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Discrete Inequalities on LCT

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Abstract

Linear canonical transform (LCT) is widely used in physical optics, mathematics and information processing. This paper investigates the generalized uncertainty principles, which plays an important role in physics, of LCT for concentrated data in limited supports. The discrete generalized uncertainty relation, whose bounds are related to LCT parameters and data lengths, is derived in theory. The uncertainty principle discloses that the data in LCT domains may have much higher concentration than that in traditional domains.

Keywords

Linear Canonical Transform (LCT), Uncertainty Inequality

1. Introduction

In physics, the uncertainty principle plays an important role in elementary fields, and data concentration is often considered carefully via the uncertainty principle [1]-[8]. In continuous signals, the supports are assumed to be $(-\infty,+\infty)$, based on which various uncertainty relations [1] [2] [9]-[21] have been presented. However, in practice, both the supports of time and frequency are often limited. In such case, the support $(-\infty,+\infty)$ fails to hold true. In limited supports, some papers such as [22]-[25] have discussed the uncertainty principle in conventional time-frequency domains for continuous and discrete cases and some conclusions are achieved. However, none of them has covered the linear canonical transform (LCT) in terms of Heisenberg uncertainty principles that have been widely used in various fields [4]-[6]. Therefore, there has a great need to discuss the uncertainty relations in LCT domains. As the generalization of the traditional FT, FRFT [5] [6] [26]-[28] and so on, LCT has some special properties with more transform parameters (or freedoms) and sometimes yields the better result [29]. Readers can see more details on LCT in [6] and so on.

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2. Preliminaries

2.1. Definition of LCT

Before discussing the uncertainty principle, we will introduce some relevant preliminaries. Here, we first briefly review the definition of LCT. For given continuous signal $x(t) \in L^1(R) \cap L^2(R)$ and $\|x(t)\|_2 = 1$, its LCT [6] is defined as

$$X_{(a,b,c,d)}(u) = F^{(a,b,c,d)}(x(t)) = \int_{-\infty}^{\infty} x(t) K_{(a,b,c,d)}(u,t)\,\mathrm{d}t$$
$$= \begin{cases} \sqrt{\dfrac{1}{2b\pi}} \cdot \mathrm{e}^{\frac{idu^2}{2b}} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{-iut}{b}} \mathrm{e}^{\frac{iat^2}{2b}} x(t)\,\mathrm{d}t, & b \neq 0, ad - bc = 1; \\ \sqrt{d} \cdot \mathrm{e}^{\frac{icdu^2}{2}} x(du), & b = 0. \end{cases} \tag{1}$$

where $n \in Z$ and i is the complex unit, (a,b,c,d) are the transform parameters defined as that in [6]. In addition, $F^{(a2,b2,c2,d2)} F^{(a1,b1,c1,d1)}(x(t)) = F^{(a,b,c,d)}(x(t))$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$. If $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, then $F^{(a1,b1,c1,d1)}(\bullet)$ and $F^{(a2,b2,c2,d2)}(\bullet)$ are the LCT transform pairs, *i.e.*, $F^{(a2,b2,c2,d2)} F^{(a1,b1,c1,d1)}(x(t)) = x(t)$. Also, if $(a,b,c,d) = (0,1,-1,0)$, we have the following equations:

$$F^{(0,1,-1,0)}(x(t)) = X(u)$$
$$= \int_{-\infty}^{\infty} x(t) K_{(0,1,-1,0)}(u,t)\,\mathrm{d}t \quad \text{and} \quad x(t) = \sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} \mathrm{e}^{\frac{iut}{b}} X(u)\,\mathrm{d}t.$$
$$= \sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{\infty} \mathrm{e}^{\frac{-iut}{b}} x(t)\,\mathrm{d}t,$$

However, unlike the discrete FT, there are a few definitions for the DLCT (discrete LCT), but not only one. In this paper, we will employ the definition defined as follows [6]:

$$\hat{x}_{(a,b,c,d)}(k) = \sum_{n=1}^{N} \sqrt{1/iNb} \cdot \mathrm{e}^{\frac{idk^2}{2b}} \mathrm{e}^{\frac{-ikn}{Nb}} \mathrm{e}^{\frac{in^2 a}{2bN^2}} \tilde{x}(n)$$
$$= \sum_{n=1}^{N} u_{(a,b,c,d)}(k,n) \cdot \tilde{x}(n), \quad 1 \le n, k \le N. \tag{2}$$

Clearly, if $(a,b,c,d) = (1,0,0,1)$, (2) reduces to the traditional discrete FT [6]. Also, we can rewrite definition (2) as $\hat{X}_A = U_A \tilde{X}$ with $A = (a,b,c,d)$ and $A^{-1}A = I$, where $U_A = [u_A(k,n)]_{N\times N}$, $\tilde{X} = [\tilde{x}(n)]_{N\times 1}$.

For DLCT, we have the following property [5] [6]:

$$\|\hat{X}_A\|_2 = \|U_A \tilde{X}\|_2 = 1.$$

More details on DLCT can be found in [6].

2.2. Frequency-Limiting Operators

***Definition* 1:** Let $x(t)$ be a complex-valued signal with $\|x(t)\|_{L^2(R)} = 1$ and its LCT $X_A(u)$, if there is a function $G_A(u)$ vanishing outside W_A (W_A is a measurable set) such that $\|X_A(u) - G_A(u)\|_{L^2(R)} \le \varepsilon_A$, then $X_A(u)$ is ε_A-concentrated.

Specially, if $A = (1,0,0,1)$, then definition 1 reduces to the case in time domain [22] [23]. If $A = (0,1,-1,0)$, then definition 1 reduces to the case in traditional frequency domain [22] [23].

***Definition* 2:** Generalized frequency-limiting operator P_{W_A} is defined as

$$\left(P_{W_A}x\right)(t) \equiv \int_{W_A} X_A(u)K_{A^{-1}}(u,t)\,\mathrm{d}u\,,\quad X_A(u) = F_A\left(x(t)\right).$$

If $A=(1,0,0,1)$, then definition 2 is the time-limiting operator [22] [23]. If $A=(0,1,-1,0)$, then definition 2 is the traditional frequency-limiting operator [22] [23]. Definitions 1 and 2 disclose the relation between ε_A and W_A. For the discrete case, we have the following definitions.

***Definition* 3:** Let $\tilde{x}(n)\in l^2(R)$ $(n=1,\cdots,N)$ be a discrete sequence with $\left\|\tilde{x}(n)\right\|_{l^2(R)}=1$ and its DLCT $\hat{x}_A(k)$, if there is a sequence $\hat{x}'_A(k)$ satisfying $\left\|\hat{x}'_\alpha(k)\right\|_0 = N_A$ such that $\left\|\hat{x}_A(k)-\hat{x}'_A(k)\right\|_{l^2(R)} \le \varepsilon_A$, then $\hat{x}(k)$ is ε_A-concentrated. Here, $\left\|\cdot\right\|_0$ is the 0-norm operator that counts the non-zero elements.

***Definition* 4:** Generalized discrete frequency-limiting operator P_{N_A} is defined as $\left(P_{N_A}\tilde{x}\right)(n)=\sum_{k=1}^{N}\chi_{N_A}\hat{x}(k)u_{A^{-1}}(k,n)$ with $\hat{x}(k)$ is the DLCT of $\tilde{x}(n)$ and χ_{N_A} is the character function on $N_A\left(N_A\le N\right)$.

Clearly, definitions 3 and 4 are the discrete extensions of definitions 1 and 2. They have the similar physical meaning. These definitions are introduced for the first time, the traditional cases [22] [23] are only their special cases. Definition 3 and 4 disclose the relation between ε_A and N_A.

3. The Uncertainty Relations

3.1. The Uncertainty Principle

First let us introduce a lemma.

***Lemma* 3:** $\left\|P_{N_A}P_{N_B}\right\|_F = \sqrt{\dfrac{N_A\cdot N_B}{N\left|a_1b_2-a_2b_1\right|}}$

where $\left\|\cdot\right\|_F$ is the Frobenius matrix norm.

Proof: From the definition of the operator $P_{N_A}P_{N_B}$ in definition 4, we have

$$\left(P_{N_A}P_{N_B}\tilde{x}\right)(n)=\sum_{k=1}^{N}\chi_{N_A}u_A(k,n)\sum_{v=1}^{N}\chi_{N_B}\hat{x}(v)u_{B^{-1}}(k,v).$$

Exchange the locations of the sum operators, we obtain

$$\begin{aligned}\left(P_{N_A}P_{N_B}\tilde{x}\right)(n)&=\sum_{v=1}^{N}\sum_{k=1}^{N}\chi_{N_A}\chi_{N_B}u_A(k,n)\hat{x}(v)u_{B^{-1}}(k,v)\\&=\sum_{k=1}^{N}\chi_{N_A}\sum_{v=1}^{N}\chi_{N_B}\hat{x}(v)u_{AB^{-1}}(n,v),\end{aligned}$$

Hence, according to the definition of the Frobenius matrix norm [1] and the definition of DLCT, we have

$$\begin{aligned}\left\|P_{N_A}P_{N_B}\right\|_F&=\left(\sum_{k=1}^{N}\chi_{N_A}\sum_{v=1}^{N}\chi_{N_B}\left|u_{AB^{-1}}(n,v)\right|^2\right)^{1/2}\\&=\sqrt{\frac{N_A\cdot N_B}{N\left|a_1b_2-a_2b_1\right|}},\end{aligned}$$

In the similar manner with the continuous case, we can obtain $\dfrac{\left\|P_{N_A}P_{N_B}\tilde{x}(n)\right\|_{l^2(R)}}{\left\|\tilde{x}(n)\right\|_{l^2(R)}}\ge 1-\left(\varepsilon_A+\varepsilon_B\right)$. Since $\left\|P_{N_A}P_{N_B}\right\|_F \ge \left\|P_{N_A}P_{N_B}\right\|_{l^2(R)} = \sup\limits_{x(n)\in l^2(R)}\dfrac{\left\|P_{N_A}P_{N_B}\tilde{x}(n)\right\|_{l^2(R)}}{\left\|\tilde{x}(n)\right\|_{l^2(R)}}$, we have

$\sqrt{\dfrac{N_A \cdot N_B}{N\left|a_1 b_2 - a_2 b_1\right|}} = \left\| P_{N_A} P_{N_B} \right\|_F \geq \dfrac{\left\| P_{N_A} P_{N_B} \tilde{x}(n) \right\|_{l^2(R)}}{\left\| \tilde{x}(n) \right\|_{l^2(R)}} \geq 1 - (\varepsilon_A + \varepsilon_B)$, thus, we get

$N_A \cdot N_B \geq N \cdot (1 - \varepsilon_A - \varepsilon_B)^2 \left|a_1 b_2 - a_2 b_1\right|$. Therefore, we can obtain the following theorem 2.

***Theorem* 2:** Let $\hat{x}_A(k)$ $(\hat{x}_B(k))$ be the DLCT of the time sequence $\tilde{x}(n) \in l^2(R)$ $(n = 1, \cdots, N)$ for transform parameter A (B), with $\hat{x}_A(k)$ $(\hat{x}_B(k))$ ε_A (ε_B)-concentrated on index set N $(\varepsilon_A \varepsilon_B \neq 0)$. Let N_A (N_B) be the numbers of nonzero entries in $\hat{x}_A(k)$ ($\hat{x}_B(k)$ respectively). Then

$$\begin{cases} N_A \cdot N_B \geq N \cdot (1 - \varepsilon_A - \varepsilon_B)^2 \left|a_1 b_2 - a_2 b_1\right|, & a_1/b_1 \neq a_2/b_2; \\ N_A \cdot N_B \geq 1, & a_1/b_1 = a_2/b_2. \end{cases}$$

3.2. Extensions

Set $\varepsilon_A = \varepsilon_B = 0$ in theorem 2, we can obtain the following theorem 3 directly.

***Theorem* 3:** Let $\hat{x}_A(k)$ $(\hat{x}_B(k))$ be the DLCT of the time sequence $\tilde{x}(n) \in l^2(R)$ $(n = 1, \cdots, N)$ with length *N*. N_A (N_B) counts the numbers of nonzero entries in $\hat{x}_A(k)$ ($\hat{x}_B(k)$ respectively). Then

$$\begin{cases} N_A \cdot N_B \geq N \cdot \left|a_1 b_2 - a_2 b_1\right|, & a_1/b_1 \neq a_2/b_2; \\ N_A \cdot N_B \geq 1, & a_1/b_1 = a_2/b_2. \end{cases}$$

Clearly, theorem 3 is a special case of theorem 2. Also, this theorem can be derived via theorem 1 in [25]. Differently, we obtain this result in a different way. Here we note that since $\left\|\tilde{x}(n)\right\|_{l^2(R)} = 1$, there is at least one non-zero element in every LCT domain for $a_1/b_1 = a_2/b_2$. Therefore, $N_A \cdot N_B \geq 1$ for $a_1/b_1 = a_2/b_2$. Through setting special value for $B = (1,0,0,1)$ in theorem 3, we have

***Corollary* 1:** $\begin{cases} N_A \cdot N_0 \geq N \cdot \left|b_1\right| & b_1 \neq 0 \\ N_A \cdot N_0 \geq 1 & b_1 = 0 \end{cases}$ with $N_0 = N_{(1,0,0,1)}$.

We can obtain the following more general uncertainty relation associated with DLCT.

***Theorem* 4:** Let $\hat{x}_{A_l}(k)$ $(l = 1, 2, \cdots, L)$ be the DLCT of the time sequence $\tilde{x}(n) \in l^2(R)$ ($n = 1, \cdots, N$ and $N > L$) with length N and $\left\|\tilde{x}(n)\right\|_{l^2(R)} = 1$. N_{α_l} counts the number of nonzero elements in $\hat{x}_{A_l}(k)$. Then

$$\frac{N_{A_1} + N_{A_2} + \cdots + N_{A_L}}{L} \geq \sqrt{N \cdot \xi} \quad \text{with} \quad \xi = \inf_{\substack{1 \leq l_1, l_2 \leq L \\ l_1 \neq l_2}} \left\{ \left| a_{l_1} b_{l_2} - a_{l_2} b_{l_1} \right| \right\}$$

Proof: From the assumption and the definition of DLCT [6], we know

$$\tilde{x}(n) = \sum_{k_1=1}^{N} u_{(A_1)^{-1}}(n, k_1) \hat{x}_{A_1}(k_1) = \sum_{k_2=1}^{N} u_{(A_2)^{-1}}(n, k_2) \hat{x}_{A_2}(k_2) = \cdots = \sum_{k_L=1}^{N} u_{(A_L)^{-1}}(n, k_L) \hat{x}_{A_L}(k_L) \quad \text{for} \quad n = 1, 2, \cdots, N.$$

where $u_{(A_l)^{-1}}(n, k_l) = \sqrt{-1/ib_l N} \cdot \mathrm{e}^{\frac{-ia_l k_l^2 \cot\alpha_l}{2b_l}} \mathrm{e}^{\frac{ik_l n}{Nb_l}} \mathrm{e}^{\frac{-in^2 d_l}{2b_l N^2}}$ $(l = 1, 2, \cdots, L)$. Therefore, let $\tilde{X} = \left[\tilde{x}(1), \tilde{x}(2), \cdots, \tilde{x}(N)\right]^{\mathrm{T}}$, have [25]

$$\tilde{X}^{\mathrm{T}} \tilde{X} = \left[\hat{x}_{A_{l_1}}(1), \hat{x}_{A_{l_1}}(2), \cdots, \hat{x}_{A_{l_1}}(N)\right] \begin{bmatrix} u^{\mathrm{T}}_{(A_{l_1})^{-1}}(1,:) \\ u^{\mathrm{T}}_{(A_{l_1})^{-1}}(2,:) \\ \vdots \\ u^{\mathrm{T}}_{(A_{l_1})^{-1}}(N,:) \end{bmatrix} \left[u_{(A_{l_2})^{-1}}(1,:), u_{(A_{l_2})^{-1}}(2,:), \cdots, u_{(A_{l_2})^{-1}}(N,:)\right] \begin{bmatrix} \hat{x}_{(A_{l_2})^{-1}}(1) \\ \hat{x}_{(A_{l_2})^{-1}}(2) \\ \vdots \\ \hat{x}_{(A_{l_2})^{-1}}(N) \end{bmatrix}$$

where $u^{\mathrm{T}}_{(A_{l_1})^{-1}}(n,:)=\left[u_{(A_{l_1})^{-1}}(n,1),u_{(A_{l_1})^{-1}}(n,2),\cdots,u_{(A_{l_1})^{-1}}(n,N)\right]$ and $u_{(A_{l_2})^{-1}}(n,:)=\left[u_{(A_{l_2})^{-1}}(n,1),u_{(A_{l_2})^{-1}}(n,2),\cdots,u_{(A_{l_2})^{-1}}(n,N)\right]^{\mathrm{T}}$ with $n=1,2,\cdots,N$ and $l_1,l_2=1,2,\cdots,L$ with $l_1\neq l_2$.

Hence, we obtain

$$\begin{aligned}\tilde{X}^{\mathrm{T}}\tilde{X}&=\left[\hat{x}_{A_{l_1}}(1),\hat{x}_{A_{l_1}}(2),\cdots,\hat{x}_{A_{l_1}}(N)\right]\\&=\sum_{n=1}^{N}\sum_{k=1}^{N}\hat{x}_{A_{l_1}}(n)\left\langle u_{(A_{l_1})^{-1}}(n,:),u_{(A_{l_2})^{-1}}(k,:)\right\rangle\hat{x}_{A_{l_2}}(k),\end{aligned}$$

Set $M_{(l_1,l_2)}=\sup\limits_{n,k}\left(\left|\left\langle u_{(A_{l_1})^{-1}}(n,:),u_{(A_{l_2})^{-1}}(k,:)\right\rangle\right|\right)$, then

$$\begin{aligned}\tilde{X}^{\mathrm{T}}\tilde{X}&\leq\sum_{n=1}^{N}\sum_{k=1}^{N}\left|\hat{x}_{A_{l_1}}(n)\left\langle u_{(A_{l_1})^{-1}}(n,:),u_{(A_{l_2})^{-1}}(k,:)\right\rangle\hat{x}_{A_{l_2}}(k)\right|\\&\leq M_{(l_1,l_2)}\sum_{s_1=1}^{N_{A_{l_1}}}\sum_{s_2=1}^{N_{A_{l_2}}}\left|\hat{x}_{A_{l_1}}(s_1)\right|\cdot\left|\hat{x}_{A_{l_2}}(s_2)\right|.\end{aligned}$$

Using the triangle inequality, we have

$$\left|\hat{x}_{A_{l_1}}(s_1)\right|\cdot\left|\hat{x}_{A_{l_2}}(s_2)\right|\leq\frac{\left|\hat{x}_{A_{l_1}}(s_1)\right|^2+\left|\hat{x}_{A_{l_2}}(s_2)\right|^2}{2}$$

hence

From $\left\|\tilde{x}(n)\right\|_2=1$ and Parseval's principle [6], we obtain:

$$\sum_{s_1=1}^{N_{A_{l_1}}}\frac{\left|\hat{x}_{A_{l_1}}(s_1)\right|^2}{2}=\sum_{s_2=1}^{N_{\alpha l_2}}\frac{\left|\hat{x}_{A_{l_2}}(s_2)\right|^2}{2}=\frac{1}{2}.$$

Hence

$$\tilde{X}^{\mathrm{T}}\tilde{X}\leq M_{(l_1,l_2)}\cdot\left(\sum_{s_2=1}^{N_{A_{l_2}}}\frac{1}{2}+\sum_{s_1=1}^{N_{A_{l_1}}}\frac{1}{2}\right)=M_{(l_1,l_2)}\cdot\frac{N_{A_{l_1}}+N_{A_{l_2}}}{2}.$$

Therefore, we obtain

$$\begin{aligned}\tilde{X}^{\mathrm{T}}\tilde{X}&\leq M_{(1,2)}\cdot\frac{N_1+N_2}{2},\\\tilde{X}^{\mathrm{T}}\tilde{X}&\leq M_{(1,3)}\cdot\frac{N_1+N_3}{2},\\&\vdots\\\tilde{X}^{\mathrm{T}}\tilde{X}&\leq M_{(L-1,L)}\cdot\frac{N_{L-1}+N_L}{2}.\end{aligned}$$

Adding all the above inequalities, we have $\Gamma_L^2\cdot\tilde{X}^{\mathrm{T}}\tilde{X}\leq\sup\limits_{\substack{1\leq l_1,l_2\leq L\\ l_1\neq l_2}}\left\{M_{(l_1,l_2)}\right\}\frac{(L-1)\cdot(N_1+N_2+\cdots+N_L)}{2}$ with $\Gamma_L^2=\frac{L\cdot(L-1)}{2\times1}$. Similarly, from $\left\|\tilde{x}(n)\right\|_2=1$ and Parseval's principle [6], we obtain $\tilde{X}^{\mathrm{T}}\tilde{X}=1$, hence

$$\frac{(L-1)\cdot(N_1+N_2+\cdots+N_L)}{2}\geq\frac{\Gamma_L^2\cdot}{\sup\limits_{\substack{1\leq l_1,l_2\leq L\\ l_1\neq l_2}}\left\{M_{(l_1,l_2)}\right\}}.$$

From the definition and property of DLCT [6] we have

$$\sup_{\substack{1\leq l_1,l_2\leq L\\ l_1\neq l_2}}\left\{M_{(l_1,l_2)}\right\}=\sup_{\substack{1\leq s_1,s_2\leq N\\ 1\leq l_1,l_2\leq L,l_1\neq l_2}}\left(\left|K_{(A_{l_1})^{-1}A_{l_2}}(s_1,s_2)\right|\right)$$

$$=\sup_{\substack{1\leq s_1,s_2\leq N\\ 1\leq l_1,l_2\leq L,l_1\neq l_2}}\left(\left|\frac{1}{\sqrt{N\cdot\left|a_{l_1}b_{l_2}-a_{l_2}b_{l_1}\right|}}\right|\right)=\frac{1}{\sqrt{N\cdot\xi}}.$$

with $\xi=\inf\limits_{\substack{1\leq l_1,l_2\leq L\\ l_1\neq l_2}}\left\{\left|a_{l_1}b_{l_2}-a_{l_2}b_{l_1}\right|\right\}$.

Hence, we finally obtain $\frac{N_1+N_2+\cdots+N_L}{L}\geq\sqrt{N\cdot\xi}$ with $\xi=\inf\limits_{\substack{1\leq l_1,l_2\leq L\\ l_1\neq l_2}}\left\{\left|a_{l_1}b_{l_2}-a_{l_2}b_{l_1}\right|\right\}$. This theorem is the extension of theorem 3 and discloses the uncertainty relation between multiple signals.

4. Conclusion

In practice, for the discrete data, not only the supports are limited, but also they are sequences of data points whose number of non-zero elements is countable accurately. This paper discussed the generalized uncertainty relations on LCT in terms of data concentration. We show that the uncertainty bounds are related to the LCT parameters and the support lengths. These uncertainty relations will enrich the ensemble of uncertainty principles and yield the potential illumination for physics.

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