-\theta_2) \\ B_1+B_2+B_3 & \text{with probabaility } \theta_1\theta_2(1-\theta_3) \\ \vdots & \vdots \\ \sum_{i=1}^{k-1} B_i & \text{with probabaility } \theta_1\theta_2\cdots\theta_{k-2}(1-\theta_{k-1}) \\ \sum_{i=1}^{k} B_i & \text{with probabaility } \theta_1\theta_2\cdots\theta_{k-1} \end{cases} \quad (2.1)$$

where $\theta_0 = 1$ and $\theta_k = 0$. Then the LST of B is given by

$$B^*(s) = \sum_{j=1}^{k}\left(\prod_{i=1}^{j}\theta_{i-1}B_i^*(s)\right)(1-\theta_j) \quad (2.2)$$

and

$$E(B) = \sum_{j=1}^{k}\left(\prod_{i=1}^{j}\theta_{i-1}\right)E(B_j) \quad (2.3)$$

As we know, in the queueing systems the utilization factor is $\lambda E(B)$ and denoted by ρ. This measure says if the system is in steady state or not. In this article, we study the model in equiblirium. It happens when $\rho<1$.

Definition 2.2. The elapsed of i-th phases service $(ps)_i$ at time "t" is denoted by $B_i^0(t)$ for $i=1,2,\cdots,k$. We introduce the random variable $Y(t)$ as follow

$$Y(t) = \begin{cases} 0 & \text{when the servser is idle at time t.} \\ i & \text{when the server is in the i-th phase at time t} \end{cases}$$

Thus, we have a bivariate Markov process $\{N_Q(t), L(t)\}$, where $L(t)=0$ if $Y(t)=0$ and $L(t)=B_i^0(t)$ if $Y(t)=i$, for $i=1,2,\cdots,k$. Now, we define probabilities as

$$P_{i,n}(x,t) = P[N_Q(t)=n, L(t) = B_i^0(t); x < B_i^0(t) \le x+\mathrm{d}x],$$
$$x>0,\ n\ge 0$$

for $i=1,2,\cdots,k$ and

$$R_0(t) = P\left[N_Q(t)=0, L(t)=0\right]$$

We know that $B_i(0)=0$ and $B_i(\infty)=1$, for $i=1,2,\cdots,k$. Also $B_i(x)$ is continuous at $x=0$. Then we have the hazard rate functions of B_i as

$$\mu_i(x)\mathrm{d}x = \frac{\mathrm{d}B_i(x)}{1-B_i(x)} \quad (2.4)$$

where $i=1,2,\cdots,k$, and $\mu_i(x)$ is the conditional probability of completion of i-th phase of service during the time interval $(x, x+\mathrm{d}x)$, given that the elapsed service time is x.

By assuming that the system is in the steady state, we let

$$R_0 = \lim_{t\to\infty} R_0(t)$$
$$P_{i,n}(x)\mathrm{d}x = \lim_{t\to\infty} P_{i,n}(x,t)\mathrm{d}x,\ x>0,\ n>0 \quad (2.5)$$

for $i=1,2,\cdots,k$.

In the next section, we find the PGF of these probabilities.

3. The PGF of the System Size

Now, for $i=1,2,\cdots,k$, the PGF of the probabilities that explained by (2-5), are defined as

$$P_i(x,z) = \sum_{n=0}^{\infty} z^n P_{i,n}(x)\ |z|\le 1,\ x>0 \quad (3.1)$$

$$P_i(0,z) = \sum_{n=0}^{\infty} z^n P_{i,n}(0)\ \ |z|\le 1 \quad (3.2)$$

For finding the steady-state PGF from Kolmogorov forward equations, for $i=1,2,\cdots,k$, we can write the steady-state equations as

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{i,n}(x) + [\lambda+\mu_i(x)]P_{i,n}(x) = \lambda P_{i,n-1}(x) \quad (3.3)$$
$$n\ge 1,\ x>0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{i,0}(x) + [\lambda+\mu_i(x)]P_{i,0}(x) = 0 \quad (3.4)$$

Also

$$\lambda R_0 = \sum_{i=0}^{k}(1-\theta_i)q_i\int_0^{\infty}\mu_i(x)P_{i,0}(x)\mathrm{d}x \quad (3.5)$$

It is clear that $P_{i,-1}=0$, for $i=1,2,\cdots,k$. Now, at $x=0$, the boundary conditions are

$$P_{1,0}(0) = \lambda R_0 + \sum_{i=1}^{k}(1-\theta_i)\left\{p_i\int_0^{\infty}\mu_i(x)P_{i,0}(x)\mathrm{d}x + q_i\int_0^{\infty}\mu_i(x)P_{i,1}(x)\mathrm{d}x\right\} \quad (3.6)$$

$$P_{1,n}(0) = \sum_{i=1}^{k}(1-\theta_i)\left\{p_i\int_0^{\infty}\mu_i(x)P_{i,n}(x)\mathrm{d}x + q_i\int_0^{\infty}\mu_i(x)P_{i,n+1}(x)\mathrm{d}x\right\} \quad (3.7)$$

$$P_{i,n}(0) = \theta_{i-1}\int_0^{\infty}\mu_{i-1}(x)P_{i-1,n}\mathrm{d}x \quad (3.8)$$

Note that, the normalizing condition is

$$R_0+\sum_{i=1}^{k}\sum_{n=0}^{\infty}\int_0^{\infty}P_{i,n}(x)\mathrm{d}x=1$$

which we use this condition to find the relation between R_0 and $P_i(1)$.

In the next Lemma, we derive the relation between $P_i(x,z)$ and $P_i(0,z)$.

Lemma 3.1. From relation (3-3), we have

$$P_i(x,z)=P_i(0,z)\left[1-B_i(x)\right]e^{-\lambda(1-z)x}\quad x>0 \tag{3.9}$$

Proof. See [4]. □

Proposition 3.2. By the z-transform of B_i that is

$$B_i^*(\lambda-\lambda z)=\int_0^{\infty}e^{-\lambda(1-z)}\mathrm{d}B_i(x)$$

we have

$$zP_1(0,z)=\lambda R_0(z-1)+\sum_{i=1}^{k}P_i(0,z)(1-\theta_i)(zp_i+q_i)B_i^*(\lambda-\lambda z) \tag{3.10}$$

Proof. By multiplying the relation (3.7) in z^n and summation from $n=1$ to ∞, and using (3.6), the proof is completed. □

Proposition 3.3. For $i=1,\cdots,k$, we have

$$P_i(0,z)=\theta_{i-1}P_{i-1}(0,z)B_{i-1}^*(\lambda-\lambda z) \tag{3.11}$$

Proof. By multiplying (3.8) in z^n and summation on $n=0$ to ∞, we can obtain (3.11). □

Corollary 3.4. By proposition 3.3, we have

$$P_i(0,z)=P_1(0,z)A_{i-1}^*(\lambda-\lambda z) \tag{3.12}$$

where

$$A_j^*(\lambda-\lambda z)=\prod_{l=1}^{j}B_l^*(\lambda-\lambda z) \tag{3.13}$$

Now, by (3.10) and (3.13), we obtain

$$P_1(0,z)=\frac{\lambda(z-1)R_0}{z-\sum_{i=1}^{k}\left\{\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(\lambda-\lambda z)(zp_i+q_i)(1-\theta_i)\right)} \tag{3.14}$$

Corollary 3.5. If $P_i(z)=\int_0^{\infty}P_i(x,z)\mathrm{d}x$, for $i=1,\cdots,k$, we have

$$P_1(z)=\frac{R_0\left[1-B_1^*(\lambda-\lambda z)\right]}{\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(\lambda-\lambda z)(zp_i+q_i)(1-\theta_i)\right]-z} \tag{3.15}$$

and for $i=2,\cdots,k$

$$P_i(z)=\frac{R_0\left[1-B_i^*(\lambda-\lambda z)\right]A_{i-1}^*(\lambda-\lambda z)\left(\prod_{l=0}^{i-1}\theta_l\right)}{\sum_{i=1}^{k}\left[\left(\left(\prod_{l=0}^{i-1}\theta_l\right)\right)A_i^*(\lambda-\lambda z)(zp_i+q_i)(1-\theta_i)\right]-z} \tag{3.16}$$

Now, if $A_0^*=1$, $\theta_0=1$ and $P(z)=\sum_{i=1}^{k}P_i(z)$ is the PGF of queue size distribution at a random epoch, then from (3.15) and (3.16), we have

$$P(z)=P_1(z)+\sum_{i=2}^{k}P_i(z)=\frac{R_0\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(\lambda-\lambda z)\right]A_{i-1}^*(\lambda-\lambda z)}{\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(\lambda-\lambda z)(zp_i+q_i)(1-\theta_i)\right]-z} \tag{3.17}$$

and the PGF of the queue size at the departure epoch is

$$P_s(z)=R_0+zP(z)=R_0\left[1+\frac{z\sum_{i=0}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(\lambda-\lambda z)\right]A_{i-1}^*(\lambda-\lambda z)}{\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(\lambda-\lambda z)(zp_i+q_i)(1-\theta_i)\right]-z}\right] \tag{3.18}$$

In the next section, we find some measures of effectiveness as the means system size, response time and busy period.

4. The Measures of Effectiveness

This section includes three sub-sections. In the sub-Section 4.1, we find the mean system size. In the sub-Section 4.2, the mean response time is obtained and in the last sub-section we calculate the mean busy period.

4.1. The Mean System Size

If L_Q be the mean number of customers in the queue, then we have

$$L_Q=\left.\frac{\mathrm{d}P_s(z)}{\mathrm{d}z}\right|z=1$$

For calculating L_Q we use the following lemma.

Lemma 4.1.1. By (3.13) and for $j=1,2,\cdots,k$, we have

$$1)\ \lim_{z\to1}\frac{d}{dz}A_j^*(\lambda-\lambda z)=\lambda\sum_{i=1}^{j}E(B_i) \tag{4.1.1}$$

$$2)\ \lim_{z\to1}\frac{\mathrm{d}^2}{\mathrm{d}^2z}A_j^*(\lambda-\lambda z)=\lambda^2\left\{\left[\sum_{i=1}^{j}E(B_i)\right]^2+\sum_{i=1}^{j}\mathrm{var}(B_i)\right\} \tag{4.1.2}$$

Now, we write $P_S(z)$ in form of $P_S(z)=R_0\left[\frac{f(z)}{g(z)}\right]$, where

$$f(z)=z\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(\lambda-\lambda z)\right]A_{i-1}^*(\lambda-\lambda z)+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(\lambda-\lambda z)(zp_i+q_i)(1-\theta_i)-z$$

and

$$g(z)=\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(\lambda-\lambda z)(zp_i+q_i)(1-\theta_i)-z$$

Since $\lim_{z\to 1} f(z)=\lim_{z\to 1} g(z)=0$, then by using the L'Hopital rule , we have

$$L_Q=R_0\frac{f''(1)g'(1)-f'(1)g''(1)}{2\left[g'(1)\right]^2}$$

where

$$\begin{aligned}h'(s)=&-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(s)\right]A_{i-1}^*(s)+(\lambda-s)\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(-\frac{\mathrm{d}}{\mathrm{d}s}B_i^*(s)\right)A_{i-1}^*(s)\\&+(\lambda-s)\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(s)\right]\frac{\mathrm{d}}{\mathrm{d}s}A_{i-1}^*(s)+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\frac{\mathrm{d}}{\mathrm{d}s}A_i^*(s)(\lambda-sp_i)(1-\theta_i)\\&-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(s)\,p_i(1-\theta_i)+1\end{aligned}$$

$$h'(0)=\lambda\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)E(B_i)-\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\sum_{j=1}^{i}E(B_j)\right)\cdot\lambda\cdot(1-\theta_i)\right]-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\cdot p_i(1-\theta_i)+1$$

$$\begin{aligned}f''(z)=&\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[-\frac{\mathrm{d}}{\mathrm{d}z}B_i^*(\lambda-\lambda z)\right]A_{i-1}^*(\lambda-\lambda z)+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(\lambda-\lambda z)\right]\frac{\mathrm{d}}{\mathrm{d}z}A_{i-1}^*(\lambda-\lambda z)\\&+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[-\frac{\mathrm{d}}{\mathrm{d}z}B_i^*(\lambda-\lambda z)\right]A_{i-1}^*(\lambda-\lambda z)+z\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2}B_i^*(\lambda-\lambda z)\right]A_{i-1}^*(\lambda-\lambda z)\\&+z\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[-\frac{\mathrm{d}}{\mathrm{d}z}B_i^*(\lambda-\lambda z)\right]\frac{\mathrm{d}}{\mathrm{d}z}A_{i-1}^*(\lambda-\lambda z)+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(\lambda-\lambda z)\right]\frac{\mathrm{d}}{\mathrm{d}z}A_{i-1}^*(\lambda-\lambda z)\\&+z\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[-\frac{\mathrm{d}}{\mathrm{d}z}B_i^*(\lambda-\lambda z)\right]\frac{\mathrm{d}}{\mathrm{d}z}A_{i-1}^*(\lambda-\lambda z)+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(\lambda-\lambda z)\right]\frac{\mathrm{d}^2}{\mathrm{d}z^2}A_{i-1}^*(\lambda-\lambda z)\\&+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}A_i^*(\lambda-\lambda z)\right)(zp_i+q_i)(1-\theta_i)+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}}{\mathrm{d}z}A_i^*(\lambda-\lambda z)\right)p_i(1-\theta_i)\\&+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}}{\mathrm{d}z}A_i^*(\lambda-\lambda z)\right)p_i(1-\theta_i)\end{aligned}$$

$$\begin{aligned}f''(1)=&-2\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)E(B_i)-\lambda^2\sum_{i=1}^{k}\left(\prod_{l=0}\theta_l\right)E\left(B_i^2\right)-2\lambda^2\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[E(B_i)\sum_{j=1}^{i-1}E(B_j)\right]\\&+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\lambda^2\left(\left[\sum_{j=1}^{i}E(B_j)\right]^2+\sum_{j=1}^{i}\operatorname{var}(B_j)\right)(1-\theta_i)+2\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\lambda E(B_i)\right)p_i(1-\theta_i)\end{aligned}$$

$$g'(z)=\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}}{\mathrm{d}z}A_i^*(\lambda-\lambda z)\right)(zp_i+q_i)(1-\theta_i)+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(\lambda-\lambda z)\,p_i(1-\theta_i)-1$$

$$g'(1)=\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\lambda\sum_{j=1}^{i}E(B_j)\right)(1-\theta_i)+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)p_i(1-\theta_i)-1$$

$$g''(z)=\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}A_i^*(\lambda-\lambda z)\right)(zp_i+q_i)(1-\theta_i)+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}}{\mathrm{d}z}A_i^*(\lambda-\lambda z)\right)p_i(1-\theta_i)$$
$$+\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}}{\mathrm{d}z}A_i^*(\lambda-\lambda z)\right)p_i(1-\theta_i)$$

$$g''(1)=\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\lambda^2\left(\left[\sum_{j=1}^{i}E(B_j)\right]^2+\sum_{j=1}^{i}\mathrm{var}(B_j)\right)(1-\theta_i)+2\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)(\lambda E(B_i))p_i(1-\theta_i)$$

hence $L_Q=R_0\dfrac{L_1}{L_2}$, in which

$$L_1=\left\{-2\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda E(B_i)\right)-\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda^2E(B_i^2)\right)\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda^2E(B_i)\left[\sum_{j=0}^{i-1}E(B_j)\right]\right)\right.$$
$$+\left\{\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda^2\left[\left[\sum_{j=0}^{i}E(B_j)\right]^2+\left[\sum_{j=0}^{i}\mathrm{var}(B_j)\right]\right](1-\theta_i)\right)+2\lambda\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i}\theta_l\right]p_i(1-\theta_i)\left[\sum_{j=0}^{i-1}E(B_j)\right]\right)\right\}$$
$$*\left\{\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda\left(\sum_{j=0}^{i}E(B_j)\right)(1-\theta_i)\right)+\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]p_i(1-\theta_i)\right)-1\right\}$$
$$-\left\{\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda E(B_j)\right)+\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda\left(\sum_{j=0}^{i}E(B_j)\right)(1-\theta_i)\right)+\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]p_i(1-\theta_i)\right)-1\right\}$$
$$*\left\{\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda^2\left[\left[\sum_{j=0}^{i}E(B_j)\right]^2+\left[\sum_{j=0}^{i}\mathrm{var}(B_j)\right]\right](1-\theta_i)\right)+2\lambda\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]p_i(1-\theta_i)\left[\sum_{j=0}^{i}E(B_j)\right]\right)\right\}$$

and

$$L_2=2\left\{\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda\left[\sum_{j=1}^{i}E(B_j)\right](1-\theta_i)\right)+\left(\sum_{i=1}^{k}\prod_{l=0}^{i-1}\theta_l]p_i(1-\theta_i)\right)-1\right\}^2$$

4.2. The Mean Response Time

Here, we show the response time variable with W_R. For finding the mean of W_R, first we need to obtain the LST of the distribution of waiting time in the queue, then by using this, we find the LST of the response time distribution. Then we can find the mean response time.

From classical formula in the queueing theory, we know that $W_Q^*(\lambda-\lambda z)B^*(\lambda-\lambda z)=P_S(z)$, where

$$B^*(\lambda-\lambda z)=\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)(1-\theta_i)A_{i-1}^*(\lambda-\lambda z)$$

Now, if we put $s=\lambda(1-z)$ and using the (3.18), we have

$$W_Q^*(s)=\frac{P_S\left(1-\frac{s}{\lambda}\right)}{B^*(s)}=R_0\left[1+\frac{\left(\frac{\lambda-s}{\lambda}\right)\sum_{i=0}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(s)\right]A_{i-1}^*(s)}{\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(s)\left(p_i-\frac{p_is}{\lambda}+q_i\right)(1-\theta_i)\right]-\left(\frac{\lambda-s}{\lambda}\right)}\right]\Bigg/B^*(s)$$
$$=R_0\left[1+\frac{(\lambda-s)\sum_{i=0}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(s)\right]A_{i-1}^*(s)}{\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(s)(\lambda-p_is)(1-\theta_i)\right]-(\lambda-s)}\right]\Bigg/B^*(s)$$

On the other hand, the response time is defined by $W_R=T_Q+B$, where T_Q is queueing time and B is the service time. Then, by using the convolution property, the LST of W_R is

$$W_R^*(S) = W_Q^*(S)\cdot B^*(S) = P_S\left(1-\frac{S}{\lambda}\right)$$

in which by some simple computation we find that

$$W_R^*(s) = R_0\left[\frac{h(s)}{k(s)}\right]$$

where

$$h(s) = (\lambda - s)\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(s)\right]A_{i-1}^*(s) + \sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(s)(\lambda - sp_i)(1-\theta_i) - (\lambda - s)$$

and

$$k(s) = \sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(s)(\lambda - sp_i)(1-\theta_i) - (\lambda - s)$$

The mean response time, for an arbitrary customer, is

$$E(W_R) = -\left.\frac{\mathrm{d}W_R^*(S)}{\mathrm{d}S}\right|_{S=0}$$

But $h(0) = k(0) = 0$, then, by using the L'Hopital rule, we have

$$E(W_R) = -R_0\frac{h''(0)k'(0) - k''(0)h'(0)}{2\left[k'(0)\right]^2} \tag{4.2.1}$$

On the other hand, $\frac{\mathrm{d}}{\mathrm{d}s}B_i^*(s) = -E(B_i)$ and $\frac{\mathrm{d}^2}{\mathrm{d}s^2}B_i^*(s) = E(B_i^2)$, then

$$\begin{aligned}
h'(s) = &-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(s)\right]A_{i-1}^*(s) + (\lambda - s)\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(-\frac{\mathrm{d}}{\mathrm{d}s}B_i^*(s)\right)A_{i-1}^*(s) \\
&+(\lambda - s)\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(s)\right]\frac{\mathrm{d}}{\mathrm{d}s}A_{i-1}^*(s) + \sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\frac{\mathrm{d}}{\mathrm{d}s}A_i^*(s)(\lambda - sp_i)(1-\theta_i) \\
&-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(s)\,p_i(1-\theta_i) + 1,
\end{aligned}$$

$$h'(0) = \lambda\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)E(B_i) - \sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\sum_{j=1}^{i}E(B_j)\right)\cdot\lambda\cdot(1-\theta_i)\right] - \sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\cdot p_i(1-\theta_i) + 1,$$

$$\begin{aligned}
h''(s) = &-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[-\frac{\mathrm{d}}{\mathrm{d}s}B_i^*(s)\right]A_{i-1}^*(s) - \sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(s)\right]\frac{\mathrm{d}}{\mathrm{d}s}A_{i-1}^*(s) \\
&-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(-\frac{\mathrm{d}}{\mathrm{d}}B_i^*(s)\right)A_{i-1}^*(s) + (\lambda - s)\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(-\frac{\mathrm{d}^2}{\mathrm{d}s^2}B_i^*(s)\right)A_{i-1}^*(s) \\
&+(\lambda - s)\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(-\frac{\mathrm{d}}{\mathrm{d}s}B_i^*(s)\right)\frac{\mathrm{d}}{\mathrm{d}s}A_{i-1}^*(s) - \sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(s)\right]\frac{\mathrm{d}}{\mathrm{d}s}A_{i-1}^*(s) \\
&+(\lambda - s)\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left(-\frac{\mathrm{d}}{\mathrm{d}s}B_i^*(s)\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}A_{i-1}^*(s)\right) + (\lambda - s)\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\left[1-B_i^*(s)\right]\frac{\mathrm{d}^2}{\mathrm{d}s^2}A_{i-1}^*(s) \\
&+\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}A_i^*(s)\right)(\lambda - sp_i)(1-\theta_i)\right] - \sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}A_i^*(s)\right).p_i\,(1-\theta_i)\right] \\
&-\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}A_i^*(s)\right)\cdot p_i\cdot(1-\theta_i)\right],
\end{aligned}$$

$$h''(0)=-2\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)E(B_i)-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\lambda E\left(B_i^2\right)+2\lambda\sum_{i=1}^{k}[\left(\prod_{l=0}^{i-1}\theta_l\right)E(B_i)\left[\sum_{j=0}^{i-1}E(B_j)\right]$$

$$+\lambda\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)E(B_i)\left[\left(\sum_{j=1}^{i}E(B_j)\right)^2+\sum_{j=1}^{i}\operatorname{var}(B_j)(1-\theta_i)\right]$$

$$+2\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)p_i(1-\theta_i)\left[\sum_{j=1}^{i}E(B_j)\right],$$

$$k'(s)=\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\frac{\mathrm{d}}{\mathrm{d}s}A_i^*(s)(\lambda-sp_i)(1-\theta_i)-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)A_i^*(s)\,p_i(1-\theta_i)+1$$

$$k'(0)=-\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\sum_{j=1}^{i}E(B_j)\right)\cdot\lambda\cdot(1-\theta_i)\right]-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\cdot p_i(1-\theta_i)+1,$$

$$k''(s)=\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}A_i^*(s)\right)(\lambda-sp_i)(1-\theta_i)\right]-\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}A_i^*(s)\right).p_i\,(1-\theta_i)\right]$$

$$-\sum_{i=1}^{k}[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\frac{\mathrm{d}}{\mathrm{d}s}A_i^*(s)\right)\cdot p_i\cdot(1-\theta_i)],$$

$$k''(0)=\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\left[\sum_{j=1}^{i}E(B_j)\right]^2+\sum_{j=1}^{i}\operatorname{var}(B_j)\right)\cdot\lambda\cdot(1-\theta_i)\right]+\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\sum_{j=1}^{i}E(B_j)\right)\cdot p_i\cdot(1-\theta_i)\right]$$

$$+\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\sum_{j=1}^{i}E(B_j)\right)\cdot p_i\cdot(1-\theta_i)\right].$$

with replacing in (4.2.6), it results that

$$E(W_R)=-R_0\frac{W_1}{W_2}$$

where

$$\begin{aligned}W_1=&\left\{-2\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]E(B_i)\right)-\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda E\left(B_i^2\right)\right)-2\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda E(B_i)\left[\sum_{j=0}^{i-1}E(B_j)\right]\right)\right.\\
&+\left(\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left[\sum_{j=1}^{i}E(B_j)\right]^2+\sum_{j=1}^{i}\operatorname{var}(B_j)\cdot\lambda\cdot(1-\theta_i)\right)\right]\right)+2\left.\left(\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\sum_{j=1}^{i}E(B_j)\right)\cdot p_i\cdot(1-\theta_i)\right]\right)\right\}\\
&*\left\{-\left(\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\sum_{j=1}^{i}E(B_j)\right)\cdot p_i\cdot(1-\theta_i)\right]\right)-\left(\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\cdot p_i(1-\theta_i)\right)+1\right\}\\
&-\left\{\left(\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\left[\sum_{j=1}^{i}E(B_j)\right]^2+\sum_{j=1}^{i}\operatorname{var}(B_j)\right)\cdot\lambda\cdot(1-\theta_i)\right]\right)+2\left(\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\sum_{j=1}^{i}E(B_j)\right)\cdot p_i\cdot(1-\theta_i)\right]\right)\right\}\\
&*\left\{\left(\sum_{i=1}^{k}\left[\prod_{l=0}^{i-1}\theta_l\right]\lambda E(B_i)\right)-\left(\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\sum_{j=1}^{i}E(B_j)\right)\cdot(1-\theta_i)\right]\right)-\left(\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\cdot p_i(1-\theta_i)\right)+1\right\}\end{aligned}$$

and

$$W_2=2\left(-\sum_{i=1}^{k}\left[\left(\prod_{l=0}^{i-1}\theta_l\right)\left(\sum_{j=1}^{i}E(B_j)\right)\cdot\lambda\cdot(1-\theta_i)\right]-\sum_{i=1}^{k}\left(\prod_{l=0}^{i-1}\theta_l\right)\cdot p_i(1-\theta_i)+1\right)^2$$

4.3. The Mean Busy Period

In this section, according to the definition of the busy period, we obtain the mean busy period of the model that we have studied here. Suppose that, T_b and T_0 are the busy and idle periods, respectively. Now, according to the renewal theory, these variables are renewal processes and we have

$$\Pr(T_b) = \frac{E(T_b)}{E(T_0) - E(T_b)}$$

From this, it results

$$\frac{E(T_0) + E(T_b)}{E(T_b)} = \frac{1}{\Pr(T_b)}$$

or

$$\frac{E(T_0)}{E(T_b)} + 1 = \frac{1}{\Pr(T_b)}$$

or

$$\frac{E(T_0)}{E(T_b)} = \frac{1}{\Pr(T_b)} - 1 = \frac{1 - \Pr(T_b)}{\Pr(T_b)}$$

thus

$$\frac{E(T_b)}{E(T_0)} = \frac{\Pr(T_b)}{1 - \Pr(T_b)}$$

Now, suppose that $P_i^{(b)}$ is the probability that the server is servicing in i-th phase that is P [the server is busy with $(ps)_i$] $= P_i(1) = \lambda \frac{\theta_{i-1} E(B_i)}{q_i}$, for $1 \le i \le k$

we have

$$\Pr(T_b) = \sum_{i=1}^{k} p_i^{(b)} = \lambda \sum_{i=0}^{k-1} \frac{\theta_i E(B_{i+1})}{q_i}$$

Also, we know $E(T_0) = \frac{1}{\lambda}$, then

$$E(T_b) = E(T_0) \frac{\Pr(T_b)}{1 - \Pr(T_b)} = \frac{1}{\lambda} \times \frac{\lambda \sum_{i=0}^{k-1} \frac{\theta_i E(B_{i+1})}{q_i}}{1 - \sum_{i=0}^{k-1} \frac{\theta_i E(B_{i+1})}{q_i}}$$

4.4. Special Cases

In this article we have obtained some results for an $M/G/1$ queue with k-phases of heterogeneous services and Bernoulli feedback design. Now, we consider some special cases of this model that are agreement with the models which have been studied by [8,10]. These special cases are followed.

Special case 4.4.1. If $\theta_0 = \theta_1 = \cdots = \theta_{k-2} = 1$, $\theta_{k-1} = \theta$ and the last phase be the vacation for server and without feedback, then $P_S(z)$, L and $E(T_b)$ are the same as those have been obtained by [10].

Special case 4.4.2. If $\theta_0 = 1, \theta_1 = \theta, \theta_2 = \cdots = \theta_{k-1} = 0$ and $p_1 = p_2 = p$, then $P_S(z)$, L, $E(T_b)$ and $E(W_R)$ are equal to the [4].

5. Conclusion

In this paper, we considered an $M/G/1$ queueing model with single server, Poisson input, k-phases of heterogeneous services and Bernoulli feedback design. In this model, as soon as the i-th (for $i = 1, \cdots, k - 1$) phase of service of a customer is completed , it may leave the system or immediately go for $(i + 1)$-th phase of optional service. However, after receiving each phase of unsuccessful service by a unit, then it may immediately join to the end of tail of the original queue as feedback customer to take service again. We analyzed this mode via obtaining the steady-state probability generating function (PGF) of queue size at the random epoch and at the service completion epoch. Then, we derived the Laplace-Stieltjes Transform (LST) of the distribution of response time, the means of response time, number of customers in the system and busy period-model, the server provides first phase of regular service to all the customers.

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