

# Constructing Entanglers in 2-Players–N-Strategies Quantum Game

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## Abstract

In quantum games based on 2-player- $N$ -strategies classical games, each player has a quNit (a normalized vector in an  $N$ -dimensional Hilbert space  $\mathcal{H}_N$ ) upon which he applies his strategy (a matrix  $U \in \text{SU}(N)$ ). The players draw their payoffs from a state  $|\Psi\rangle = J^\dagger U_1 \otimes U_2 J |\Psi_0\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$ . Here  $|\Psi_0\rangle$  and  $J$  (both determined by the game's referee) are respectively an unentangled 2-quNit (pure) state and a unitary operator such that  $|\Psi_1\rangle \equiv J |\Psi_0\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$  is partially entangled. The existence of pure strategy Nash equilibrium in the quantum game is intimately related to the degree of entanglement of  $|\Psi_1\rangle$ . Hence, it is practical to design the entangler  $J = J(\beta)$  to be dependent on a *single* real parameter  $\beta$  that controls the degree of entanglement of  $|\Psi_1\rangle$ , such that its von-Neumann entropy  $S_N(\beta)$  is continuous and obtains any value in  $[0, \log N]$ . Designing  $J(\beta)$  for  $N = 2$  is quite standard. Extension to  $N > 2$  is not obvious, and here we suggest an algorithm to achieve it. Such construction provides a special quantum gate that should be a useful tool not only in quantum games but, more generally, as a special gate in manipulating quantum information protocols.

## Keywords

Quantum Games, Qubits, Qutrits, quNits, Controlled Entanglement, von Neumann Entropy

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## 1. Introduction

The theory of quantum games is an evolving discipline that, similar to quantum information [1] [2], explores the implications of quantum mechanics to fields outside physics proper, such as economics, finance, auctions, gam-

bling, etc. [3]-[9]. One way of constructing a quantum game is to start from a standard (classical) game and to “quantize” it by formulating appropriate rules and letting the players employ quantum information tools such as qubits and quantum gates (or strategies in the quantum game nomenclature). This procedure has been applied on classical strategic games that describe an interactive decision-making in which each player chooses his strategy only once, and all choices are taken simultaneously. A simple example is a quantum game based on 2-player-2-strategies classical game usually defined by a game table (for example, the prisoner dilemma). We refer to it briefly as a 2-2 game.

## 2. Quantum Games: The Role of Entanglement

There is an extensive work on the quantized version of classical strategic 2-2 games, most of them are based on the protocol specified in [5]. It requires application of an entanglement (unitary) operator  $J(\beta)$  (where  $\beta$  is a real parameter), which acts on a non-entangled 2-qubit (pure) state resulting in an entangled state whose degree of entanglement is measured by its von-Neumann entropy  $S_2(\beta)$ . A desired property of  $J(\beta)$  is that  $S_2(\beta)$  is a continuous function of  $\beta$  that varies (preferably monotonically) between 0 (no entanglement) and  $\log 2$  (maximal entanglement). The reason for exploring partially entangled 2-qubit states is that the existence of pure strategy Nash equilibrium in the 2-2 quantum game crucially depends on the degree of entanglement (see below).

Controlling the entropy by a *single parameter* that all values between 0 and  $\log 2$  are obtained is referred to here as *single parameter completeness*. The relevance of this problem to quantum information in general is quite obvious. An important procedure in quantum information is to design a quantum gate that generates Bell states [1] [2]. The gate, operating on a non-entangled two qubit state, results in a Bell state that is *maximally entangled*. Therefore, designing the gate  $J(\beta)$  that does the job for an arbitrary two qubit state resulting in an entangled state whose degree of entanglement is controlled by a single parameter should be very useful.

## 3. Classical Commensurability

Another practical property required from  $J(\beta)$  is that it can easily be constructed from the *classical strategies*. In a 2-2 game, the classical strategy of a player is  $i\sigma_y \in \text{SU}(2)$ , and an appropriate construction is then

$J(\beta) = e^{-i\frac{\beta}{2}\sigma_y \otimes \sigma_y}$ . Its action on an unentangled 2-qubit state (e.g.  $|00\rangle$ ) yields,

$$J(\beta)|00\rangle = e^{-i\frac{\beta}{2}\sigma_y \otimes \sigma_y} |00\rangle = \cos\frac{\beta}{2}|00\rangle - i\sin\frac{\beta}{2}|11\rangle, \quad (1)$$

where the left (right) factor in the Kronecker product refers to player 1 (2). In this way,  $|\Psi_1\rangle = J(\beta)|00\rangle$  appears in a *Schmidt decomposed form*, enabling an easy computation of the corresponding entanglement entropy of the 2-qubit state on the RHS as

$$S_2(\beta) = -\left(\cos^2\frac{\beta}{2}\log\cos^2\frac{\beta}{2} + \sin^2\frac{\beta}{2}\log\sin^2\frac{\beta}{2}\right). \quad (2)$$

Thus,  $S_2(\beta)$  is a continuous function of  $\beta$  and gets all values in  $[0, \log 2]$ , namely  $J(\beta)$  as defined in Equation (1) satisfies single parameter completeness. Other properties (of less significance) are that  $S_2(\beta)$  is periodic with period  $\pi$ , symmetric about the mid-point  $\frac{\pi}{2}$ , with  $S_2(0) = 0$  and  $S_2\left(\frac{\pi}{2}\right) = \log 2$ .

The entangler defined in Equation (1) has a property referred to as *classical commensurability*,  $[J(\beta), \sigma_y \otimes \sigma_y] = 0$ . Following the rules of the game specified in [5], it means that players in a 2-2 quantum game can, if they wish, use their classical strategies as a special case and if they do so, they collect the corresponding classical payoffs. In most cases, however, the classical strategies do not constitute a pure-strategy Nash Equilibrium (NE) (defined below). Generically, as we explain below, classical commensurability does not hold for  $N > 2$  [10].

## 4. The Present Work

In the present work we examine the issue of constructing  $J(\beta)$  for a  $2-N$  quantum game based on a 2-

players- $N$ -strategies classical game. We suggest a natural extension of Equation (1) for constructing an operator  $J(\beta)$  that turns a non-entangled 2-quNit state into an entangled one. For  $N=3,4$  the corresponding von-Neumann entropy  $S_N(\beta)$  varies continuously between 0 and  $\log N$  so that single parameter completeness is satisfied. Unfortunately, this method does not work for  $N > 4$  because in that case  $S_N(\beta) < \log N$ . To alleviate this deficiency we suggest another method (albeit less intuitive) to design  $J(\beta)$  that satisfies single parameter completeness for any  $N$ . In what follows we will first introduce the classical 2-3 game using quantum information language and formulate its quantum version (extension to  $N > 3$  is straightforward). Most of this introductory exposition is well established and is included here merely for self consistence. Then, in the second step, we shall address the issue of constructing  $J(\beta)$ .

## 5. Two-Players-Three-Strategies Classical Games: Trits

Consider a two-players classical game with three strategies for each player. For example, two prisoners may have three options, marked as three values of a trit  $C=|1\rangle$ ,  $S=|2\rangle$  and  $D=|3\rangle$  for Confess, Stay quiet or Don't confess. The two prisoner "system" can be found in nine two-trit states  $|ij\rangle$ ,  $i=1,2,3$ ,  $j=1,2,3$ , corresponding to the nine entries of the game table. The protocol of the classical game with 2-players and 3-strategies is as follows: The referee (judge) calls the players (prisoners) and tells them he assumes that they are in an initial two-trit state  $|11\rangle$  meaning (C,C) namely both confess. He then asks them to decide whether to leave their respective trit state as it is on  $|1\rangle$  or to change it either to  $|2\rangle$  (meaning S) or to  $|3\rangle$  (meaning D). These replacement operations (specified explicitly below) are the players' classical strategies. If the initial state suggested by the referee is  $|11\rangle$  the strategies of the two players include  $\mathbf{1}_3$  (leaving the trit at  $|1\rangle$  as it is),  $S_{12}$  (swapping of  $|1\rangle$  and  $|2\rangle$ ) namely, replacing C by S) and  $S_{13}$  (swapping  $|1\rangle$  and  $|3\rangle$ ) namely replacing C by D). These three operations generate the group  $\mathbf{S}_3$  of permutations of three elements  $\begin{pmatrix} 123 \\ ijk \end{pmatrix}$ . Explicitly,

$$\mathbf{1}_3 = (123), \quad S_{12} = (213), \quad S_{13} = (321), \quad S_{13}S_{12}S_{13} = S_{23} = (132), \quad S_{13}S_{12} = S_{312} = (312), \quad S_{12}S_{13} = S_{231} = (231)$$

We shall indicate below that a quantum strategy is a gate represented by an  $SU(3)$  matrix. A reasonable requirement for the procedure of "quantizing" a classical game is that the classical strategies obtain as special case of the quantum ones. For that purpose we need to construct a representation of the permutation group  $\mathbf{S}_3$  in terms of unitary matrices with unit determinant. This can be achieved by choosing

$$S_{12} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad S_{13} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad S_{23} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$S_{312} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_{231} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{1}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (3)$$

For example, suppose player 1 and 2 choose respective strategies  $s_1 = S_{12}$  and  $s_2 = S_{13}$ . This brings the system into a state  $s_1 \otimes s_2 |11\rangle = |23\rangle = |SD\rangle$ . Then the respective payoff of player  $k$ ,  $k=1,2$  will be  $u_k(s_1, s_2) = u_k(S, D) = u_k(2, 3)$ , where  $u_k(i, j)$  is the payoff of player  $k$  at entry  $(i, j)$  of the game table.

## 6. The Analogous Quantum Game: 1 and 2 Quirt States

We now briefly explain the structure of the corresponding quantum game. Its main ingredients are qutrits, quantum strategies, and entanglement operations. Both versions use the same game table but the payoff rules are somewhat different.

Consider the three dimensional Hilbert space  $\mathcal{H}_3$  with orthonormal basis vectors  $|1\rangle$ ,  $|2\rangle$  and  $|3\rangle$ . A qu-trit is a vector  $|\psi\rangle = v_1|1\rangle + v_2|2\rangle + v_3|3\rangle \in \mathcal{H}_3$  of unit norm,  $|v_1|^2 + |v_2|^2 + |v_3|^2 = 1$ . A general 2-qu-trit state is a normalized vector in  $\mathcal{H}_3 \otimes \mathcal{H}_3$ ,

$$|\Psi\rangle = \sum_{i,j=1}^3 v_{ij} |ij\rangle, \quad v_{ij} \in \mathbb{C}, \quad \sum_{i,j=1}^3 |v_{ij}|^2 = 1 \quad (4)$$

A maximally entangled two-qutrit state is written as

$$|\Psi\rangle_{\text{ME}} = \frac{1}{\sqrt{3}}(u_1|11\rangle + u_2|22\rangle + u_3|33\rangle), \quad u_i \in \mathbb{C}, \quad |u_i| = 1, \quad (5)$$

given in a Schmidt decomposed form. Its entanglement degree is measured by the von Neumann entropy

$$S_3 = -\frac{1}{3} \sum_{i=1}^3 |u_i|^2 \log(|u_i|^2/3) = \log 3.$$

## 7. 2-3 Quantum Game Strategies

A strategy of a player in a 2-3 quantum game is an  $SU(3)$  matrix by which he operates on his qutrit (that is a quantum gate). A strategy  $U(\boldsymbol{\gamma}) \in SU(3)$  depends on eight Euler angles  $\boldsymbol{\gamma} \equiv \{\alpha_1, \alpha_2, \dots, \alpha_8\}$ . The explicit expression of  $U(\boldsymbol{\gamma})$  in terms of Gellman matrices  $\{\lambda_m\}$ , ( $m=1, 2, \dots, 8$ ) is well known. For quantum game theory, a practical parametrization of  $U(\boldsymbol{\gamma})$  is suggested in [10].

## 8. 2-3 Quantum Game Procedure

The referee suggests an initial non-entangled two-qutrit state  $|\Psi_0\rangle$  (e.g, the analog of the classical two trit state  $|\Psi_0\rangle = |11\rangle$ ). Before letting the players apply their quantum strategies, the referee operates on  $|\Psi_0\rangle$  with a unitary operator  $J(\beta)$  such that  $|\Psi_1\rangle = J(\beta)|\Psi_0\rangle$  is entangled (otherwise the game remains classical). Construction of the operator  $J(\beta)$  (our central goal) is detailed below. At this stage of the game, the players apply their respective strategies  $U_1 \otimes U_2$ . Finally, the referee applies the operator  $J^\dagger(\beta)$ , leading to the final state

$$|\Psi\rangle = \overbrace{J^\dagger(\beta)}^{\text{referee}} \overbrace{U(\boldsymbol{\gamma}_1) \otimes U(\boldsymbol{\gamma}_2)}^{\text{players}} \overbrace{J(\beta)|11\rangle}^{\text{referee}} = \sum_{i,j=1,3} v_{ij} |ij\rangle, \quad (6)$$

where  $\boldsymbol{\gamma}_k$  is the octet of 8 Euler angles defining the  $SU(3)$  matrix  $U(\boldsymbol{\gamma}_k)$  (that is the strategy of player  $k=1, 2$ ). The payoff  $P_k$  of player  $k=1, 2$  is given by

$$P_k(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2) = \sum_{i,j=1,3} u_k(i, j) |v_{ij}(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)|^2, \quad k=1, 2, \quad (7)$$

where  $u_k(i, j)$  are the payoffs at entry  $(i, j)$  of the *classical* game table. Like in the classical game, each player chooses a strategy with the goal of maximizing his payoff.

## 9. Pure Strategy Nash Equilibrium (NE)

Because the set of 8 Euler angles  $\boldsymbol{\gamma}$  uniquely determines the player's strategy  $U(\boldsymbol{\gamma}) \in SU(3)$ , a pure strategy NE in the 2-3 quantum game is a pair of strategies  $(\boldsymbol{\gamma}_1^*, \boldsymbol{\gamma}_2^*)$  (each represents 8 angles) such that

$$\begin{aligned} P_1(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2^*) &\leq P_1(\boldsymbol{\gamma}_1^*, \boldsymbol{\gamma}_2^*), \quad \forall \boldsymbol{\gamma}_1, \\ P_2(\boldsymbol{\gamma}_1^*, \boldsymbol{\gamma}_2) &\leq P_2(\boldsymbol{\gamma}_1^*, \boldsymbol{\gamma}_2^*), \quad \forall \boldsymbol{\gamma}_2. \end{aligned} \quad (8)$$

The question of whether pure strategy NE exists in a 2-2 quantum game and its relation to the degree of entanglement (controlled by  $\beta$ ) has been discussed in numerous works [11]-[17]. In brief, if there is NE in the classical game that is not *Pareto efficient* [18], then there is a critical value  $\beta_c$  above which there is no pure strategy NE in the quantum game. As  $\beta$  approaches  $\beta_c$  from below, the respective payoffs in the quantum game at NE approach the Pareto point of cooperation [13] [17]. This is the main reason why, right from the onset, we stress the relevance of partially entangled 2-quNit states where  $S_N(\beta) < \log N$ .

## 10. Absence of Classical Commensurability

We now explain why in a 2-3 quantum game there is no classical commensurability [10]. Recall that classical commensurability means that if the players use classical strategies they respectively get their classical payoffs. For a classical strategy  $\bar{\boldsymbol{\gamma}}$  we have  $U(\bar{\boldsymbol{\gamma}}) \in \mathbf{S}_3$ . From Equation (6) it means that  $J$  should commute with all outer products of the classical strategies. If the initial state is  $|\Psi_0\rangle = |11\rangle$  the 9 outer products of classical

strategies are  $s_1 \otimes s_2$ , where  $s_k \in \{\mathbf{1}_3, S_{12}, S_{13}\}$  (see Equation (3)). Classical commensurability then requires

$$[J, S_{12} \otimes S_{13}] = [J, S_{13} \otimes S_{12}] = 0. \quad (9)$$

This is possible only if  $J$  is a function of  $A \otimes A$  where  $A$  is a  $3 \times 3$  matrix satisfying  $[A, S_{12}] = [A, S_{13}] = 0$ , and  $A$  is not just a multiple of  $\mathbf{1}$ . But this is impossible because  $S_{12}$  and  $S_{13}$  generate an irreducible representation of the permutation group  $\mathbf{S}_3$  and hence, according to Schure's lemma  $A$  is just a multiple of  $\mathbf{1}_3$ . These arguments naturally hold for any  $N > 2$ .

## 11. Designing the Entangler $J(\beta)$

The main result of the present study concerns the analysis and construction of an entanglement operator  $J(\beta)$  for  $N > 2$ . We carry it out for  $N = 3$  and then extend it straightforwardly to any  $N$ . For  $N = 3$  we require that  $J(\beta)|11\rangle$  yields an entangled 2-qutrit state with  $\beta$  specifying the degree of entanglement that achieves any value between 0 and  $\log 3$ . Following the 2-2 game framework specified in Equation (1), we try to construct  $J$  by exponentiating a combination of *classical strategies*. In order to get the "diagonal" 2-qutrit states  $|22\rangle$  and  $|33\rangle$  from the qutrit state  $|11\rangle$ , we have to operate on  $|11\rangle$  with  $Z \equiv S_{12} \otimes S_{12} + S_{13} \otimes S_{13}$ . Therefore, we define

$$J(\beta) = e^{\frac{i\beta}{2}Z} = e^{\frac{i\beta}{2}(S_{12} \otimes S_{12} + S_{13} \otimes S_{13})}. \quad (10)$$

Calculation of the exponent yields

$$J(\beta)|11\rangle = \frac{1}{3} \left[ \left( 2e^{-\frac{i\beta}{2}} + e^{i\beta} \right) |11\rangle + \left( e^{i\beta} - e^{-\frac{i\beta}{2}} \right) (|22\rangle + |33\rangle) \right]. \quad (11)$$

Maximal entanglement obtains when the absolute values of all three coefficients are equal, namely

$$\left| 2e^{-\frac{i\beta}{2}} + e^{i\beta} \right| = \left| e^{i\beta} - e^{-\frac{i\beta}{2}} \right| \Rightarrow \beta = \frac{4\pi}{9}, \frac{8\pi}{9}. \quad (12)$$

Here  $S_3(\beta)$  raises monotonically from 0 to its first maximum  $\log 3$ , hence we have found the desired entanglement operator that satisfies single parameter completeness. **Figure 1(a)** displays the von Neumann entropy  $S_3(\beta)$  of the entangled 2-qutrit state (11) as function of  $\beta$ . Here again it possesses other properties, namely  $S_3(\beta)$  is a periodic function of  $\beta$  with period  $\frac{4\pi}{3}$  and it is symmetric about the mid-point  $\frac{2\pi}{3}$  where it

has a local *minimum*. It has two maxima for  $\beta = \frac{4\pi}{9}, \frac{8\pi}{9}$  where it equals  $\log 3$ . Inspecting Equation (10),

we see that, in quantum games, the entanglement is not obtained in terms of spin rotations but, rather, in terms of permutation exponentials that are  $SU(3)$  matrices (for  $N = 2$  these are the same).

## 12. Extension to Arbitrary $N$

Let  $S_{ij} \in \mathbf{S}_N$  denote the  $N \times N$  matrix representing the permutation  $i \leftrightarrow j$  and let  $|ij\rangle$ ,  $i, j = 1, 2, \dots, N$  be an unentangled 2-quNit state. To get an entangled state from  $|11\rangle$  we define and assert that

$$J(\beta)|11\rangle = e^{\frac{i\beta}{2} \sum_{j=2}^N S_{1j} \otimes S_{1j}} |11\rangle = \frac{e^{-\frac{i\beta}{2}}}{N} \left[ \left( e^{\frac{iN\beta}{2}} + N - 1 \right) |11\rangle + \left( e^{\frac{iN\beta}{2}} - 1 \right) \sum_{j=2}^N |jj\rangle \right]. \quad (13)$$

To proceed, let us define the absolute value squared of the coefficients

$$f_1(\beta) = \frac{\left| e^{\frac{iN\beta}{2}} + N - 1 \right|^2}{N^2}, \quad f_2(\beta) = \frac{\left| e^{\frac{iN\beta}{2}} - 1 \right|^2}{N^2}. \quad (14)$$

It is easy to verify that: 1)  $0 \leq f_i(\beta) \leq 1$ ; 2)  $f_1(\beta) + (N-1)f_2(\beta) = 1, \forall \beta$ ; 3)  $f_i(\beta)$  is periodic with pe-

riod  $\frac{4\pi}{N}$  and symmetric about the mid-point  $\frac{2\pi}{N}$ . The entanglement entropy is

$$S_N(\beta) = -[f_1(\beta)\log f_1(\beta) + (N-1)f_2(\beta)\log f_2(\beta)]. \quad (15)$$

Maximal entanglement  $S_N(\beta) = \log N$  obtains for  $\beta_0$  that is the solution of the equality

$$\frac{\left| e^{\frac{iN\beta_0}{2}} + N - 1 \right|^2}{N^2} = \frac{\left| e^{\frac{iN\beta_0}{2}} - 1 \right|^2}{N^2} = \frac{1}{N}. \quad (16)$$

For  $N = 3$  the two solutions are specified in Equation (12). For  $N = 4$ , there is a single solution at  $\beta_0 = \pi/2$ , as shown in **Figure 1(b)**. Thus, for  $N = 3, 4$  we have achieved our goal of constructing an entanglement operator  $J(\beta)$  such that the degree of entanglement  $S_N(\beta)$  of the 2-quNit state  $J(\beta)|11\rangle$  varies continuously reaching all values in the interval  $[0, \log N]$ , so that single parameter completeness is satisfied.

For  $N > 4$  there is no solution  $\beta_0$  of Equation (16), and maximal entanglement is not achieved. It might be argued that 2-N quantum games with  $N > 4$  are much rarer than those with smaller  $N$  but we believe that the construction of  $J(\beta)$  that satisfies single parameter completeness also for  $N > 4$  is useful in other areas (outside the ballpark of quantum game theory), so we carry it out for the sake of completeness.

The method suggested here is not based on permutation exponentials as in Equation (10). It consists of the following steps.

1) Assume a lexicographic order of the  $N^2$  basis states  $\{|ij\rangle\}$  such that the diagonal states  $\{|ii\rangle\}$  appear in the first  $N$  places. Choose a unitary  $N^2 \times N^2$  matrix of the form  $\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}$  where  $R$  is an  $N \times N$  uniry matrix

with equal first column elements  $R_{i1} = \frac{1}{\sqrt{N}}$  and  $I$  is the  $N(N-1) \times N(N-1)$  unit matrix. The problem is then reduced to the  $N$  dimensional subspace spanned by  $\{|ii\rangle\}$ . By construction,

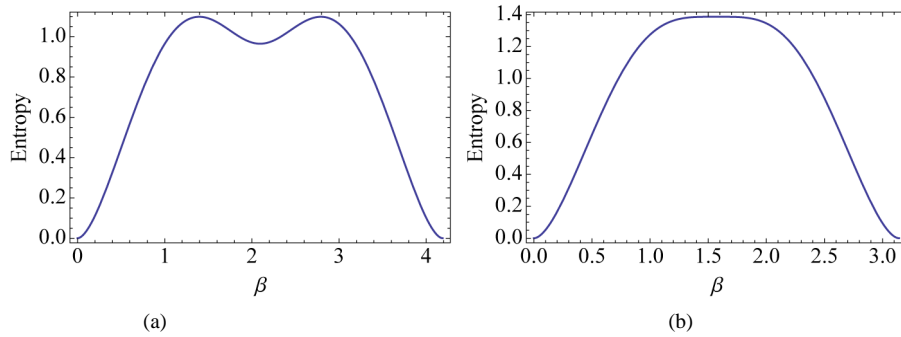
$$R|11\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N |ii\rangle$$

that is a maximally entangled state.

2) Diagonalize  $R$  as  $R = V\Lambda V^{-1}$  where  $V$  is the matrix of eigenvectors of  $R$ , and

$$\Lambda = \text{diag}\{e^{in_1}, e^{in_2}, \dots, e^{in_N}\}$$

is the diagonal matrix of (unimodular) eigenvalues of  $R$  with eigenphases  $\{\eta_i\}$ .



**Figure 1.** von Neumann entropy  $S_N(\beta)$  defined in Equation (15), for the 2-quNit state  $J(\beta)|11\rangle$  defined in Equation (13): (a)  $N=3$ , (b)  $N=4$ .  $S_N(\beta)$  varies continuously reaching all values in the interval  $[0, \log N]$ , so that single parameter completeness is satisfied.

Here  $S_N(\beta)$  is periodic with period  $\frac{4\pi}{N}$ .

3) Now consider the matrix

$$J(\beta) \equiv V \Lambda(\beta) V^{-1}, \quad \text{with } \Lambda(\beta) \equiv \text{diag}\{e^{i\beta\eta_1}, e^{i\beta\eta_2}, \dots, e^{i\beta\eta_N}\}. \quad (17)$$

By construction,  $J(0) = \mathbf{1}_{N \times N}$  and  $J(1) = R$ . Hence, the state  $J(\beta)|11\rangle = \sum_{i=1}^N [J(\beta)]_{ii} |ii\rangle$  is partially entangled. Since it is given in a Schmidt decomposed form, the corresponding von Neumann entropy  $S_N(\beta)$  is easily calculable.  $S_N(\beta)$  is continuous in  $[0, \infty)$  with  $S_N(0) = 0$  and  $S_N(1) = \log N$ , namely single parameter completeness is satisfied. Generically, the eigen phases are not rational multiples of  $\pi$  so that  $S_N(\beta)$  is not periodic, but this lack of periodicity is of no special significance.

### 13. Illustration for $N = 5$

A convenient way to build an appropriate unitary  $N \times N$  matrix  $R$  is to start from a simple non-singular matrix  $A$  and then orthogonalize it within the Gram-Schmidt procedure. For example,

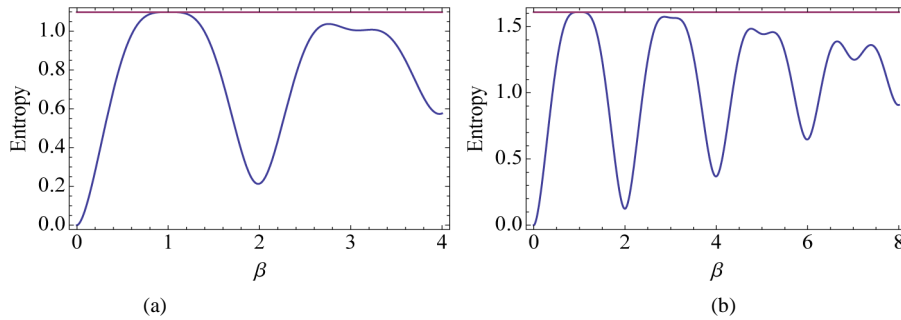
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{\sqrt{3}}{\sqrt{10}} & \frac{\sqrt{3}}{\sqrt{14}} & \frac{3}{2\sqrt{14}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{5}} & \frac{\sqrt{3}}{\sqrt{10}} & -\frac{\sqrt{3}}{\sqrt{14}} & -\frac{3}{2\sqrt{14}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{5}} & -\frac{\sqrt{2}}{\sqrt{15}} & 2\sqrt{\frac{2}{21}} & -\frac{3}{2\sqrt{14}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{5}} & -\frac{\sqrt{2}}{\sqrt{15}} & -\frac{\sqrt{2}}{\sqrt{21}} & \frac{5}{2\sqrt{14}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{5}} & -\frac{\sqrt{2}}{\sqrt{15}} & -\frac{\sqrt{2}}{\sqrt{21}} & -\frac{1}{\sqrt{14}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Proceeding with the list of steps prescribed above we can easily construct  $J(\beta)$  and compute the von Neumann entropy of the state  $|\Psi_1\rangle = J(\beta)|11\rangle$  upon which the players apply their strategies according to the game protocol specified in Equation (6). The result is given in [Figure 2](#).

As explained in the figure's caption, the degree of entanglement is controlled by a single parameter and  $S_N(\beta)$  is a continuous function of  $\beta$  reaching any value in the interval  $[0, \log N]$ . Thus we have achieved our goal of constructing an entangler  $J(\beta)$  that turns a non-entangled 2-quNit state into an entangled one given in a Schmidt decomposed form with single parameter completeness satisfied.

### 14. Summary

In conclusion, we suggest two methods to design an entanglement operator  $J(\beta)$  that turns a non-entangled



**Figure 2.** von Neumann entropy for 2-quNit state  $|\Psi_1\rangle = J(\beta)|11\rangle$ , where  $J(\beta)$  is defined in Equation (17). (a), (b) correspond to  $N = 3, 5$ . By constructing,  $S_N(0) = 0$  and  $S_N(1) = \log N$  (horizontal lines). Since  $S_N(\beta)$  is a continuous function of  $\beta$  it reaches any value in the interval  $[0, \log N]$ . That is, single parameter completeness is achieved for  $N > 4$ .



2-quNit state to a partially entangled state whose von Neumann entropy is fully controlled by a single real parameter. The first method is intuitively clear and simple, based on exponential of classical strategies, Equation (10), and results in the von Neumann entropy, as displayed in **Figure 1**. This method does not work for  $N > 4$  because the resulting entropy does not reach the maximally entangled value  $\log N$ . For that reason we suggest another method that is somewhat less transparent but works for any  $N$ . The resulting entropy as function of  $\beta$  is displayed in **Figure 2**.

The entangler  $J(\beta)$  constructed here generalizes, in two directions, the familiar quantum gate used in quantum information science to create Bell states from non-entangled two-qubit state. First, it is applicable to any two-quNit state, and second, it contains a single continuous parameter that controls the degree of entanglement of the resultant state.

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