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Hamiltonian, Path Integral and BRST Formulations of the Restricted Gauge Theory of QCD_2

Usha Kulshreshtha[1*], Daya Shankar Kulshreshtha[2#], James P. Vary[3]

[1]Department of Physics, Kirori Mal College, University of Delhi, Delhi, India
[2]Department of Physics and Astrophysics, University of Delhi, Delhi, India
[3]Department of Physics and Astronomy, Iowa State University, Ames, Iowa, USA
Email: *ushakulsh@gmail.com, dskulsh@gmail.com, jvary@iastate.edu

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Abstract

We study the Hamiltonian, path integral and Becchi-Rouet-Stora and Tyutin (BRST) formulations of the restricted gauge theory of QCD_2 à la Cho *et al.* under appropriate gauge-fixing conditions.

Keywords

Hamiltonian Quantization, Path Integral Quantization, BRST Quantization, Quantum Chromodynamics, QCD_2, Field Theories in Lower Dimensions, Gauge-Invariant Theories, Gauge-Fixing

1. Introduction

In this work we consider the restricted gauge theory of quantum chromodynamics (QCD) in one-space one-time dimension (QCD_2) à la Cho *et al.* [1]-[14], studied rather widely [2]-[23], and study its quantization using Hamiltonian [24], path integral [25] [26] [27] [28] and Becchi-Rouet-Stora and Tyutin (BRST) [29] [30] [31], formulations [24]-[31], in the usual instant-form (IF) of dynamics (on the hyperplanes: $x^0 = t = \text{constant}$) [32] [33]. We recap the basis of this theory in the next section where we also highlight the motivations for the present study. The theory is seen to be gauge-invariant (GI) possessing a set of first-class constraints [14]. We quantize this theory under appropriate gauge-fixing conditions (GFC's) using the Hamiltonian and path integral formulations [24] [25] [26] [27]

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[28].

However, in the usual Hamiltonian and path integral quantization [24] [25] [26] [27] [28] of a theory under some GFC's, the gauge-invariance of the theory gets broken because of the gauge-fixing. In view of this, in order to achieve the quantization of the theory such that the gauge-invariance of the theory is maintained even under gauge-fixing, we go to a more generalized procedure called the BRST quantization [29] [30] [31], [27] [28] [29] [30] [31], where the extended gauge symmetry of the theory (called the BRST symmetry) is maintained even under gauge-fixing. In fact, this also necessitates a study of the BRST quantization of the theory to achieve a kind of complete quantization of the theory.

The paper is organized as follows. In the next section, we briefly recap the basics of the so-called restricted gauge theory of QCD_2 à la Cho *et al.* [1]-[13]. In Section 3, we study its Hamiltonian and path integral formulations. Its BRST formulation is studied in Section 4. Finally the summary and discussion is given in Section 5.

2. Restricted Gauge Theory of QCD_2 à la Cho *et al.*: A Recap of Basics

In this section, we recap the basics of the restricted gauge theory of QCD_2 à la Cho *et al.* [1]-[14] and others [1]-[23]. The theory makes use of the so-called "Cho-decomposition", which is, in fact, the gauge independent decomposition of the non-Abelian potential into the restricted potential and the valence potential and it helps in the clarification of the topological structure of the non-Abelian gauge theory, and it also takes care of the topological characters in the dynamics.

The non-Abelian gauge theory has rich topological structures manifested by the non-Abelian monopoles, the multiple vacua and the instantons and one needs to take into account these topological characters in the non-Abelian dynamics. Since the decomposition of the non-Abelian connection contains these topological degrees explicitly, it can naturally take care of them in the non-Abelian dynamics.

An important consequence of the decomposition is that it allows one to view QCD as the restricted gauge theory (made of the restricted potential) which is coupled to a gauge-covariant colored vector field (the valence potential). The restricted potential is defined in such a way that it allows a covariantly constant unit isovector $\hat{n}$ everywhere in space-time, which enables one to define the gauge-independent color direction everywhere in space-time and, at the same time, allows one to define the magnetic potential of the non-Abelian monopoles. Furthermore it has the full SU(2) gauge degrees of freedom, in spite of the fact that it is restricted. Consequently, the restricted QCD made of the restricted potential describes a very interesting dual dynamics of its own, and plays a crucial role in the understanding of QCD.

On the other hand, the restricted QCD is a constrained system, due to the presence of the topological field $\hat{n}$ which is constrained to have the unit norm. A natural way to accommodate the topological degrees into the theory is to introduce a topological field $\hat{n}$ of unit norm, and to decompose the connection into the Abelian projection part which leaves $\hat{n}$ a covariant constant and the remaining part which forms a covariant vector field:

$$\begin{aligned} \mathbf{A}_\mu &= A_\mu \hat{n} - \frac{1}{g}\hat{n}\times\partial_\mu \hat{n} + \mathbf{X}_\mu = \hat{A}_\mu + \mathbf{X}_\mu \\ A_\mu &= \hat{n}\cdot\mathbf{A}_\mu \\ \hat{n}^2 &= 1 \end{aligned} \tag{1a}$$

where A^μ is the "electric" potential and the Abelian projection A^μ is precisely the connection which leaves $\hat{n}$ invariant under the parallel transport and makes $\hat{n}$ a covariant constant:

$$\hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + g\hat{A}_\mu \times \hat{n} = 0 \tag{2}$$

Also, under the infinitesimal gauge-transformation:

$$\begin{aligned} &\delta\mathbf{A}_\mu = \frac{1}{g}D_\mu\vec{\alpha}\ ,\ \delta A_\mu = \frac{1}{g}\hat{n}\cdot\partial_\mu\vec{\alpha}\ ,\ \delta\hat{A}_\mu = \frac{1}{g}\hat{D}_\mu\vec{\alpha} \\ &\delta\mathbf{X}_\mu = -\vec{\alpha}\times\mathbf{X}_\mu\ ,\ \delta\hat{n} = -\vec{\alpha}\times\hat{n} \end{aligned} \tag{3}$$

This shows that A^μ by itself describes an $SU(2)$ connection which enjoys the full $SU(2)$ gauge degrees of freedom. The restricted potential A^μ is defined by the Abelian projection and the connection space (the space of all gauge potentials) forms an affine space. Indeed the affine nature of the connection space guarantees that one can describe an arbitrary potential simply by adding a gauge-covariant piece X_μ to the restricted potential.

The above mentioned decomposition is known as the Cho-decomposition or the Cho-Faddeev-Niemi decomposition. It was introduced [2]-[23] in an attempt to demonstrate the monopole condensation in QCD. The decomposition itself and the importance of this decomposition in clarifying the non-Abelian dynamics in QCD_2 has been studied by many authors and for further details we refer to the work of Refs. [2]-[23].

The restricted potential A^μ actually has a dual structure and the field strength made of the restricted potential is decomposed as:

$$\hat{F}_{\mu\nu} = \left(F_{\mu\nu} + H_{\mu\nu}\right)\hat{n},\ F_{\mu\nu} = \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \tag{4a}$$

$$H_{\mu\nu} = -\frac{1}{g}\hat{n}\cdot\left(\partial_\mu\hat{n}\times\partial_\nu\hat{n}\right) = \left(\partial_\mu\tilde{C}_\nu - \partial_\nu\tilde{C}_\mu\right) \tag{4b}$$

where C_μ is the "magnetic" potential. Further, following the work of Refs. [2]-[23], it is possible to introduce the magnetic potential as above (at least locally section-wise) in view of the following identity:

$$\partial_\mu\tilde{H}_{\mu\nu} = 0,\ \tilde{H}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}H_{\rho\sigma} \tag{5}$$

which allows one to identify the non-Abelian monopole potential by:

$$\mathbf{C}_\mu = -\frac{1}{g}\hat{n}\times\partial_\mu\hat{n} \tag{6}$$

in terms of which the magnetic field is expressed as

$$\mathbf{H}_{\mu\nu} = \partial_\mu\mathbf{C}_\nu - \partial_\nu\mathbf{C}_\mu + g\mathbf{C}_\mu\times\mathbf{C}_\nu = -\frac{1}{g}\partial_\mu\hat{n}\times\partial_\nu\hat{n} = H_{\mu\nu}\hat{n} \tag{7}$$

With the above connection (-albeit decomposition) one has:

$$\begin{aligned}
&\mathbf{F}_{\mu\nu} = \left[\left(F_{\mu\nu}+H_{\mu\nu}\right)\hat{n} + \hat{D}_\mu\mathbf{X}_\nu - \hat{D}_\nu\mathbf{X}_\mu + g\mathbf{X}_\mu\times\mathbf{X}_\nu\right]\\
&\hat{n}\cdot\mathbf{X}_\mu = 0\\
&\hat{n}\cdot\hat{D}_\mu\mathbf{X}_\nu = 0
\end{aligned} \tag{8}$$

and for the Yang-Mills Lagrangian density one has: [1]-[15]:

$$\begin{aligned}
\tilde{\mathcal{L}} = \Bigg[&-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \frac{1}{4}\left(\hat{D}_\mu\mathbf{X}_\nu - \hat{D}_\nu\mathbf{X}_\mu\right)^2 - \frac{g}{2}\hat{F}_{\mu\nu}\cdot\left(\mathbf{X}_\mu\times\mathbf{X}_\nu\right)\\
&-\frac{g^2}{4}\left(\mathbf{X}_\mu\times\mathbf{X}_\nu\right)^2 + \lambda\left(\hat{n}^2-1\right) + \lambda_\mu\hat{n}\cdot\mathbf{X}_\mu\Bigg]
\end{aligned} \tag{9}$$

where λ and λ_μ are the Lagrange multiplier fields and $g^{\mu\nu} := g_{\mu\nu} = diag(+1,-1)$. The Lagrangian density of the so-called restricted gauge theory made of the Abelian projection without $\mathbf{X}_\mu$ is therefore defined by [2]-[14]:

$$\mathcal{L} = \left[-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \lambda\left(\hat{n}^2-1\right)\right] \tag{10}$$

The theory defined by the above Lagrangian density has a full $SU(2)$ gauge invariance and it describes the dual dynamics of QCD with the dynamical degrees of the maximal Abelian subgroup U(1) as the electric component and the topological degrees of SU(2) as the magnetic component. It therefore represents an important model in the QCD theory namely, in QCD_2 and deserves to be studied more properly. One of the important steps in this direction is to construct the quantum theory corresponding to this classical theory of QCD_2 by quantizing the theory. This provides motivation for the present studies and in fact, necessitates our presents studies. In the next section, we consider the Hamiltonian and path integral formulations of this constrained theory.

3. Hamiltonian and Path Integral Formulations

We now study the Hamiltonian and path integral quantization of the above restricted gauge theory of QCD_2 (made of the Abelian projection without $\mathbf{X}_\mu$) defined by the Lagrangian density [2]:

$$\begin{aligned}
\mathcal{L} &= \left[-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \lambda\left(\hat{n}^2-1\right)\right] = \left[-\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \lambda\left(\hat{n}^2-1\right)\right]\\
&= \left[-\frac{1}{4}\left(F_{\mu\nu}+H_{\mu\nu}\right)\left(F^{\mu\nu}+H^{\mu\nu}\right) + \lambda\left(\hat{n}^2-1\right)\right]\\
&= \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F_{\mu\nu}H^{\mu\nu} - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} + \lambda\left(\hat{n}^2-1\right)\right]
\end{aligned} \tag{11}$$

In the instant-form (IF) of dynamics, the above Lagrangian density reads:

$$\mathcal{L} = \left[\frac{1}{2}(\partial_0 A_1 - \partial_1 A_0)^2 + \frac{1}{g}(\partial_0 A_1 - \partial_1 A_0)\Phi + \frac{1}{2g^2}\Phi^2 + \lambda\left(\hat{n}^2 - 1\right)\right] \tag{12a}$$

$$\Phi = \left[\partial_0 \hat{n} \cdot (\hat{n} \times \partial_1 \hat{n})\right] \tag{12b}$$

Here Φ is another topological scalar field constructed out of the topological field $\hat{n}$ (of unit norm) and its space derivative ($\partial_1 \hat{n}$) as well as its time derivative ($\partial_0 \hat{n}$), as defined above. The canonical momenta obtained for the above theory are:

$$\Pi_\lambda = \frac{\partial \mathcal{L}}{\partial(\partial_0 \lambda)} = 0,\ \Pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0 \tag{13a}$$

$$E\left(:= \Pi^1\right) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_1)} = (\partial_0 A_1 - \partial_1 A_0) + \frac{1}{g}\Phi \tag{13b}$$

$$\hat{\Pi}_{\hat{n}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \hat{n})} = -\frac{1}{g}(\partial_0 A_1 - \partial_1 A_0)(\hat{n} \times \partial_1 \hat{n}) \tag{13c}$$

Here Π_λ, Π^0, $E\left(:= \Pi^1\right)$, and $\hat{\Pi}_{\hat{n}}$ are the momenta conjugate canonically to λ, A_0, A_1, and $\hat{n}$ respectively. The above equations however, imply that the theory possesses three primary constraints:

$$\chi_1 = \Pi_\lambda \approx 0$$

$$\Omega_1 = \Pi^0 \approx 0$$

$$\psi_1 = \hat{n} \cdot \hat{\Pi}_{\hat{n}} \approx 0 \tag{14}$$

The symbol $\approx$ here denotes a weak equality in the sense of Dirac [24]. The canonical Hamiltonian density of the theory $\mathcal{H}_c$ is:

$$\mathcal{H}_c^N = \left[\frac{1}{2}E^2 + E\partial_1 A_0 - \lambda\left(\hat{n}^2 - 1\right)\right] \tag{15}$$

After including the primary constraint χ_1, Ω_1, and ψ_1 in the canonical Hamiltonian density $\mathcal{H}_c$ with the help of Lagrange multiplier field $u(x,t)$, $v(x,t)$ and $w(x,t)$ which is treated as dynamical, the total Hamiltonian density of the theory $\mathcal{H}_T$ could be written as:

$$\mathcal{H}_T^N = \left[\frac{1}{2}E^2 + E\partial_1 A_0 - \lambda\left(\hat{n}^2 - 1\right) + \chi_1 u + \Omega_1 v + \psi_1 w\right] \tag{16}$$

The Hamilton's equations of motion of the theory that preserve the constraints of the theory in the course of time could be obtained from the total Hamiltonian: $H_T = \int \mathcal{H}_T \mathrm{d}x^1$ and are omitted here for the sake of brevity. Demanding the preservation of the primary constraints χ_1 and Ω_1 for all time leads to the secondary Gauss-law constraints χ_2 and Ω_2 respectively:

$$\chi_2 = \left(\hat{n}^2 - 1\right) \approx 0$$

$$\Omega_2 = \partial_1 E \approx 0 \tag{17}$$

The preservation of ψ_1 for all times does not lead to any secondary constraint. The preservation of χ_2 and Ω_2 also does not lead to any new constraints. The theory is therefore seen to possess a set of five constraints:

$$\begin{aligned}
\rho_1 &= \Omega_1 = \Pi^0 \approx 0 \\
\rho_2 &= \Omega_2 = \partial_1 E \approx 0 \\
\rho_3 &= \chi_1 = \Pi_\lambda \approx 0 \\
\rho_4 &= \chi_2 = \left(\hat{n}^2 - 1\right) \approx 0 \\
\rho_5 &= \psi_1 = \hat{n} \cdot \hat{\Pi}_{\hat{n}} \approx 0
\end{aligned} \tag{18}$$

Matrix $R_{\alpha\beta}$ of the Poisson brackets of the above constraints ρ_i among themselves is clearly singular implying that the set of these constraints ρ_i is first-class and that the theory under consideration is GI. In fact, the theory is seen to be invariant under the local vector gauge transformations:

$$\begin{aligned}
&\delta A_\mu = \frac{1}{g}\hat{n}\cdot\partial_\mu\vec{\alpha} = \frac{1}{g}\partial_\mu\alpha \,, \delta\lambda = \frac{1}{g}\partial_0\alpha \,, \delta\hat{n} = \hat{n}\times\vec{\alpha} = 0 \\
&\delta\Pi^0 = \delta E = \delta\Pi_\lambda = \delta\hat{\Pi}_{\hat{n}} = \delta\Pi_u = \delta\Pi_v = \delta\Pi_w = 0
\end{aligned} \tag{19}$$

where gauge parameter $\alpha(x,t)$ is an arbitrary function of its arguments. The components of the vector gauge current $\left(J^\mu \equiv J^0, J^1\right)$ are:

$$\begin{aligned}
J^0 &= \int \mathrm{d}x^0 \mathrm{d}x^1 j^0 = \int \mathrm{d}x^0 \mathrm{d}x^1 \left[\frac{1}{g}\partial_1\alpha\left(\partial_0 A_1 - \partial_1 A_0 + \frac{1}{g}\Phi\right)\right] \\
J^1 &= \int \mathrm{d}x^0 \mathrm{d}x^1 j^1 = \int \mathrm{d}x^0 \mathrm{d}x^1 \left[-\frac{1}{g}\partial_0\alpha\left(\partial_0 A_1 - \partial_1 A_0 + \frac{1}{g}\Phi\right)\right]
\end{aligned} \tag{20}$$

The theory is clearly gauge-invariant and could now be quantized under appropriate gauge-fixing conditions (GFC's), which could e.g. be chosen as (which by no means is an unique choice):

$$\zeta_1 = \lambda \approx 0,\ \zeta_2 = A_1 \approx 0,\ \zeta_3 = A_0 \approx 0 \tag{21}$$

It may be important to mention here that any set of GFC's could be chosen here such that the resulting set of constraints of the theory (including the set of GFC's) becomes a set of second-class constraints so that the matrix of the total set of constraints becomes non-singular and consequently could be inverted.

The total set of constraints of the theory under these GFC's then becomes:

$$\xi_1 = \rho_1 = \Omega_1 = \Pi^0 \approx 0 \tag{22a}$$

$$\xi_2 = \rho_2 = \Omega_2 = \partial_1 E \approx 0 \tag{22b}$$

$$\xi_3 = \rho_3 = \chi_1 = \Pi_\lambda \approx 0 \tag{22c}$$

$$\xi_4 = \rho_4 = \chi_2 = \left(\hat{n}^2 - 1\right) \approx 0 \tag{22d}$$

$$\xi_5 = \rho_5 = \psi_1 = \hat{n}\cdot\hat{\Pi}_{\hat{n}} \approx 0 \tag{22e}$$

$$\xi_6 = \zeta_1 = \lambda \approx 0 \tag{22f}$$

$$\xi_7 = \zeta_2 = A_1 \approx 0 \tag{22g}$$

$$\xi_8 = \zeta_3 = A_0 \approx 0 \tag{22h}$$

The non-vanishing matrix elements of the matrix $M_{\alpha\beta}$ of the Poisson Brackets's among these above constraints ξ_i are:

$$M_{18} = -M_{81} = -\delta(x-y) \tag{23a}$$

$$M_{27} = +M_{72} = -\partial_1\delta(x-y) \tag{23b}$$

$$M_{36} = -M_{63} = -\delta(x-y) \tag{23c}$$

$$M_{45} = -M_{54} = +2\hat{n}^2\delta(x-y) \tag{23d}$$

The above matrix $M_{\alpha\beta}$ is clearly non-singular implying that the constraints ξ_i form a set of second-class constraints. The theory could therefore be quantized using the Dirac's Hamiltonian Formulation as well as using the path integral formulation. The square root of the absolute value of the determinant of this matrix $M_{\alpha\beta}$ is:

$$\left[\left\|det\left(M_{\alpha\beta}\right)\right\|\right]^{\frac{1}{2}} = \left[2\hat{n}^2\left(\delta(x-y)\right)^3\partial_1\delta(x-y)\right] \tag{24}$$

The non-vanishing equal-time Dirac brackets (DB's) of the theory are [24]:

$$\left\{\Pi_{\hat{n}}^a(x,t), \Pi_{\hat{n}}^b(y,t)\right\}_{DB} = \frac{1}{\hat{n}^2}\left(n^a\Pi_{\hat{n}}^b - \Pi_{\hat{n}}^a n^b\right)\delta(x-y) \tag{25a}$$

$$\left\{n^a(x,t), \Pi_{\hat{n}}^b(y,t)\right\}_{DB} = \frac{1}{\hat{n}^2}\left(n^a\Pi_{\hat{n}}^b - \Pi_{\hat{n}}^a n^b\right)\delta(x-y) \tag{25b}$$

$$\left\{\Pi_{\hat{n}}^a(x,t), n^b(y,t)\right\}_{DB} = \frac{-1}{\hat{n}^2}\left(n^a\Pi_{\hat{n}}^b - \Pi_{\hat{n}}^a n^b\right)\delta(x-y) \tag{25c}$$

Here one needs to remember that while making a transition from equal-time Dirac brackets to the equal-time commutation relations using the Dirac quantization rule, one needs to take in to account the problem of operator ordering (which occurs here because the results of the equal-time commutation relations involve the product of the operators). Also, the roman indices a and b here, are the color indices of the gauge theory of QCD_2.

Also for the later use, for considering the BRST formulation of the theory we convert the total Hamiltonian density of the theory into the first order Lagrangian density $\mathcal{L}_{IO}$:

$$\begin{aligned}\mathcal{L}_{IO} &= \Big[\Pi_\lambda\partial_0\lambda + \Pi^0\partial_0 A_0 + E\partial_0 A_1 + \hat{\Pi}_{\hat{n}}\cdot\partial_0\hat{n} \\ &\quad + \Pi_u\partial_0 u + \Pi_v\partial_0 v + \Pi_w\partial_0 w - \mathcal{H}_T\Big] \\ &= \left[\frac{1}{2}\left(\partial_0 A_1 - \partial_1 A_0\right)^2 - \frac{1}{2g^2}\Phi^2 + \lambda\left(\hat{n}^2 - 1\right)\right]\end{aligned} \tag{26}$$

For considering the path integral formulation, the transition to quantum theory is made again by writing the vacuum to vacuum transition amplitude for the theory, called the generating functional $Z[J_k]$ of the theory, following again the Senjanovic procedure for a theory possessing a set of second-class constraints [25] [26] [27] [28], appropriate for our present theory, considered under the gauge-fixing conditions ξ_i, in the presence of the external sources: J_k as follows [25] [26] [27] [28]:

$$
\begin{aligned}
Z[J_k] = \int [\mathrm{d}\mu] \exp\Big[i \int \mathrm{d}x^0 \mathrm{d}x^1 \Big[& J_k \Phi^k + \Pi_\lambda \partial_0 \lambda + \Pi^0 \partial_0 A_0 + E \partial_0 A_1 \\
& + \hat{\Pi}_{\hat{n}} \cdot \partial_0 \hat{n} + \Pi_u \partial_0 u + \Pi_v \partial_0 v + \Pi_w \partial_0 w - \mathcal{H}_T \Big]\Big]
\end{aligned} \tag{27}
$$

where the phase space variables of the theory are: $\Phi^k \equiv (\lambda, A_0, A_1, \hat{n}, u, v, w)$ with the corresponding respective canonical conjugate momenta: $\Pi_k \equiv (\Pi_\lambda, \Pi_0, E, \hat{\Pi}_{\hat{n}}, \Pi_u, \Pi_v, \Pi_w)$. The functional measure $[\mathrm{d}\mu]$ of the generating functional $Z[J_k]$ under this gauge-fixing is obtained as:

$$
\begin{aligned}
[\mathrm{d}\mu] = \Big[& \Big[2\hat{n}^2 (\delta(x-y))^3 \partial_1 \delta(x-y) \Big] [\mathrm{d}\lambda][\mathrm{d}A_0][\mathrm{d}A_1][\mathrm{d}\hat{n}][\mathrm{d}u][\mathrm{d}v][\mathrm{d}w][\mathrm{d}\Pi_\lambda] \\
& [\mathrm{d}\Pi^0][\mathrm{d}E][\mathrm{d}\hat{\Pi}_{\hat{n}}][\mathrm{d}p_u][\mathrm{d}p_v][\mathrm{d}p_w] \delta[\Pi^0 \approx 0] \delta[\partial_1 E \approx 0] \delta[\Pi_\lambda \approx 0] \\
& \delta[(\hat{n}^2 - 1) \approx 0] \delta[\hat{n} \cdot \hat{\Pi}_{\hat{n}} \approx 0] \delta[\lambda \approx 0] \delta[A_1 \approx 0] \delta[A_0 \approx 0] \Big]
\end{aligned} \tag{28}
$$

This completes the Hamiltonian and path integral formulations of the theory. The BRST formulation of the theory is considered in the next section.

4. BRST Formulation

In the following, we study the BRST formulation of the theory. For the BRST formulation of the model, we rewrite the theory as a quantum system that possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our gauge-invariant theory and replace the notion of gauge-transformation, which shifts operators by c-number functions, by a BRST transformation, which mixes operators with Bose and Fermi statistics. We then introduce new anti-commuting variables c and $\bar{c}$ (Grassman numbers on the classical level and operators in the quantized theory) and a commuting variable b such that [27] [28] [29] [30] [31]:

$$\hat{\delta}\lambda = \frac{1}{g}\partial_0 c, \hat{\delta}A_0 = \frac{1}{g}\partial_0 c, \hat{\delta}A_1 = \frac{1}{g}\partial_1 c, \hat{\delta}u = \frac{1}{g}\partial_0\partial_0 c \tag{29a}$$

$$\hat{\delta}\hat{\Pi}_{\hat{n}} = \hat{\delta}\Pi^0 = \hat{\delta}E = \hat{\delta}\Pi_\lambda = \hat{\delta}\Pi_u = \hat{\delta}\Pi_v = \hat{\delta}\Pi_w = 0 \tag{29b}$$

$$\hat{\delta}\hat{n} = \hat{\delta}v = \hat{\delta}w = 0, \hat{\delta}c = 0, \hat{\delta}\bar{c} = b, \hat{\delta}b = 0 \tag{29c}$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST-invariant function of the dynamical phase space variables of the theory to be a function f such that $\hat{\delta} f = 0$. Now the BRST gauge-fixed quantum Lagrangian density $\mathcal{L}_{BRST}$ for the theory could be obtained by adding to the first-order Lagrangian density $\mathcal{L}_{IO}$, a trivial BRST-invariant function (e.g.) as follows:

$$
\begin{aligned}
\mathcal{L}_{BRST} = \Big[& \frac{1}{2}(\partial_0 A_1 - \partial_1 A_0)^2 - \frac{1}{2g^2}\Phi^2 + \lambda(\hat{n}^2 - 1) \\
& + \hat{\delta}\Big[\bar{c}\Big(g\partial_0 A_0 + g\partial_1 A_0 - g\partial_0 A_1 + \frac{1}{2} b \Big)\Big]\Big]
\end{aligned} \tag{30}
$$

The last term in the above equation is the extra BRST-invariant gauge-fixing term. After one integration by parts, the above equation could now be written as:

$$\mathcal{L}_{BRST} := \left[\frac{1}{2}\left(\partial_0 A_1 - \partial_1 A_0\right)^2 - \frac{1}{2g^2}\Phi^2 + \lambda\left(\hat{n}^2 - 1\right) + \frac{1}{2}b^2 + gb\left(\partial_0 A_0 - E + \frac{1}{g}\Phi\right) + \left(\partial_0 \bar{c}\right)\left(\partial_0 c\right)\right] \tag{31}$$

The last term in the above equation is the BRST-invariant gauge-fixing term. Proceeding classically, the Euler Lagrange equation for b reads:

$$-b = \left(g\partial_0 A_0 + g\partial_1 A_0 - g\partial_0 A_1\right) \tag{32}$$

which in turn (with the requirement $\hat{\delta} b = 0$) then implies:

$$\partial_0 \partial_0 c = 0 \tag{33}$$

The above equation is also an Euler-Lagrange equation (ELE) obtained by the variation of $\mathcal{L}_{BRST}$ with respect to $\bar{c}$. We define the bosonic momenta in the usual manner:

$$\Pi_0 := \frac{\partial}{\partial\left(\partial_0 A_0\right)}\mathcal{L}_{BRST} = gb \tag{34}$$

but for the fermionic momenta with directional derivatives we set

$$\Pi_c := \mathcal{L}_{BRST}\frac{\overleftarrow{\partial}}{\partial\left(\partial_0 c\right)} = \partial_0 \bar{c}; \quad \Pi_{\bar{c}} := \frac{\overrightarrow{\partial}}{\partial\left(\partial_0 \bar{c}\right)}\mathcal{L}_{BRST} = \partial_0 c \tag{35}$$

implying that the variable canonically conjugate to c is ($\partial_0 \bar{c}$) and the variable conjugate to $\bar{c}$ is ($\partial_0 c$). For writing the quantum Hamiltonian density from the Lagrangian density in the usual manner we remember that the former has to be Hermitian so that:

$$\mathcal{H}_{BRST} = \left[\Pi_u \partial_0 u + \Pi_v \partial_0 v + \hat{n}\cdot\hat{\Pi}_{\hat{n}} w + \frac{1}{2}E^2 + E\partial_1 A_0 - \lambda\left(\hat{n}^2 - 1\right) + \frac{1}{2}g^2 b^2 + \Pi_c \Pi_{\bar{c}} - gb\left(\partial_0 A_0 - \partial_0 A_1 + \partial_1 A_0\right)\right] \tag{36}$$

We can check the consistency of our definitions of the the fermionic momenta by looking at the Hamilton's equations for the fermionic variables:

$$\partial_0 c = \frac{\overrightarrow{\partial}}{\partial \Pi_c}\mathcal{H}_{BRST}; \quad \partial_0 \bar{c} = \mathcal{H}_{BRST}\frac{\overleftarrow{\partial}}{\partial \Pi_{\bar{c}}} \tag{37}$$

We thus see that

$$\partial_0 c = \frac{\overrightarrow{\partial}}{\partial \Pi_c}\mathcal{H}_{BRST} = \Pi_{\bar{c}}; \quad \partial_0 \bar{c} = \mathcal{H}_{BRST}\frac{\overleftarrow{\partial}}{\partial \Pi_{\bar{c}}} = \Pi_c \tag{38}$$

is in agreement with our definitions of the Fermionic momenta. Also, for the operators $c, \bar{c}, \partial_0 c$ and $\partial_0 \bar{c}$, one needs to satisfy the anti-commutation relations of $\partial_0 c$ with $\bar{c}$ or of $\partial_0 \bar{c}$ with c, but not of c, with $\bar{c}$. In general, c and $\bar{c}$ are independent canonical variables and one assumes that [25] [26] [27] [28]:

$$\left\{\Pi_c, \Pi_{\bar{c}}\right\} = \left\{\bar{c}, c\right\} = 0; \quad \partial_0\left\{\bar{c}, c\right\} = 0; \quad \left\{\partial_0 \bar{c}, c\right\} = (-1)\left\{\partial_0 c, \bar{c}\right\} \tag{39}$$

where $\{,\}$ means an anti-commutator. We thus see that the anti-commulators in the above equation are non-trivial and need to be fixed. In order to fix these, we demand that c satisfy the Heisenberg equation:

$$\left[c, \mathcal{H}_{BRST}\right] = i\partial_0 c \tag{40}$$

and using the property $c^2 = \overline{c}^2 = 0$ one obtains

$$\left[c, \mathcal{H}_{BRST}\right] = \{\partial_0 \overline{c}, c\}\partial_0 c \tag{41}$$

The last three equations then imply :

$$\{\partial_0 \overline{c}, c\} = (-1)\{\partial_0 c, \overline{c}\} = i \tag{42}$$

Here the minus sign in the above equation is nontrivial and implies the existence of states with negative norm in the space of state vectors of the theory.

The BRST charge operator Q is the generator of the BRST transformations. It is nilpotent and satisfies $Q^2 = 0$. It mixes operators which satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anti-commutators with Fermi operators for the present theory satisfy:

$$[\lambda, Q] = (-c),\ [A_0, Q] = \partial_0 c,\ [A_1, Q] = \partial_1 c \tag{43a}$$

$$[\hat{n}, Q] = \hat{n}\partial_0 c,\ \left[\hat{\Pi}_{\hat{n}}, Q\right] = \left(2\hat{n}c - \hat{\Pi}_{\hat{n}}\partial_0 c\right), \tag{43b}$$

$$\{\overline{c}, Q\} = \left(\Pi_0 + \Pi_\lambda + \hat{n}\cdot\hat{\Pi}_{\hat{n}}\right) \tag{43c}$$

$$\{\partial_0 \overline{c}, Q\} = (-1)\left(\partial_1 E + \hat{n}^2 - 1\right) \tag{43d}$$

All other commutators and anti-commutators of the theory involving Q and the other phase space variables of the theory are seen to vanish. In view of this, the BRST charge operator of the present theory could be written as:

$$Q = \int \mathrm{d}x^1 \left[ic\left(\partial_1 E + \hat{n}^2 - 1\right) - i\partial_0 c\left(\Pi^0 + \Pi_\lambda + \hat{n}\cdot\hat{\Pi}_{\hat{n}}\right)\right] \tag{44}$$

This equation implies that the set of states satisfying the conditions:

$$\Pi^0 |\psi\rangle = 0, \partial_1 E|\psi\rangle = 0, \Pi_\lambda |\psi\rangle = 0, \left(\hat{n}^2 - 1\right)|\psi\rangle = 0, \left(\hat{n}\cdot\hat{\Pi}_{\hat{n}}\right)|\psi\rangle = 0 \tag{45}$$

belong to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, *i.e.*, it belongs to the set of BRST-invariant states.

In order to understand the condition needed for recovering the physical states of the theory we rewrite the operators c and $\overline{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider Euler-Lagrange equation for the variable c derived earlier. The solution of this equation gives (for the instant-form time $x^0 \equiv t$) the Heisenberg operators $c(t)$ and correspondingly $\overline{c}(t)$ in terms of the fermionic annihilation and creation operators as:

$$c(t) = G(t) + F(t),\ \ \overline{c}(t) = G^\dagger(t) + F^\dagger(t) \tag{46}$$

which at the instant-form time $t = 0$ imply

$$c \equiv c(0) = F,\ \ \overline{c}(t) \equiv \overline{c}(0) = F^\dagger \tag{47a}$$

$$\partial_0 c(t) \equiv \partial_0 c(0) = G, \quad \partial_0 \bar{c}(t) \equiv \partial_0 \bar{c}(0) = G^\dagger \tag{47b}$$

By imposing the conditions (obtained earlier):

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\partial_0 \bar{c}, \partial_0 c\} = 0 \tag{48a}$$

$$\{\partial_0 \bar{c}, c\} = (-1)\{\partial_0 c, \bar{c}\} = i \tag{48b}$$

we then obtain

$$F^2 = \left(F^\dagger\right)^2 = \left\{F^\dagger, F\right\} = \left\{G^\dagger, G\right\} = 0, \left\{G^\dagger, F\right\} = (-1)\left\{G, F^\dagger\right\} = i \tag{49}$$

Now let $|0\rangle$ denote the fermionic vacuum for which

$$G|0\rangle = F|0\rangle = 0 \tag{50}$$

Defining $|0\rangle$ to have norm one, the last three equations imply

$$\langle 0|FG^\dagger|0\rangle = i, \quad \langle 0|GF^\dagger|0\rangle = -i \tag{51}$$

so that

$$G^\dagger|0\rangle \neq 0, \quad F^\dagger|0\rangle \neq 0 \tag{52}$$

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathcal{H}_{BRST}$ is however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space. In terms of annihilation and creation operators $\mathcal{H}_{BRST}$ is:

$$\begin{aligned}\mathcal{H}_{BRST} = \Big[&\Pi_u \partial_0 u + \Pi_v \partial_0 v + \hat{n} \cdot \hat{\Pi}_{\hat{n}} w + \frac{1}{2} E^2 + E \partial_1 A_0 - \lambda\left(\hat{n}^2 - 1\right) \\ &+ \frac{1}{2}\left(\Pi^0\right)^2 - \Pi^0\left(\partial_0 A_0 - \partial_0 A_1 + \partial_1 A_0\right) + G^\dagger G \Big]\end{aligned} \tag{53}$$

and the BRST charge operator of the present theory could be written as:

$$Q = \int \mathrm{d}x^1 \left[iF\left(\partial_1 E + \hat{n}^2 - 1\right) - iG\left(\Pi^0 + \Pi_\lambda + \hat{n} \cdot \hat{\Pi}_{\hat{n}}\right)\right] \tag{54}$$

Now because $Q|\psi\rangle = 0$, the set of states annihiliated by Q contains not only the set for which the constraints of the theory hold but also additional states for which

$$F|\psi\rangle = G|\psi\rangle = 0$$

$$\Pi^0|\psi\rangle \neq 0, \partial_1 E|\psi\rangle \neq 0, \Pi_\lambda|\psi\rangle \neq 0, \left(\hat{n}^2 - 1\right)|\psi\rangle \neq 0, \left(\hat{n} \cdot \hat{\Pi}_{\hat{n}}\right)|\psi\rangle \neq 0 \tag{55}$$

Now because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set for which the constraints of the theory hold but also additional states for which the constraints of the theory do not hold. However in our considerations, the Hamiltonian is also invariant under the anti-BRST transformations given by:

$$\hat{\bar{\delta}}\lambda = -\frac{1}{g}\partial_0 \bar{c}, \hat{\bar{\delta}} A_0 = -\frac{1}{g}\partial_0 \bar{c}, \hat{\bar{\delta}} A_1 = -\frac{1}{g}\partial_1 \bar{c}, \hat{\bar{\delta}} u = -\frac{1}{g}\partial_0 \partial_0 \bar{c} \tag{56a}$$

$$\hat{\bar{\delta}}\hat{\Pi}_{\hat{n}} = \hat{\bar{\delta}}\Pi^0 = \hat{\bar{\delta}} E = \hat{\bar{\delta}}\Pi_\lambda = \hat{\bar{\delta}}\Pi_u = \hat{\bar{\delta}}\Pi_v = \hat{\bar{\delta}}\Pi_w = 0 \tag{56b}$$

$$\bar{\hat{\delta}}\hat{n} = \bar{\hat{\delta}}v = \bar{\hat{\delta}}w = 0, \bar{\hat{\delta}}\bar{c} = 0, \bar{\hat{\delta}}c = -b, \bar{\hat{\delta}}b = 0 \tag{56c}$$

with the generator or anti-BRST charge:

$$\bar{Q} = \int \mathrm{d}x^1 \left[-i\bar{c}\left(\partial_1 E + \hat{n}^2 - 1\right) + i\partial_0\bar{c}\left(\Pi^0 + \Pi_\lambda + \hat{n}\cdot\hat{\Pi}_{\hat{n}}\right)\right] \tag{57}$$

or

$$\bar{Q} = \int \mathrm{d}x^1 \left[-iF^\dagger\left(\partial_1 E + \hat{n}^2 - 1\right) + iG^\dagger\left(\Pi^0 + \Pi_\lambda + \hat{n}\cdot\hat{\Pi}_{\hat{n}}\right)\right] \tag{58}$$

We also have

$$\partial_0 Q = \left[Q, \mathcal{H}_{BRST}\right] = 0, \quad \partial_0 \bar{Q} = \left[\bar{Q}, \mathcal{H}_{BRST}\right] = 0 \tag{59}$$

with

$$\mathcal{H}_{BRST} = \int \mathrm{d}x^1 \mathcal{H}_{BRST} \tag{60}$$

and we further impose the dual condition that both Q and $\bar{Q}$ annihilate physical states, implying that:

$$Q|\psi\rangle = 0 \quad \text{and} \quad \bar{Q}|\psi\rangle = 0 \tag{61}$$

The states for which the constraints of the theory hold, satisfy both of these conditions and are in fact, the only states satisfying both of these conditions, since with

$$G^\dagger G = (-1) G G^\dagger \tag{62}$$

there are no states of this operator with $G^\dagger|\psi\rangle = 0$ and $F^\dagger|\psi\rangle = 0$, and hence no free eigenstates of the fermionic part of $\mathcal{H}_{BRST}$ that are annihilated by each of G, $G^\dagger$, F, and $F^\dagger$. Thus the only states satisfying $Q|\psi\rangle = 0$ and $\bar{Q}|\psi\rangle = 0$ are those that satisfy the constraints of the theory.

Now because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which the constraints of the theory hold but also additional states for which the constraints of the theory do not hold. This situation is, however, easily avoided by additionally imposing on the theory, the dual condition: $Q|\psi\rangle = 0$ and $\bar{Q}|\psi\rangle = 0$. By imposing both of these conditions on the theory simultaneously, one finds that the states for which the constraints of the theory hold are the only states satisfying both of these conditions. This is traced to the conditions on the fermionic variables c and $\bar{c}$ which constrain the solutions such that one cannot have simultaneously c, $\partial_0 c$ and $\bar{c}$, $\partial_0\bar{c}$, applied to $|\psi\rangle$ giving zero. Thus the only states satisfying $Q|\psi\rangle = 0$ and $\bar{Q}|\psi\rangle = 0$ are those that satisfy the constraints of the theory and they belong to the set of BRST-invariant as well as to the set of anti-BRST-invariant states.

Alternatively, one can understand the above point in terms of fermionic annihiliation and creation operators as follows. The condition $Q|\psi\rangle = 0$ implies the that the set of states annihilated by Q contains not only the states for which the constraints of the theory hold but also additional states for which the constraints do not hold. However, $\bar{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\bar{Q}$ contains only the states for which the constraints hold, simply because $G^\dagger|\psi\rangle \neq 0$ and $F^\dagger|\psi\rangle \neq 0$. Thus in this alternative way also, we see that the states satisfying $Q|\psi\rangle = \bar{Q}|\psi\rangle = 0$ are only those states which satisfy the constraints of the theory and we also see that these states belong to the set of BRST-invariant states as well as to the set of anti-BRST invariant states. This completes the BRST formulation of the theory.

5. Summary and Discussion

In the present work, we have considered the restricted gauge theory of quantum chromodynamics (QCD) in one-space one-time dimension (QCD_2) à la Cho *et al.* [1]-[14]. We have summarized the basics of the theory in Section 2 where the motivations of our present studies have also been discussed and are being omitted here the sake of brevity. The theory under our present investigation is seen to be GI and we have studied its quantization using Hamiltonian [24], path integral [25] [26] [27] [28] and Becchi-Rouet-Stora and Tyutin (BRST) [29] [30] [31], formulations [24]-[31], in the usual instant-form (IF) of dynamics (on the hyperplanes: $x^0 = t = \text{constant}$) [32] [33], under appropriate gauge-fixing conditions.

The restricted gauge theory of QCD_2 à la Cho *et al.* [1]-[14] and others [1]-[23] makes use of the so-called "Cho-decomposition", which is, in fact, the gauge independent decomposition of the non-Abelian potential into the restricted potential and the valence potential and it helps in the clarification of the topological structure of the non-Abelian gauge theory. This decomposition allows one to view QCD as the restricted gauge theory (made of the restricted potential) which is coupled to a gauge-covariant colored vector field (the valence potential). The restricted potential is defined in such a way that it allows a covariantly constant unit isovector $\hat{n}$ everywhere in space-time, which enables one to define the gauge-independent color direction everywhere in space-time and at the same time allows one to define the magnetic potential of the non-Abelian monopoles. It even has full SU(2) gauge degrees of freedom, in spite of the fact that it is restricted. Consequently, the restricted QCD made of the restricted potential describes a very interesting dual dynamics of its own, and plays a crucial role in the understanding of QCD. This restricted gauge theory of QCD is therefore very important and it is important to study its quantization using the standard constraint quantization methods, including the Hamiltonian, path integral and BRST quantizations, as we have done in the present work.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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