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The Properties and Fast Algorithm of Quaternion Linear Canonical Transform

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Abstract

The quaternion linear canonical transform (QLCT) is defined in this paper, with proofs given for its reversibility property, its linear property, its odd-even invariant property and additivity property. Meanwhile, the quaternion convolution (QCV), quaternion correlation (QCR) and product theorem of LCT are deduced. Their physical interpretation is given as classical convolution, correlation and product theorem. Moreover, the fast algorithm of QLCT (FQLCT) is obtained, whose calculation complexity for different signals is similar to FFT. In addition, the paper presents the relationship between the convolution and correlation in LCT domains, and the convolution and correlation can be calculated via product theorem in Fourier transform domain using FFT.

Keywords

Quaternion Signals (Hyper-Complex Signals), LCT, Convolution, Correlation

1. Introduction

The linear canonical transform (LCT) is a new tool that comes into being in signal processing [1]-[32]. The LCT is the generalization of the FRFT and so on [2] [3] [12]. Up till now there have been a lot of papers involving the FRFT and the LCT, such as papers [1]-[10]. However, none of them has involved the LCT of quaternion signals (or Hyper-complex signals) even if there has been similar work on FRFT [5]. Quaternion signals can be taken as the generalization of scalar, complex signals and vector, and after the introduction of quaternion signals by Hamilton in 1843 [11] it has become one basic tool for multi-channel and multi-dimensional space. For example, grey image [30] can be taken as scalar, and the analytic signal after Hilbert transformation [12] [13] [14] [15] [16] [29] is complex signal. The color image can be taken as one vector [17] [18], a quaternion number whose real part is zero. In [19], the transform, convolution and correlation have been addressed in fractional Fourier transform (FRFT) domain. In this paper we first propose the definition of the QLCT, QCV and QCR in the LCT domain for quaternion signals, which are the generalization of those in [5]. Meanwhile, some properties and the fast algorithm of QLCT are discussed. We also discover the relationship of QCV and QCR in the LCT domain for quaternion signals. We found that QCV and QCR can be implemented via product theorem in the QLCT domain. Thus we not only yield the generalized frame for scalar, complex signal, vector and quaternion signal [17] [20] in the QLCT domain, but also give one new idea and one theoretical base for future engineering use.

In the rest of this paper, we will introduce the definition of QLCT in Section II. We will show the properties in Section III. In Section IV, FRQCV and FRQCR will be addressed. Section V is the fast algorithm. The last section concludes our paper.

2. Definitions of QLCT

For convenience of discussion, we first give some notations used in the following of this paper. $f(x,y)$ denotes 2D signal in time domain; F is classical Fourier transform operator; $F^{L(a,b,c,d)}$ (F^{L} in short) is 1D LCT operator, and $F^{L}(u)$ is the 1D LCT of $f(x,y)$; F^{L_1,L_2} is the 2D LCT operator of $f(x,y)$; F^{Q} is classical quaternion Fourier transform operator, and $F^{Q}(u,v)$ is quaternion Fourier transform of $f(x,y)$; I is equivalence operator; P is odd-even operator; "*" is classical convolution operator; "$^{-}$" is conjugation operator. "N" is integer set; "R" is real set. Define the product operator of two LCTs' transform parameter systems:

$$L_1L_2 = L_1(a_1,b_1,c_1,d_1)\cdot L_2(a_2,b_2,c_2,d_2) = L(a,b,c,d)$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$. Quaternion signals are also called Hypercomplex signals, which are the generalization of complex signals. Complex signals have two components: the real part and the imaginary part. However, one quaternion signal has four parts, one real component and three imaginary parts:

$$q = q_r + iq_i + jq_j + kq_k \tag{1}$$

where $q_r, q_i, q_j, q_k \in R$, i, j, k are three imaginary units, which satisfy the following relations: $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. $q_a = q_r + iq_i$, $q_b = q_i + iq_k$. If $q_r = 0$, then $q = iq_i + jq_j + kq_k$ is called vector, and q_r is called scalar. q_a and q_b are complex signals. Since the sequences of i, j and k will affect the result, the definition of QLCT would take them into account.

Definition 1: For any quaternion signal
$f(x,y) = f_r(x,y) + if_i(x,y) + jf_j(x,y) + kf_k(x,y)$ ($f_r(x,y)$, $f_i(x,y)$,

$f_j(x,y)$, $f_k(x,y)$ are real ones), the QLCT of $f(x,y)$ is $F_{i,j}^{L_1,L_2}(u,v)$

$$\begin{aligned} F_{i,j}^{L_1,L_2}(u,v) &= F_{i,j}^{L_1,L_2}\{f(x,y)\} = F_{i,j}^{L_1,L_2}\{f(x,y)\}(u,v) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_{L_1,i}(x,u)\, f(x,y)\, K_{L_2,j}(y,v)\,\mathrm{d}x\mathrm{d}y \end{aligned} \tag{2-1}$$

where, $K_{L_1,i}(x,u) = \sqrt{\frac{1}{2\pi b_1 i}} \exp\left(i\frac{(a_1+d_1)u^2}{2b_1} - i\frac{ux}{b_1} \right)$,

$K_{L_2,j}(y,v) = \sqrt{\frac{1}{2\pi b_2 j}} \exp\left(j\frac{(a_2+d_2)v^2}{2b_2} - j\frac{vy}{b_2} \right)$. Meanwhile, in the following of this paper we assume $a_1d_1 - b_1c_1 = 1$, $a_2d_2 - b_2c_2 = 1$ and $b_1, b_2 \neq 0$.

The reversibility transform is defined as

$$\begin{aligned} F_{i,j}^{-L_1,-L_2}(u,v) &= F_{i,j}^{-L_1,-L_2}\{f(x,y)\} = F_{i,j}^{-L_1,-L_2}\{f(x,y)\}(u,v) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_{-L_1,i}(x,u)\, f(x,y)\, K_{-L_2,j}(y,v)\,\mathrm{d}x\mathrm{d}y \end{aligned} \tag{2-2}$$

where, $K_{-L_1,i}(x,u) = \sqrt{\frac{1}{-2\pi b_1 i}} \exp\left(-i\frac{(a_1+d_1)u^2}{2b_1} + i\frac{ux}{b_1} \right)$,

$K_{-L_2,j}(y,v) = \sqrt{\frac{1}{-2\pi b_2 j}} \exp\left(-j\frac{(a_2+d_2)v^2}{2b_2} + j\frac{vy}{b_2} \right)$.

If $(a_1,b_1,c_1,d_1) = (a_2,b_2,c_2,d_2) = (0,-1,1,0)$, definition 1 is quaternion Fourier transform; if $(a_1,b_1,c_1,d_1) = (0,-1,1,0), (a_2,b_2,c_2,d_2) = (1,0,0,1)$, definition 1 is classical 1D Fourier transform of $f(x,y)$ for variable *x*; if $(a_1,b_1,c_1,d_1) = (1,0,0,1)$, $(a_2,b_2,c_2,d_2) = (0,-1,1,0)$, definition 1 is classical 1D Fourier transform of $f(x,y)$ for variable *y*; if $(a_1,b_1,c_1,d_1) = (a_2,b_2,c_2,d_2) = (1,0,0,1)$, definition 1 is equivalence transform of $f(x,y)$. As shown above, definition 1 is the generalization of the fractional quaternion Fourier transform and the quaternion Fourier transform [18] [19] [20] [21]. The reversibility (or reconstruction) is one important property for one transform, especially for the processing in another domain. The following gives the proof of the reversibility property.

Theorem 1: One quaternion $f(x,y)$ can be reconstructed from $F_{i,j}^{L_1,L_2}(u,v)$ via QLCT.

Proof: The proof is trivial and omitted here.

3. The Properties of QLCT

In the following section we list the properties and present the proof.

Property 1: For any one quaternion signal $f_n(x,y)(n \in \aleph)$, the following relationship is true: $F_{i,j}^{L_1,L_2}\{\sum a_n f_n(x,y)\} = \sum a_n \cdot F_{i,j}^{L_1,L_2}\{f_n(x,y)\}$ $(a_n \in \Re)$.

Proof: Since QLCT is one linear transform, property 1 can be easily obtained from definition 1.

Property 2: $F_{i,j}^{L_3,L_4} F_{i,j}^{L_1,L_2} = F_{i,j}^{L_1,L_2} F_{i,j}^{L_3,L_4} = F_{i,j}^{L_1L_3,L_2L_4}$. Proof: For any one quaternion signal $f(x,y)$, from definition 1 we can obtain

$$F_{i,j}^{L_3,L_4}F_{i,j}^{L_1,L_2}\{f(x,y)\}$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}K_{L_3,i}(u,s)\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}K_{L_1,i}(x,u)f(x,y)K_{L_2,j}(y,v)\mathrm{d}x\mathrm{d}y\right\}K_{L_4,j}(v,w)\mathrm{d}u\mathrm{d}v$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}K_{L_3,i}(u,s)K_{L_1,i}(x,u)f(x,y)K_{L_2,j}(y,v)K_{L_4,j}(v,w)\mathrm{d}u\mathrm{d}v\right\}\mathrm{d}x\mathrm{d}y$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\left[\int_{-\infty}^{+\infty}K_{L_3,i}(u,s)K_{L_1,i}(x,u)\mathrm{d}u\right]f(x,y)\left[\int_{-\infty}^{+\infty}K_{L_2,j}(y,v)K_{L_4,j}(v,w)\mathrm{d}v\right]\right\}\mathrm{d}x\mathrm{d}y \tag{3}$$

For 1D signal the right formula is true [2]:

$$\int_{-\infty}^{+\infty}K_{L_2}(u,u')K_{L_1}(u',u'')\mathrm{d}u''=K_{L_2L_1}(u,u'') \tag{4}$$

Substitute (4) into (3):

$$F_{i,j}^{L_3,L_4}F_{i,j}^{L_1,L_2}\{f(x,y)\}$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{K_{L_3L_1,i}(x,s)f(x,y)K_{L_4L_2,j}(y,w)\}\mathrm{d}x\mathrm{d}y=F_{i,j}^{L_1L_3,L_2L_4}\{f(x,y)\}$$

Therefore,

$$F_{i,j}^{L_3,L_4}F_{i,j}^{L_1,L_2}=F_{i,j}^{L_1L_3,L_2L_4} \tag{5}$$

The result can be obtained similarly:

$$F_{i,j}^{L_1,L_2}F_{i,j}^{L_3,L_4}=F_{i,j}^{L_1L_3,L_2L_4} \tag{6}$$

From (5) (6): $F_{i,j}^{L_3,L_4}F_{i,j}^{L_1,L_2}=F_{i,j}^{L_1,L_2}F_{i,j}^{L_3,L_4}=F_{i,j}^{L_1L_3,L_2L_4}$

Property 3: $F_{i,j}^{L_3,L_4}F_{i,j}^{L_1,L_2}=F_{i,j}^{L_1,L_2}F_{i,j}^{L_3,L_4}$, $F_{i,j}^{L_5,L_6}\left(F_{i,j}^{L_3,L_4}F_{i,j}^{L_1,L_2}\right)=\left(F_{i,j}^{L_5,L_6}F_{i,j}^{L_3,L_4}\right)F_{i,j}^{L_1,L_2}$.

Proof: This property can be obtained from property 2.

Property 4: If $F_{i,j}^{L_1,L_2}\{f(x,y)\}=F_{i,j}^{L_1,L_2}(u,v)$, then $F_{i,j}^{L_1,L_2}\{f(-x,-y)\}=F_{i,j}^{L_1,L_2}(-u,-v)$, $F_{i,j}^{L_1,L_2}\{f(-x,y)\}=F_{i,j}^{L_1,L_2}(-u,v)$, $F_{i,j}^{L_1,L_2}\{f(x,-y)\}=F_{i,j}^{L_1,L_2}(u,-v)$.

Proof: Let $A_1=\sqrt{2\pi b_1 i}$, $A_2=\sqrt{2\pi b_2 j}$, $C_1=\dfrac{d_1}{2b_1}$, $C_2=\dfrac{d_2}{2b_2}$, and insert them into (2):

$$F_{i,j}^{L_1,L_2}\{f(-x,-y)\}=A_1\mathrm{e}^{iu^2C_1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{ux}{b_1}}\mathrm{e}^{ix^2C_1}f(-x,-y)\mathrm{e}^{jy^2C_1}\mathrm{e}^{-j\frac{vy}{b_2}}\mathrm{d}x\mathrm{d}y\cdot A_2\mathrm{e}^{jv^2C_2} \tag{7}$$

Let $s=-x,z=-y$, and substitute them in (7):

$$F_{i,j}^{L_1,L_2}\{f(-x,-y)\}=F_{i,j}^{L_1,L_2}\{f(s,z)\}$$
$$=A_1\mathrm{e}^{iu^2C_1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{u\cdot(-s)}{b_1}}\mathrm{e}^{is^2C_1}f(s,z)\mathrm{e}^{jz^2C_2}\mathrm{e}^{-j\frac{v\cdot(-z)}{b_2}}\mathrm{d}s\mathrm{d}z\cdot A_2\mathrm{e}^{jv^2C_2}$$
$$=A_1\mathrm{e}^{i(-u)^2C_1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{(-u)\cdot s}{b_1}}\mathrm{e}^{is^2C_1}f(s,z)\mathrm{e}^{jz^2C_2}\mathrm{e}^{-j\frac{(-v)\cdot z}{b_2}}\mathrm{d}s\mathrm{d}z\cdot A_2\mathrm{e}^{j(-v)^2C_2}$$
$$=F_{i,j}^{L_1,L_2}(-u,-v)$$

It can be obtained as well: $F_{i,j}^{L_1,L_2}\left\{f\left(-x,y\right)\right\}=F_{i,j}^{L_1,L_2}\left(-u,v\right)$ and $F_{i,j}^{L_1,L_2}\left\{f\left(x,-y\right)\right\}=F_{i,j}^{L_1,L_2}\left(u,-v\right)$.

We can draw the conclusion that transformed signal of the odd is odd, and even is even.

Property 5: If $n\in\aleph$, then $\left(F_{i,j}^{L_1,L_2}\right)^n=F_{i,j}^{(L_1)^n,(L_2)^n}$.

Proof: From property 2,

$$\underbrace{F_{i,j}^{L_1,L_2}F_{i,j}^{L_1,L_2}\cdots F_{i,j}^{L_1,L_2}}_{n}=F_{i,j}^{\overbrace{L_1\times\cdots\times L_1}^{n},\overbrace{L_2\times\cdots\times L_2}^{n}}=F_{i,j}^{(p_1)^n,(p_2)^n}$$

then

$$\left(F_{i,j}^{L_1,L_2}\right)^n=F_{i,j}^{(L_1)^n,(L_2)^n}.$$

QLCT doesn't satisfy Parseval's principle. Meanwhile, it is hard to find one obvious relationship between QLCT and Wigner-Ville time-frequency plane. Some other properties [2] cannot find physical interpretation in QLCT domains.

4. FRQCV and FRQCR

Convolution and correlation play an important role in signal processing, especially for linear system design and filter design, etc. The convolution in time domain is to the product in Fourier transform domain, that is to say, the classical convolution in time domain can be implemented in Fourier transform domain via FFT, which is beneficial for real-time engineering use. In classical time-frequency analysis correlation is special convolution in that the original signals are implemented via conjugation and so on. This is very important for engineering use [17] [20] [24]. The key to this paper is to discover the relationships in fractional quaternion Fourier transform domain between them so that we can find the physical interpretation as that of the classical Fourier transform. Paper [26] yielded fractional convolution and product theorem for 1D signals first, however, it didn't give the similar physical interpretation as that of the classical theorem. Later papers [27] [28] [29] obtained similar result as the classical theorems. However, they are only for 1D signals. In this section the QCV and QCR of the LCT would be discussed, and can be implemented via FFT.

4.1. Fractional Convolution and Product Theorem

In the following, four theorems are yielded, and theorem 2 and 3 are suitable for scalar and complex signals, and theorem 4 and 5 are suitable for scalar, complex signals, vector and quaternion signals.

Theorem 2: For any real scalar or complex signal $f\left(x,y\right)$ and convolution kernel $h\left(x,y\right)$,

$$\begin{aligned}&g\left(x,y\right)=f\left(x,y\right)\bar{*}h\left(x,y\right)=\left(f\bar{*}h\right)\left(x,y\right)\\&\triangleq A_{1,2}\mathrm{e}^{-i\left(x^2C_1+y^2C_2\right)}\left(\mathrm{e}^{i\left(x^2C_1+y^2C_2\right)}f\left(x,y\right)\right)*\left(h\left(x,y\right)\mathrm{e}^{i\left(x^2C_1+y^2C_2\right)}\right)\end{aligned}$$

where $B_1 = 1/b_1$, $B_2 = 1/b_2$, $A_{1,2} = \frac{1}{2\pi i\sqrt{b_1 b_2}}$, $C_1 = \frac{a_1}{2b_1}$, $C_2 = \frac{a_2}{2b_2}$, then:

$$F^{L_1,L_2}\{g(x,y)\} = \mathrm{e}^{-i(u^2C_1+v^2C_2)}F^{L_1,L_2}\{f(x,y)\}\cdot F^{L_1,L_2}\{h(x,y)\} \tag{8}$$

Proof:

$$\begin{aligned} F^{L_1,L_2}\{g(x,y)\} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K_{L_1,L_2}(x,y,u,v)\,g(x,y)\,\mathrm{d}x\mathrm{d}y \\ &= A_{1,2}\mathrm{e}^{i(u^2C_1+v^2C_2)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{i(x^2C_1+y^2C_2)}A_{1,2}\mathrm{e}^{-i(x^2C_1+y^2C_2)}\mathrm{e}^{-i(xuB_1+yvB_2)} \\ &\quad\cdot\left(\mathrm{e}^{i(x^2C_1+y^2C_2)}f(x,y)\right)*\left(h(x,y)\mathrm{e}^{i(x^2C_1+y^2C_2)}\right)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{9}$$

Substitute (9) with $s = x-\tau, z = y-\eta$:

$$\begin{aligned} F^{L_1,L_2}\{g(x,y)\} &= A_{1,2}^2\mathrm{e}^{i(u^2C_1+v^2C_2)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{i(\tau^2C_1+\eta^2C_2)}f(\tau,\eta)\mathrm{e}^{-i(\tau uB_1+\eta vB_2)} \\ &\quad\cdot\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(s,z)\mathrm{e}^{i(s^2C_1+z^2C_2)}\mathrm{e}^{-i(suB_1+zvB_2)}\mathrm{d}s\mathrm{d}z\right\}\mathrm{d}\tau\mathrm{d}\eta \\ &= A_{1,2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{i(\tau^2C_1+\eta^2C_2)}f(\tau,\eta)\mathrm{e}^{-i(\tau uB_1+\eta vB_2)}\mathrm{d}\tau\mathrm{d}\eta\cdot F^{L_1,L_2}\{h(x,y)\} \\ &= \mathrm{e}^{-i(u^2C_1+v^2C_2)}F^{L_1,L_2}\{f(x,y)\}\cdot F^{L_1,L_2}\{h(x,y)\} \end{aligned}$$

From theorem 2 it can be concluded that the convolution of scalar or complex signal is to the product, frequency-modulated by a chirp, of them in linear canonical transform.

Theorem 3: For any real scalar or complex signal $f(x,y)$ and convolution kernel $h(x,y)$,

$$\begin{aligned} g(x,y) &= \left(f \overset{=}{*} h\right)(x,y) \\ &\triangleq \left(\mathrm{e}^{i(x^2C_1+y^2C_2)}\right)\Big/2\pi\cdot\left(\mathrm{e}^{-i(x^2C_1+y^2C_2)}f(x,y)\right)*\left(h(x,y)\mathrm{e}^{-i(x^2C_1+y^2C_2)}\right) \end{aligned}$$

where $B_1 = 1/b_1$, $B_2 = 1/b_2$, $A_{1,2} = \frac{1}{2\pi i\sqrt{b_1 b_2}}$, $A_{1,2}^{-1,-1} = \frac{1}{2\pi i\sqrt{(-b_1)(-b_2)}}$,

$C_1 = \frac{a_1}{2b_1}$, $C_2 = \frac{a_2}{2b_2}$, then

$$\begin{aligned} &F^{L_1,L_2}\left\{\mathrm{e}^{i(u^2C_1+v^2C_2)}f(x,y)g(x,y)\right\} \\ &= A_{1,2}^{-1,-1}\left(f_{L_1,L_2}(u,v)\overset{=}{*}h_{L_1,L_2}(u,v)\right) \end{aligned} \tag{10}$$

Proof:

$$F^{-L_1,-L_2}\left(A_{1,2}^{-1,-1}f_{L_1,L_2}(u,v)\overset{=}{*}h_{L_1,L_2}(u,v)\right)$$

$$= \frac{A_{1,2}^{-1,-1}\mathrm{e}^{-i(x^2C_1+y^2C_2)}}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i(C_1u^2+C_2v^2)}\left\{A_{1,2}^{-1,-1}f_{L_1,L_2}(u,v)\overset{=}{*}h_{L_1,L_2}(u,v)\right\}\mathrm{e}^{i(xuB_1+yvB_2)}\mathrm{d}u\mathrm{d}v$$

Substitute $s = x-\tau, z = y-\eta$ in above equation

$$F^{-L_1,-L_2}\left\{A_{1,2}^{-1,-1} f_{L_1,L_2}(u,v)\overline{\overline{*}} h_{L_1,L_2}(u,v)\right\}$$
$$=\frac{\left(A_{1,2}^{-1,-1}\right)^2 e^{-i\left(x^2C_1+y^2C_2\right)}}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_{L_1,L_2}(s,z)e^{-i\left(C_1s^2+C_2z^2\right)}e^{i(xsB_1+yzB_2)}\mathrm{d}s\mathrm{d}z\right\}$$
$$\cdot e^{i(x\tau B_1+y\eta B_2)} f_{L_1,L_2}(\tau,\eta)e^{-i\left(C_1\tau^2+C_2\eta^2\right)}\mathrm{d}\tau\mathrm{d}\eta$$
$$=h(x,y)\frac{A_{1,2}^{-1,-1}}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(x\tau B_1+y\eta B_2)} f_{L_1,L_2}(\tau,\eta)e^{-i\left(C_1\tau^2+C_2\eta^2\right)}\mathrm{d}\tau\mathrm{d}\eta$$
$$=f(x,y)h(x,y)e^{-i\left(C_1x^2+C_2y^2\right)}$$

Therefore, $F^{L_1,L_2}\left\{e^{i\left(u^2C_1+v^2C_2\right)} f(x,y)g(x,y)\right\}=A_{1,2}^{-1,-1}\left(f_{L_1,L_2}(u,v)\overline{\overline{*}}\, h_{L_1,L_2}(u,v)\right)$.

Theorem 4: For one given quaternion function $f(x,y)=f_a(x,y)+f_b(x,y)j$ and convolution kernel function

$$h(x,y)=h_a(x,y)+h_b(x,y)j$$

where $f_a(x,y)=f_r(x,y)+if_i(x,y)$, $f_b(x,y)=f_j(x,y)+if_k(x,y)$, $h_a(x,y)=h_r(x,y)+ih_i(x,y)$, $h_b(x,y)=h_j(x,y)+ih_k(x,y)$.

Set $A_{1,2}=\frac{1}{2\pi i\sqrt{b_1b_2}}$, $A_{1,2}^{-1,-1}=\frac{1}{2\pi i\sqrt{(-b_1)(-b_2)}}$, and define

$$g(x,y)=(f\,\overline{*}\,h)(x,y)$$
$$\triangleq A_{1,2}e^{-i\left(x^2C_1+y^2C_2\right)}\left(e^{i\left(x^2C_1+y^2C_2\right)} f(x,y)\right)*\left(h(x,y)e^{i\left(x^2C_1+y^2C_2\right)}\right)$$

where, $C_1=\frac{a_1}{2b_1}$, $C_2=\frac{a_2}{2b_2}$, $B_1=1/b_1$, $B_2=1/b_2$, $\alpha=\arcsin b_1$, $\beta=\arcsin b_2$, then

$$\begin{aligned}F^{L_1,L_2}\{g(x,y)\}=e^{-i\left(u^2C_1+v^2C_2\right)}&\left\{F^{L_1,L_2}\left[f_a(x,y)\right]\cdot F^{L_1,L_2}\left[h_a(x,y)\right]\right.\\&\left.-F^{L_1,L_2}\left[f_b(x,y)\right]\cdot F^{L_1,L_2}\left[\overline{h_b}(x,y)\right]\right\}\\&+e^{i\left(u^2C_1+v^2C_2-\alpha-\beta\right)}\left\{F^{L_1,L_2}\left[f_a(x,y)\right]\cdot F^{-L_1,-L_2}\left[h_b(-x,-y)\right]\right.\\&\left.+F^{L_1,L_2}\left[f_b(x,y)\right]\cdot F^{-L_1,-L_2}\left[\overline{h_a}(-x,-y)\right]\right\}\cdot j\end{aligned}\tag{11}$$

Proof:

$$\left(e^{i\left(x^2C_1+y^2C_2\right)} f(x,y)\right)*\left(h(x,y)e^{i\left(x^2C_1+y^2C_2\right)}\right)$$
$$=\left(e^{i\left(x^2C_1+y^2C_2\right)}\left(f_a(x,y)+f_b(x,y)j\right)\right)*\left(\left(h_a(x,y)+h_b(x,y)j\right)e^{i\left(x^2C_1+y^2C_2\right)}\right)$$
$$=\left(e^{i\left(x^2C_1+y^2C_2\right)} f_a(x,y)\right)*\left(h_a(x,y)e^{i\left(x^2C_1+y^2C_2\right)}\right)$$
$$+\left(e^{i\left(x^2C_1+y^2C_2\right)} f_a(x,y)\right)*\left(h_b(x,y)e^{-i\left(x^2C_1+y^2C_2\right)}\right)\cdot j$$
$$+\left(e^{i\left(x^2C_1+y^2C_2\right)} f_b(x,y)\right)*\left(\overline{h_a}(x,y)e^{-i\left(x^2C_1+y^2C_2\right)}\right)\cdot j$$
$$-\left(e^{i\left(x^2C_1+y^2C_2\right)} f_b(x,y)\right)*\left(\overline{h_b}(x,y)e^{-i\left(x^2C_1+y^2C_2\right)}\right)$$

From theorem 2 it can be obtained

$$F^{L_1,L_2}\left\{A_{1,2}e^{-i\left(x^2C_1+y^2C_2\right)}\left(e^{i\left(x^2C_1+y^2C_2\right)}f_a(x,y)\right)*\left(h_a(x,y)e^{i\left(x^2C_1+y^2C_2\right)}\right)\right\}$$
$$=e^{-i\left(u^2C_1+v^2C_2\right)}\left(F^{L_1,L_2}\left\{f_a(x,y)\right\}\cdot F^{L_1,L_2}\left\{h_a(x,y)\right\}\right)$$
$$F^{L_1,L_2}\left\{A_{\alpha,\beta}e^{-i\left(x^2C_1+y^2C_2\right)}\left(e^{i\left(x^2C_1+y^2C_2\right)}f_b(x,y)\right)*\left(\overline{h_b}(x,y)e^{-i\left(x^2C_1+y^2C_2\right)}\right)\right\}$$
$$=e^{-i\left(u^2C_1+v^2C_2\right)}\left(F^{L_1,L_2}\left\{f_b(x,y)\right\}\cdot F^{L_1,L_2}\left\{\overline{h_b}(x,y)\right\}\right)$$

From the linear property of fractional Fourier transfor

$$\begin{aligned}&F^{L_1,L_2}\left\{g(x,y)\right\}\\&=e^{-i\left(u^2C_1+v^2C_2\right)}\left\{F^{L_1,L_2}\left[f_a(x,y)\right]\cdot F^{L_1,L_2}\left[h_a(x,y)\right]\right.\\&\quad\left.-F^{L_1,L_2}\left[f_b(x,y)\right]\cdot F^{L_1,L_2}\left[\overline{h_b}(x,y)\right]\right\}\\&\quad+e^{i\left(u^2C_1+v^2C_2-\alpha-\beta\right)}\left\{F^{L_1,L_2}\left[f_b(x,y)\right]\cdot F^{-L_1,-L_2}\left[h_b(-x,-y)\right]\right.\\&\quad\left.+F^{L_1,L_2}\left[f_b(x,y)\right]\cdot F^{-L_1,-L_2}\left[\overline{h_a}(-x,-y)\right]\right\}\cdot j\end{aligned}$$

From theorem 4 we draw the conclusion that the convolution of two quaternion signals is to the summation of product of their components, conjugated or odd-even operated, and the product is frequency modulated by chirps. Meanwhile, it must be noted that the orders of i and j in cannot be disordered.

Theorem 5: For any two quaternion signals

$$f(x,y)=f_a(x,y)+f_b(x,y)j \quad\text{and}\quad h(x,y)=h_a(x,y)+h_b(x,y)j$$

where $f_a(x,y)=f_r(x,y)+if_i(x,y)$, $f_b(x,y)=f_j(x,y)+if_k(x,y)$, $h_a(x,y)=h_r(x,y)+ih_i(x,y)$, $h_b(x,y)=h_j(x,y)+ih_k(x,y)$, set

$$A_{1,2}=\frac{1}{2\pi i\sqrt{b_1b_2}} \quad\text{and}\quad A_{1,2}^{-1,-1}=\frac{1}{2\pi i\sqrt{(-b_1)(-b_2)}},$$

$$g(x,y)=\left(f\,\overline{\overline{*}}\,h\right)(x,y)\triangleq\frac{e^{i\left(x^2C_1+y^2C_2\right)}}{2\pi}\left(e^{-i\left(x^2C_1+y^2C_2\right)}f(x,y)\right)*\left(h(x,y)e^{-i\left(x^2C_1+y^2C_2\right)}\right)$$

where $B_1=1/b_1$, $B_2=1/b_2$, $C_1=\frac{a_1}{2b_1}$, $C_2=\frac{a_2}{2b_2}$, then

$$\begin{aligned}&F^{L_1,L_2}\left\{e^{i\left(u^2C_1+v^2C_2\right)}f(x,y)g(x,y)\right\}\\&=A_{1,2}^{-1,-1}\left\{\left(f_a(x,y)\right)_{L_1,L_2}\overline{\overline{*}}\left(h_a(x,y)\right)_{L_1,L_2}\right.\\&\quad\left.-\left(f_b(x,y)\right)_{L_1,L_2}\overline{\overline{*}}\left(\overline{h_b}(x,y)\right)_{L_1,L_2}\right\}(u,v)\\&\quad+A_{1,2}^{-1,-1}\left\{\left(f_a(x,y)\right)_{L_1,L_2}\overline{\overline{*}}\left(h_b(x,y)\right)_{L_1,L_2}\right.\\&\quad\left.+\left(f_b(x,y)\right)_{L_1,L_2}\overline{\overline{*}}\left(\overline{h_a}(x,y)\right)_{L_1,L_2}\right\}(u,v)\cdot j\end{aligned}\tag{12}$$

Proof: Since

$$e^{i(u^2C_1+v^2C_2)} f(x,y)g(x,y) = e^{i(u^2C_1+v^2C_2)}\{f_a(x,y)h_a(x,y)+f_a(x,y)h_b(x,y)j + f_b(x,y)\overline{h_a}(x,y)j - f_b(x,y)\overline{h_b}(x,y)\}$$

From theorem 3, it can be obtained:

$$F^{L_1,L_2}\left\{e^{i(u^2C_1+v^2C_2)} f_a(x,y)h_a(x,y)\right\} = A_{1,2}^{-1,-1}\left\{(f_a(x,y))_{L_1,L_2} \overline{\overline{*}} (h_a(x,y))_{L_1,L_2}\right\}(u,v)$$

$$F^{L_1,L_2}\left\{e^{i(u^2C_1+v^2C_2)} f_a(x,y)h_b(x,y)j\right\} = A_{1,2}^{-1,-1}\left\{(f_a(x,y))_{L_1,L_2} \overline{\overline{*}} (h_b(x,y))_{L_1,L_2}\right\}(u,v)\cdot j$$

$$F^{L_1,L_2}\left\{e^{i(u^2C_1+v^2C_2)} f_b(x,y)\overline{h_a}(x,y)j\right\} = A_{1,2}^{-1,-1}\left\{(f_b(x,y))_{L_1,L_2} \overline{\overline{*}} (\overline{h_a}(x,y))_{L_1,L_2}\right\}(u,v)\cdot j$$

$$F^{L_1,L_2}\left\{e^{i(u^2C_1+v^2C_2)} f_b(x,y)h_b(x,y)\right\} = A_{1,2}^{-1,-1}\left\{(f_b(x,y))_{L_1,L_2} \overline{\overline{*}} (\overline{h_b}(x,y))_{L_1,L_2}\right\}(u,v)$$

From the linear property of Fourier transform:

$$F^{L_1,L_2}\left\{e^{i(u^2C_1+v^2C_2)} f(x,y)g(x,y)\right\} = A_{1,2}^{-1,-1}\left\{(f_a(x,y))_{L_1,L_2} \overline{\overline{*}} (h_a(x,y))_{L_1,L_2} - (f_b(x,y))_{L_1,L_2} \overline{\overline{*}} (\overline{h_b}(x,y))_{L_1,L_2}\right\}(u,v) + A_{1,2}^{-1,-1}\left\{(f_a(x,y))_{L_1,L_2} \overline{\overline{*}} (h_a(x,y))_{L_1,L_2} + (f_b(x,y))_{L_1,L_2} \overline{\overline{*}} (\overline{h_a}(x,y))_{L_1,L_2}\right\}(u,v)\cdot j$$

From theorem 5 we draw the conclusion that, the product, frequency modulated by a chirp, of two quaternion signals is to the summation, amplitude modulated, of their pseudo convolution.

4.2. FRQCR

Headings, or heads, are organizational devices that guide the reader through your paper. There are two types: component heads and text heads.

Theorem 6 is suitable for scalar and complex signals, and theorem 7 is suitable for scalar, complex signals, vector and quaternion signals.

Theorem 6: For two scalar (or complex) signals $f(x,y)$ and $h(x,y)$, $A_{1,2} = \frac{1}{2\pi i\sqrt{b_1b_2}}$, $C_1 = \frac{a_1}{2b_1}$, $C_2 = \frac{a_2}{2b_2}$,

$\langle f(x,y), h(x,y)\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tau,\eta)\overline{h(x+\tau, y+\eta)}\mathrm{d}\tau\mathrm{d}\eta$, and set

$$g(x,y) = f(x,y)\otimes h(x,y) = A_{1,2}e^{-i(C_1x^2+C_2y^2)}\left\langle e^{i(C_1x^2+C_2y^2)} f(x,y), e^{-i(C_1x^2+C_2y^2)}h(x,y)\right\rangle,$$ then:

$$f(x,y)\otimes h(x,y)=\left(f(-x,-y)\right)\bar{*}\left(\overline{h(x,y)}\right) \tag{13}$$

Proof: the proof is similar with that of FRQCV and is omitted here.

From theorem 6 we draw the conclusion that correlation can be implemented by convolution.

Theorem 7: For any two quaternion signals $f(x,y)$ and $h(x,y)$, $A_{1,2}=\frac{1}{2\pi i\sqrt{b_1b_2}}$, $C_1=\frac{a_1}{2b_1}$, $C_2=\frac{a_2}{2b_2}$,

$\langle f(x,y),h(x,y)\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(\tau,\eta)\overline{h(x+\tau,y+\eta)}\,\mathrm{d}\tau\mathrm{d}\eta$, and let

$$g(x,y)=f(x,y)\otimes h(x,y)$$
$$=A_{1,2}\mathrm{e}^{-i\left(C_1x^2+C_2y^2\right)}\left\langle \mathrm{e}^{i\left(C_1x^2+C_2y^2\right)}f(x,y),\mathrm{e}^{-i\left(C_1x^2+C_2y^2\right)}h(x,y)\right\rangle,$$

"$\otimes$" is correlation operator, then

$$\begin{aligned}&f(x,y)\otimes h(x,y)\\&=\left(f_a(-x,-y)\right)\bar{*}\left(\overline{h_a(x,y)}\right)+\left(f_b(-x,-y)\right)\bar{*}\left(\overline{h_b(x,y)}\right)\\&\quad-A_{1,2}\mathrm{e}^{-i\left(x^2C_1+y^2C_2\right)}\left\{\left(\mathrm{e}^{i\left(x^2C_1+y^2C_2\right)}f_a(-x,-y)\right)*\left(\overline{h_b(x,y)}\mathrm{e}^{-i\left(x^2C_1+y^2C_2\right)}\right)\right\}\cdot j\\&\quad+A_{1,2}\mathrm{e}^{-i\left(x^2C_1+y^2C_2\right)}\left\{\left(\mathrm{e}^{i\left(x^2C_1+y^2C_2\right)}f_b(-x,-y)\right)*\left(\overline{h_a(x,y)}\mathrm{e}^{-i\left(x^2C_1+y^2C_2\right)}\right)\right\}\cdot j\end{aligned} \tag{14}$$

Proof: The proof is similar with that of FRQCV and is omitted here.

From theorem 7 we draw the conclusion that the correlation of two quaternion signals is to the summation of convolution of their components, conjugated or odd-even operated. It means that correlation can be implemented by convolution via FFT.

5. Fast Algorithm of QLCT

Fast algorithm of QLCT is the key to engineering use. The following discusses the efficient implementation in great detail through the decomposition of quaternion [24] and the definition of the QLCT. For one quaternion function $f(x,y)$, from definition 1 we have

$$\begin{aligned}F_{i,j}^{L_1,L_2}\{f(x,y)\}&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}K_{L_1,i}(x,u)f(x,y)K_{L_2,j}(y,v)\mathrm{d}x\mathrm{d}y\\&=\sqrt{\frac{1}{2\pi b_1 i}}\mathrm{e}^{\frac{id_1u^2}{2b_1}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{ux}{b_1}}\mathrm{e}^{i\frac{x^2a_1}{2b_1}}f(x,y)\mathrm{e}^{j\frac{y^2a_2}{2b_2}}\mathrm{e}^{-j\frac{vy}{b_2}}\mathrm{d}x\mathrm{d}y\sqrt{\frac{1}{2\pi b_2 j}}\mathrm{e}^{\frac{jd_2v^2}{2b_2}}\\&=G_i(u)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{ux}{b_1}}g(x,y)\mathrm{e}^{-j\frac{vy}{b_2}}\mathrm{d}x\mathrm{d}y\cdot G_j(v)\end{aligned}$$

where,

$$G_i(u)=\sqrt{\frac{1}{2\pi b_1 i}}\mathrm{e}^{\frac{id_1u^2}{2b_1}},\quad G_j(v)=\sqrt{\frac{1}{2\pi b_2 j}}\mathrm{e}^{\frac{jd_2v^2}{2b_2}},$$

$$g(x,y)=\mathrm{e}^{i\frac{x^2a_1}{2b_1}}f(x,y)\mathrm{e}^{j\frac{y^2a_2}{2b_2}}=g_r(x,y)+ig_i(x,y)+jg_j(x,y)+kg_k(x,y)$$

$g_r(x,y)$, $g_i(x,y)$ are real signals.

Let

$$W(u,v)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{ux}{b_1}}g(x,y)\mathrm{e}^{-j\frac{vy}{b_2}}\mathrm{d}x\mathrm{d}y \tag{17}$$

Then,

$$\frac{W(u,v)+W(u,-v)}{2}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{ux}{b_1}}g(x,y)\cos\left(\frac{vy}{b_2}\right)\mathrm{d}x\mathrm{d}y \tag{18}$$

$$\frac{W(u,v)-W(u,-v)}{2}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{ux}{b_1}}g(x,y)\sin\left(\frac{vy}{b_2}\right)\mathrm{d}x\mathrm{d}y\cdot(-i) \tag{19}$$

Therefore,

$$\frac{W(u,v)+W(u,-v)}{2}+\frac{W(u,v)-W(u,-v)}{2}\cdot(-k)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{ux}{b_1}}g(x,y)\mathrm{e}^{-j\frac{vy}{b_2}}\mathrm{d}x\mathrm{d}y \tag{20}$$

Therefore,

$$F_{i,j}^{L_1,L_2}(u,v)=G_i(u)\frac{W(u,v)(1-k)+W(u,-v)(1+k)}{2}G_j(v) \tag{21}$$

Then the following task is to implement $W(u,v)$.

$g(x,y)$ can be expressed as

$$g(x,y)=g_r(x,y)+ig_i(x,y)+jg_j(x,y)+kg_k(x,y)=g_a(x,y)+g_b(x,y)\cdot j$$

where, $g_a(x,y)=g_r(x,y)+ig_i(x,y)$, $g_b(x,y)=g_j(x,y)+ig_k(x,y)$.

Therefore,

$$\begin{aligned}W(u,v)&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{ux}{b_1}}\mathrm{e}^{-i\frac{vy}{b_2}}g_a(x,y)\mathrm{d}x\mathrm{d}y+\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-i\frac{ux}{b_1}}\mathrm{e}^{-i\frac{vy}{b_2}}g_b(x,-y)\mathrm{d}x\mathrm{d}y\right)\cdot j\\&=F\{g_a(x,y)\}\left(\frac{u}{b_1},\frac{v}{b_2}\right)+F\{g_b(x,-y)\}\left(\frac{u}{b_1},\frac{v}{b_2}\right)\cdot j\end{aligned} \tag{22}$$

$W(u,v)$ can be Calculated by two 2D FFT and some scaling transform. The steps of calculating QLCT:

1) Calculate $g(x,y)$ from $f(x,y)$ using (16);
2) Calculate $W(u,v)$ from $g(x,y)$ using (22) and (17);
3) Calculate $G_i(u)$ and $G_j(v)$ using (16);
4) At last Calculate $F_{i,j}^{L_1,L_2}(u,v)$ using (20) and (21).

For one 2D discrete signal with size $M \times N$, one 2D-DFT needs $MN\cdot\log_2(MN)$ real number multiplications [25]. To implement $W(u,v)$, we require $2MN\cdot\log_2(MN)$ real number multiplications. Therefore, the complexity of quaternion signal $f(x,y)$ is $O(2MN\cdot\log_2(MN))$. And the complexity of scalar, complex signal and vector is: $O\left(\frac{MN\cdot\log_2(MN)}{2}\right)$, $O(MN\cdot\log_2(MN))$, $O\left(\frac{3MN\cdot\log_2(MN)}{2}\right)$.

For any discrete 2D signal $f(m,n)$ ($m \in [1,M], n \in [1,N]$), it can be expressed as:

$$f(m,n) = f_{ee}(m,n) + f_{eo}(m,n) + f_{oe}(m,n) + f_{oo}(m,n)$$

where:

$$f_{ee}(m,n) = \frac{f(m,n) + f(n,N-n) + f(M-m,n) + f(M-m,N-n)}{4}$$

$$f_{oe}(m,n) = \frac{f(m,n) + f(n,N-n) - f(M-m,n) - f(M-m,N-n)}{4}$$

$$f_{eo}(m,n) = \frac{f(m,n) - f(n,N-n) + f(M-m,n) - f(M-m,N-n)}{4}$$

$$f_{oo}(m,n) = \frac{f(m,n) - f(n,N-n) - f(M-m,n) + f(M-m,N-n)}{4}$$

If in the right side of $f(m,n) = f_{ee}(m,n) + f_{eo}(m,n) + f_{oe}(m,n) + f_{oo}(m,n)$ there is only one term, we call $f(m,n)$ symmetric;

If $f(M-m,n) = \pm f(m,n)$, we call $f(m,n)$ symmetric about *x*;

If $f(m,N-n) = \pm f(m,n)$, we call $f(m,n)$ symmetric about *y*;

If any above relationship is not true, we call $f(m,n)$ asymmetric.

The symmetry is of great importance to greatly decreasing the calculation complexity of them. Table 1 lists the calculation complexity of different types of signals. It gives the conclusion that the symmetry can decrease the calculation complexity by a few times. Meanwhile, the calculation complexity will increase with the number of components by a few times.

Meanwhile, the calculation complexity of QLCT for different signals is multiplications. Also, the complexity of QCV and QCR for the same type of signals is the same and is much less than calculation in time-domain directly.

Figure 1 shows one intuitive result. The QCR of the quaternion signal $f(x,y)$ and kernel $h(x,y)$ is calculated. We take different signals (scalar, complex, vector and quaternion) as the convolution kernel $h(x,y)$. The red lines denote the complexity of implementing QCR in time domain directly, and the blue lines denote the complexity of implementing QCR via FFT. For example,

Table 1. The calculation complexity of QLCT for different signals.

Types of signals	Asymmetric	Symmetric	Symmetric about *x* or *y*
Scalar	$O\left(\frac{MN \cdot \log_2(MN)}{2}\right)$	$O\left(\frac{MN \cdot \log_2(MN)}{8}\right)$	$O\left(\frac{MN \cdot \log_2(MN)}{4}\right)$
Complex	$O(MN \cdot \log_2(MN))$	$O\left(\frac{MN \cdot \log_2(MN)}{4}\right)$	$O\left(\frac{MN \cdot \log_2(MN)}{2}\right)$
Vector	$O\left(\frac{3MN \cdot \log_2(MN)}{2}\right)$	$O\left(\frac{3MN \cdot \log_2(MN)}{8}\right)$	$O\left(\frac{3MN \cdot \log_2(MN)}{4}\right)$
Quaternion	$O(2MN \cdot \log_2(MN))$	$O\left(\frac{MN \cdot \log_2(MN)}{2}\right)$	$O(MN \cdot \log_2(MN))$

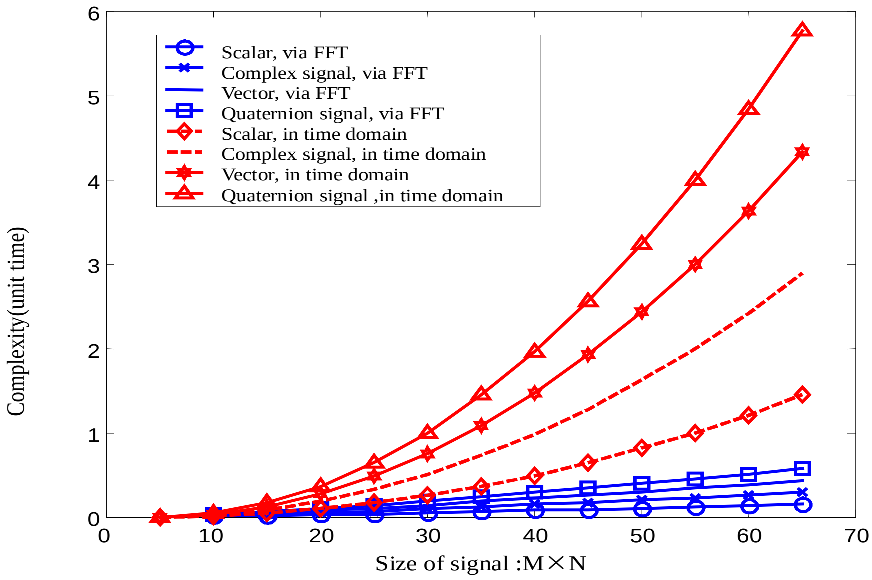

Figure 1. The comparison of complexity via FFT and calculation directly.

when the size is 60, there is one nearly-ten-times relationship. Moreover, with the increase of size the gap would become bigger and bigger.

6. Conclusion

One contribution of this paper is that the definition of QLCT is obtained, and its properties are given, and its generalization is proved. The reversibility property disclosed the efficiency of QLCT. The linear property indicated that LCT is linear transform. Another contribution of this paper is that the QCV and QCR of LCT are defined and their relationships and physical interpretation are discovered: the fractional convolution of two quaternion signals is to the summation of product of their components, conjugated or odd-even operated, and the product is frequency modulated by chirps; and the product, frequency modulated by a chirp, of two quaternion signals is to the summation, amplitude modulated, of their pseudo convolution; and the correlation of two quaternion signals is to the summation of convolution of their components, which are conjugated or odd-even operated. The last contribution is that the complexity of QLCT, QCV and QCR are given, and its Fast Algorithm is obtained through implementing them via the product theorem in transformed domain whose complexity is similar to FFT, which is of great importance to engineering use [31] [32].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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