

# Linear Control Problems of the Fuzzy Maps

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## ABSTRACT

*In the present paper, we show the some properties of the fuzzy R-solution of the control linear fuzzy differential inclusions and research the optimal time problems for it.*

**Keywords:** Fuzzy Differential Inclusions, Control Problems

## 1. Introduction

The first study of differential equations with multivalued right-hand sides was performed by A. Marchaud [1] and S. C. Zaremba [2]. In early sixties, T. Wazewski [3,4], A. F. Filippov [5] obtained fundamental results on existence and properties of the differential equations with multivalued right-hand sides (differential inclusions). One of the most important results of these articles was an establishment of the relation between differential inclusions and optimal control problems, that promoted to develop the differential inclusion theory [6–9].

Considering of the differential inclusions required to study properties of multivalued functions, *i.e.* an elaboration the whole tool of mathematical analysis for multivalued functions [6,10,11].

In works [12,13] annotate of an R-solution for differential inclusion is introduced as an absolutely continuous multivalued function. Various problems for the R-solution theory were regarded in [14–18]. The basic idea for a development of an equation for R-solutions (integral funnels) is contained in [19].

In the last years there has been forming new approach to control problems of dynamic systems, which foundation on analysis of trajectory bundle but not separate trajectories. The section of this bundle in any instant is some set and it is necessary to describe the evolution of this set. Obtaining and research dynamic equations of sets there is important problem in this case. The metric space of sets with the Hausdorff metric is natural space for description dynamic of sets. In theory of multivalued maps definitions on derivative as for single-valued maps is impossible because space of sets is nonlinear. This bound possibility description dynamic sets by differential equations. Therefore, the control differential equations with

set of initial conditions [20–22] and the control differential inclusions [8,23–34] use for it.

In recent years, the fuzzy set theory introduced by Zadeh [35] has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of regional, physical, mathematical, differential equations, and engineering sciences. Recently there have been new advances in the theory of fuzzy differential equations [36–47] and inclusions [43,48–52] as well as in the theory of control fuzzy differential equations [53–55] and inclusions [56,57].

In this article we consider the some properties of the fuzzy R-solution of the control linear fuzzy differential inclusions and research the optimal time problems for it.

## 2. The Fundamental Definitions and Designations

Let  $comp(R^n)$  ( $conv(R^n)$ ) be a set of all nonempty (convex) compact subsets from the space  $R^n$ ,

$$h(A, B) = \min_{r \geq 0} \{S_r(A) \supset B, S_r(B) \supset A\}$$

be Hausdorff distance between sets  $A$  and  $B$ ,  $S_r(A)$  is  $r$ -neighborhood of set  $A$ .

Let  $E^n$  be the set of all  $u: R^n \rightarrow [0, 1]$  such that  $u$  satisfies the following conditions:

1)  $u$  is normal, that is, there exists an  $x_0 \in R^n$  such that  $u(x_0) = 1$ ;

2)  $u$  is fuzzy convex, that is,

$$u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\};$$

3) For any  $x, y \in R^n$  and  $0 \leq \lambda \leq 1$ ;

4)  $u$  is upper semicontinuous;

5)  $[u]^0 = cl\{x \in R^n : u(x) > 0\}$  is compact.

If  $u \in E^n$ , then  $u$  is called a fuzzy number, and  $E^n$  is said to be a fuzzy number space. For  $0 < \alpha \leq 1$ , denote

$$[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}.$$

Then from 1)-4), it follows that the  $\alpha$ -level set  $[u]^\alpha \in conv(R^n)$  for all  $0 \leq \alpha \leq 1$ .

**Theorem 1.** (Negoita and Ralescu [58]). If  $u \in E^n$ , then

- 1)  $[u]^\alpha \in conv(R^n)$  for all  $\alpha \in [0,1]$ ;
- 2)  $[u]^\alpha \subset [u]^\beta$  for  $0 \leq \alpha < \beta \leq 1$ ;
- 3) If  $\{\alpha_k\} \subset [0,1]$  is a decreasing sequence converging to  $\alpha > 0$  then

$$[u]^\alpha = \bigcap_{k \geq 1} [u]^{\alpha_k}$$

Conversely, if  $\{A^\alpha : 0 \leq \alpha \leq 1\}$  is a family of convex compact subsets of  $R^n$  satisfying 1)-3), then  $[u]^\alpha = A^\alpha$  for  $0 < \alpha \leq 1$  and

$$[u]^0 = \overline{\bigcap_{0 < \alpha \leq 1} A^\alpha} \subset A^0.$$

If  $g : R^n \times R^n \rightarrow R^n$  is a function, then using Zadeh's extension principle we can extend  $\tilde{g}$  to  $E^n \times E^n \rightarrow E^n$  by the equation

$$\tilde{g}(u, v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}.$$

It is well known that

$$[\tilde{g}(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha)$$

for all  $u, v \in E^n, 0 \leq \alpha \leq 1$  and continuous function  $g$ . Further, we have

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, [ku]^\alpha = k[u]^\alpha,$$

where  $k \in R$ .

Define  $D : E^n \times E^n \rightarrow [0, \infty)$  by the relation

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} h([u]^\alpha, [v]^\alpha),$$

where  $h$  is the Hausdorff metric defined in  $comp(R^n)$ . Then  $D$  is a metric in  $E^n$ .

Further we know that [59]

- 1)  $(E^n, D)$  is a complete metric space;
- 2)  $D(u+w, v+w) = D(u, v)$  for all  $u, v, w \in E^n$ ;
- 3)  $D(\lambda u, \lambda v) = |\lambda| D(u, v)$  for all  $u, v \in E^n$  and  $\lambda \in R$ .

It can be proved that

$$D(u+v, w+z) \leq D(u, w) + D(v, z)$$

for  $u, v, w, z \in E^n$ .

**Definition 1.** A mapping  $F : [0, T] \rightarrow E^n$  is strongly measurable if for all  $\alpha \in [0,1]$  the set-valued map  $F_\alpha : [0, T] \rightarrow conv(R^n)$  defined by  $F_\alpha(t) = [F(t)]^\alpha$  is Lebesgue measurable.

**Definition 2.** A mapping  $F : [0, T] \rightarrow E^n$  is said to be integrably bounded if there is an integrable function  $h(t)$  such that  $\|x(t)\| \leq h(t)$  for every  $x(t) \in F_0(t)$ .

**Definition 3.** The integral of a fuzzy mapping  $F : [0, T] \rightarrow E^n$  is defined levelwise by  $\int_0^T F(t) dt$  is defined levelwise by  $\int_0^T F(t) dt = \int_0^T F_\alpha(t) dt$ . The set of all  $\int_0^T f(t) dt$  such that  $f : [0, T] \rightarrow R^n$  is a measurable selection for  $F_\alpha$  for all  $\alpha \in [0,1]$ .

**Definition 4.** A strongly measurable and integrably bounded mapping  $F : [0, T] \rightarrow E^n$  is said to be integrable over  $[0, T]$  if  $\int_0^T F(t) dt \in E^n$ .

Note that if  $F : [0, T] \rightarrow E^n$  is strongly measurable and integrably bounded, then  $F$  is integrable. Further if  $F : [0, T] \rightarrow E^n$  is continuous, then it is integrable.

**Theorem 2.** [36]. Let  $F, G : [0, T] \rightarrow E^n$  be integrable and  $c \in [0, T], \lambda \in R$ . Then

- 1)  $\int_0^T F(t) dt = \int_0^c F(t) dt + \int_c^T F(t) dt$ ;
- 2)  $\int_0^T F(t) + G(t) dt = \int_0^T F(t) dt + \int_0^T G(t) dt$ ;
- 3)  $\int_0^T \lambda F(t) dt = \lambda \int_0^T F(t) dt$ ;
- 4)  $D(F, G)$  is integrable;
- 5)  $D\left(\int_0^T F(t) dt, \int_0^T G(t) dt\right) \leq \int_0^T D(F(t), G(t)) dt$

Consider the following control linear fuzzy differential inclusions

$$\dot{x} \in A(t)x + G(t, w), x(t_0) = x_0, \tag{1}$$

and the following nonlinear fuzzy differential inclusions

$$\dot{x} \in F(t, x, w), x(t_0) = x_0, \tag{2}$$

where  $\dot{x}$  means  $\frac{dx}{dt}$ ;  $t \in R_+$  is the time;  $x \in R^n$  is the state;  $w \in R^m$  is the control;  $A(t)$  is  $(n \times n)$ -dimensional matrix-valued function;  $G : R_+ \times R^m \rightarrow E^n$ ,  $F : R_+ \times R^n \times R^m \rightarrow E^n$  are the set-valued functions.

Let

$$W : R_+ \rightarrow conv(R^m) \tag{3}$$

be the measurable multivalued map.

**Definition 5.** Set  $LW$  of all single-valued branches of the multivalued map  $W(\cdot)$  is the set of the possible controls.

Obviously, the control fuzzy differential Inclusion (2) turns into the ordinary fuzzy differential inclusion

$$\dot{x} \in \Phi(t, x), \quad x(t_0) = x_0, \tag{4}$$

if the control  $\tilde{w}(\cdot) \in LW$  is fixed and  $\Phi(t, x) \equiv F(t, x, \tilde{w}(t))$ .

The fuzzy differential Inclusions (3) has the fuzzy R-solution, if right-hand side of the fuzzy differential Inclusion (3) satisfies some conditions [52].

Let  $X(t)$  denotes the fuzzy R-solution of the differential Inclusion (3), then  $X(t, w)$  denotes the fuzzy R-solution of the control differential Inclusion (2) for the fixed  $w(\cdot) \in LW$ .

**Definition 6.** The set

$$Y(T) = \{X(T, w) : w(\cdot) \in LW\}$$

be called the attainable set of the fuzzy System (2).

### 3. The Some Properties of the R-Solution

In this section, we consider the some properties of the R-solution of the control fuzzy differential Inclusion (1).

Let the following condition is true.

**Condition A:**

A1.  $A(\cdot)$  is measurable on  $[t_0, T]$ ;

A2. The norm  $\|A(t)\|$  of the matrix  $A(t)$  is integrable on  $[t_0, T]$ ;

A3. The multivalued map  $W : [t_0, T] \rightarrow conv(R^m)$  is measurable on  $[t_0, T]$ ;

A4. The fuzzy map  $G : R_+ \times R^m \rightarrow E^n$  satisfies the conditions

- 1) measurable in  $t$ ;
- 2) continuous in  $w$ ;

A5. There exist  $v(\cdot) \in L_2[t_0, T]$  and  $l(\cdot) \in L_2[t_0, T]$  such that

$$|W(t)| \leq v(t), \quad |G(t, w)| \leq l(t)$$

almost everywhere on  $[t_0, T]$ ;

A6. The set  $Q(t) = \{G(t, w(t)) : w(\cdot) \in LW\}$  is compact and convex for almost every  $[t_0, T]$ , i.e.  $Q(t) \in conv(E^n)$ .

**Theorem 3.** Let the condition A is true.

Then for every  $w(\cdot) \in LW$  there exists the fuzzy R-solution  $X(\cdot, w)$  such that

1) the fuzzy map  $X(\cdot, w)$  has form

$$X(t, w) = \Phi(t)x_0 + \Phi(t) \int_{t_0}^t \Phi^{-1}(s)G(s, w(s))ds,$$

where  $t \in [t_0, T]$ ;  $\Phi(t)$  is Cauchy matrix of the differential equation  $\dot{x} = A(t)x$ ;

2)  $X(t, w) \in E^n$  for every  $t \in [t_0, T]$ ;

3) the fuzzy map  $X(\cdot, w)$  is the absolutely continuous fuzzy map on  $[t_0, T]$ .

**Proof.** The proof is easy consequence of the [31,34,52,54] and Theorem 1.

**Theorem 4.** Let the condition A is true.

Then the attainable set  $Y(T)$  is compact and convex.

**Proof.** The proof is easy consequence of the [31,34,52,54] and Theorem 1.

We obtained the basic properties of the fuzzy R-solution of System (1). Now, we consider the some control fuzzy problems.

### 4. The Optimal Time Problems

Consider the control linear fuzzy differential Inclusion (1), when

$$G(t, w) = B(t)w + F(t), \tag{4}$$

where

B1.  $B(\cdot)$  is measurable on  $[t_0, T]$ ;

B2. The norm  $\|B(t)\|$  of the matrix  $B(t)$  is integrable on  $[t_0, T]$ ;

B3. The fuzzy map  $F : [t_0, T] \rightarrow E^n$  is measurable on  $[t_0, T]$ ;

B4. There exists  $f(\cdot) \in L_2[t_0, T]$  such that

$$|F(t)| \leq f(t)$$

almost everywhere on  $[t_0, T]$ .

Consider the following optimal control problem: it is necessary to find the minimal time  $T$  and the control  $w^*(\cdot) \in LW$  such that the fuzzy R-solution of Systems (1),(4) satisfies one of the conditions:

$$X(T, w^*) \cap S_k \neq \emptyset, \tag{5}$$

$$X(T, w^*) \subset S_k, \tag{6}$$

$$X(T, w^*) \supset S_k, \tag{7}$$

where  $S_k \in E^n$  is the terminal set.

Clearly, these time optimal problems are different from the ordinary time optimal problem by that here control object has the volume.

**Definition 6.** We shall say that the pair  $(w^*(\cdot), X(\cdot, w^*))$  satisfies the maximum principle on  $[t_0, T]$ , if there exists the vector-function  $\psi(\cdot)$ , which is the solution of the

system

$$\dot{\psi} \in -A^T(t)\psi, \psi(T) \in S_1(0)$$

and the following conditions are true

1) the maximum condition

$$C(B(t)w^*(t), \psi(t)) = \max_{w \in W(t)} C(B(t)w, \psi(t))$$

almost everywhere on  $[t_0, T]$ ;

2) the transversal condition:

a) in the case (5):

$$C\left([X(T, w^*)]^1, \psi(T)\right) = -C\left([S_k]^1, -\psi(T)\right);$$

b) in the case (6): for all  $\alpha \in [0, 1]$

$$C\left([X(T, w^*)]^\alpha, \psi(T)\right) \leq C\left([S_k]^\alpha, \psi(T)\right)$$

and there exists  $\beta \in [0, 1]$  such that

$$C\left([X(T, w^*)]^\beta, \psi(T)\right) = C\left([S_k]^\beta, \psi(T)\right);$$

c) in the case (6): for all  $\beta \in [0, 1]$

$$C\left([X(T, w^*)]^\alpha, -\psi(T)\right) \leq C\left([S_k]^\alpha, -\psi(T)\right)$$

and there exists  $\beta \in [0, 1]$  such that

$$C\left([X(T, w^*)]^\beta, -\psi(T)\right) = C\left([S_k]^\beta, -\psi(T)\right).$$

Clearly, that there cases of the transversal condition of the maximum principle correspond to the three cases of the time optimal problems.

**Theorem 5. (necessary optimal condition).** Let the condition A are true and the pair  $(T, w^*(\cdot))$  is optimality.

Then the pair  $(w^*(\cdot), X(\cdot, w^*))$  satisfies the maximum principle on  $[t_0, T]$ .

**Proof.** Let  $w^*(\cdot)$  is the optimal control and  $X(\cdot, w^*)$  is the optimal R-solution of the Systems (1),(4), i.e.

1)  $X(T, w^*) \in Y(T)$ ;

2)  $X(T, w^*) \cap S_k = \emptyset$ .

From 1) and 2) we have

$$\max_{X \in [Y(T)]^1} C(X, \psi) \geq C\left([S_k]^1, -\psi\right)$$

for all  $\psi \in S_1(0)$ .

Consequently

$$p = \max_{X \in [Y(T)]^1} \min_{\psi \in S_1(0)} C(X, \psi) + C\left([S_k]^1, -\psi\right) \geq 0.$$

From  $[X(T, w^*)]^1 \cap [S_k]^1 \neq \emptyset$  we have

$$q(T, \psi) = C\left([X(T, w^*)]^1, \psi\right) + C\left([S_k]^1, -\psi\right) \geq 0$$

for all  $\psi \in S_1(0)$ .

From Theorem 1 we have that the function  $q(T, \psi)$  is continuous on  $R_+ \times S_1(0)$ .

If  $q(T, \psi) > 0$  for all  $\psi \in S_1(0)$  then we have  $q^0(T) = \min_{\psi \in S_1(0)} q(T, \psi) \geq \gamma > 0$ . Hence there exists  $\tau < T$  such that  $q^0(\tau) \geq 0$ . Consequently we have

$$C\left([X(\tau, w^*)]^1, \psi\right) + C\left([S_k]^1, -\psi\right) \geq 0$$

for all  $\psi \in S_1(0)$ , i.e.  $[X(\tau, w^*)]^1 \cap [S_k]^1 \neq \emptyset$ .

It contradicts that  $T$  is optimal time.

If  $p > 0$ ,

$$\begin{aligned} & \max_{X \in [Y(T)]^1} \min_{\psi \in S_1(0)} C(X, \psi) + C\left([S_k]^1, -\psi\right) \\ & = C(\tilde{X}, \tilde{\psi}) + C\left([S_k]^1, -\tilde{\psi}\right) \end{aligned}$$

and  $[X(T, w^*)]^1 \neq \tilde{X}$ , than we have a contradiction. Hence there exist  $\tilde{\psi} \in S_1(0)$  such that

$$\begin{aligned} C\left([X(T, w^*)]^1, \tilde{\psi}\right) & = \max_{X \in [Y(T)]^1} C(X, \tilde{\psi}), \\ C\left([X(T, w^*)]^1, \tilde{\psi}\right) & = -C\left([S_k]^1, -\tilde{\psi}\right). \end{aligned}$$

Consequently

$$\begin{aligned} & \left( \int_0^T \Phi(T)\Phi^{-1}(s)B(s)w^*(s)ds, \tilde{\psi} \right) \\ & = \max_{w(\cdot) \in LW} \left( \int_0^T \Phi(T)\Phi^{-1}(s)B(s)w(s)ds, \tilde{\psi} \right) \end{aligned}$$

Then we have

$$\begin{aligned} & (\Phi(T)\Phi^{-1}(s)B(s)w^*(s), \tilde{\psi}) \\ & = \max_{w(\cdot) \in LW} (\Phi(T)\Phi^{-1}(s)B(s)w(s), \tilde{\psi}) \end{aligned}$$

for almost everywhere  $s \in [t_0, T]$ . If

$$\psi(t) = \frac{(\Phi(T)\Phi^{-1}(t))^T \tilde{\psi}}{\|(\Phi(T)\Phi^{-1}(t))^T \tilde{\psi}\|},$$

than the theorem is proved.

**Example.** Consider the following control linear fuzzy differential inclusions

$$\dot{x} \in \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + w + F, \quad x(0) = 0,$$

where  $x = (x_1, x_2)^T$  is the state;  $w = (w_1, w_2)^T \in W = S_1(0)$  is the control;  $F \in E^2$  is the fuzzy set, where

$$v(f) = \begin{cases} 1 - 4f_1^2 - 9f_2^2, & 4f_1^2 + 9f_2^2 \leq 1 \\ 0, & 4f_1^2 + 9f_2^2 > 1 \end{cases}$$

Consider the following optimal control problem: it is necessary to find the minimal time  $T$  and the control  $w^*(\cdot) \in LW$  such that the fuzzy R-solution of system satisfies of the conditions:

$$X(T, w^*) \cap S_k \neq \emptyset$$

where  $S_k \in E^2$  is the terminal set such, that

$$\sigma(x) = \begin{cases} \sqrt{1 - (x_1 - 2\pi)^2 - (x_2 - 1)^2}, & x \in Q, x_2 \geq 1 \\ \sqrt{1 - (x_1 - 2\pi)^2} & x \in Q, -1 < x_2 < 1 \\ \sqrt{1 - (x_1 - 2\pi)^2 - (x_2 + 1)^2} & x \in Q, x_2 \leq -1 \\ 0 & x \notin Q \end{cases}$$

$$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2 : \begin{matrix} 2\pi - 1 \leq x_1 \leq 2\pi + 1 \\ \sqrt{1 - (x_1 - 2\pi)^2} - 1 \leq x_2 \\ \leq \sqrt{1 - (x_1 - 2\pi)^2} + 1 \end{matrix} \right\}$$

Obviously, the optimal pair  $T = 2\pi$  and  $w^*(t) = (\cos(t), -\sin(t))$  satisfy of the conditions of the Theorem 5:

- 1)  $(w^*(t), \psi(t)) = C(W, \psi(t))$  for a.e.  $t \in [0, 2\pi]$ ;
- 2)  $C\left(\left[X(T, w^*)\right]^1, \psi(T)\right) = -C\left([S_k]^1, -\psi(T)\right)$ ,

where  $\psi(t) = (\cos(t), -\sin(t))^T$  for a.e.  $t \in [0, 2\pi]$ ,

$$\left[X(T, w^*)\right]^1 = (T \cos(T), -T \sin(T))^T = (2\pi, 0)^T,$$

$$[S_k]^1 = \{(x_1, x_2)^T : x_1 = 2\pi, -1 \leq x_2 \leq 1\}.$$

### 5. Conclusions

In the last decades, a number of works devoted to problems of optimal control of multiple-valued trajectories (fuzzy trajectories, trajectory bundles or an ensemble of trajectories) appeared; these works fall into a subdivision of the optimal control theory, namely, the theory of process control under uncertainty and fuzzy conditions. This is conditioned by the fact that, in actual problems arising in economy and engineering in the course of construction of a mathematical model, it is practically impossible to exactly describe the behavior of an object. This is explained by the following fact. First, for some parameters of the object, it impossible to specify exact values and laws of their change, but it is possible to determine the domain of these changes. Second, for the sake of simplicity of the mathematical model being constructed, the equations that describe the behavior of the object are simplified and one should estimate the conse-

quences of such a simplification. Therefore, if is possible to divide the articles devoted to this direction into two types characterized by the following distinctive features:

- 1) There exists an incomplete or fuzzy information on the initial data;
- 2) The equations describing the behavior of the object to be controlled are assumed to be inexact, for example, they can contain some parameters whose exact values and laws of variation are unknown but the domain of their values is fuzzy.

In the second case, fuzzy differential inclusions are frequently used to describe behavior of objects. The reason is that, first this approach is most obvious and, second, theory of fuzzy and ordinary differential inclusions is well found and is rapidly developed at the present time.

In the present paper, the necessary conditions of optimal of control for a system of the latter form of equations with the fuzzy R-solutions are formulated and proved.

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