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Anisotropic Geodesic Fluid in Non-Comoving Spherical Coordinates

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Abstract

We start with a recently introduced spherically symmetric geodesic fluid model (arXiv: 1601.07030) whose energy-momentum tensor (EMT) in the comoving frame is dust-like with nontrivial energy flux. In the non-comoving energy frame (vanishing energy flux), the same EMT contains besides dust only radial pressure. We present Einstein's equations together with the matter equations in static spherically symmetric coordinates. These equations are self-contained (four equations for four unknowns). We solve them analytically except for a resulting nonlinear ordinary differential equation (ODE) for the gravitational potential. This ODE can be rewritten as a Lienard differential equation which, however, may be transformed into a rational Abel differential equation of the first kind. Finally, we list some open mathematical problems and outline possible physical applications (galactic halos, dark energy stars) and related open problems.

Keywords

Einsteins Equations, Stationary Solution, Anisotropic Geodesic Fluid, Non-Comoving Coordinates, Spherical Symmetry, Gravitational Potential, Nonlinear Ordinary Differential Equation

1. Introduction

Analytic solutions of the coupled Einstein-matter equations for the stationary anisotropic and spherically symmetric case, without supplying any external input, are rather rare. The only example we know, a conformal flat generalization of the de Sitter space-time, has been published very recently ([1], Section 6). But normally one has to provide some external input. The generic case has been discussed recently by Herrera, Ospino and Di Prisco [2]. The authors of [2] provide only three Einstein equations for five unknowns (energy density, two pressures, two metric functions). So the knowledge of two solution generating functions is required.

In the present paper, we describe another model for which we derive an analytic solution except for one remaining ordinary differential equation. We start with a recently introduced irrotational geodesic fluid model whose energy-momentum tensor (EMT) in the frame comoving with the fluid is dust-like with nontrivial energy flux [3]. Then we pass over to the non-comoving energy frame (vanishing energy flux [4]). Here the same EMT contains besides dust only radial pressure. We consider the resulting Einstein's field equations together with the matter equations in static spherically symmetric coordinates. These equations are self-contained (four equations for four unknowns). We solve them analytically except for a resulting nonlinear ordinary differential equation (ODE) for the gravitational potential. This ODE turns out to be the general relativistic generalization of a corresponding ODE derived in [5] for the nonrelativistic darkon fluid model. It has been used in [5] as a model for galactic halos.

The paper is organized as follows. We define our model in Section 2. In Section 3, we introduce non-comoving coordinates and present the corresponding Einstein equations and the matter equations. In the course of integration of these equations in Section 4, we derive a nonlinear ODE for the gravitational potential. We reformulate this ODE in Section 5 as a Lienard differential equation and transform it into a rational Abel differential equation of the first kind. In Section 6, we list some open mathematical problems. Finally, possible physical applications (galactic halos, dark energy stars) and related open problems are outlined in Section 7.

2. Fluid Model

Our model is defined by a self-gravitating, irrotational, pressure-less and stress free geodesic fluid whose EMT in the frame comoving with the fluid is dust-like with a nontrivial energy flux.

Therefore our model will be described by the following covariant set of equations (Greek indices run from 0 to 3 and we use the usual summation convention)

- Einsteins equations ($\kappa = 8\pi G, c = 1$)

$$G^{\mu\nu} = \kappa T^{\mu\nu} \tag{1}$$

with a EMT $T^{\mu\nu}$, decomposed w.r.t. the unit and time-like fluid velocity vector u^{μ}

$$T^{\mu\nu} = \rho u^{\mu} u^{\nu} + q^{\mu} u^{\nu} + q^{\nu} u^{\mu} \tag{2}$$

where ρ is the total energy density (comprising baryonic and the so called dark sector contributions) and q^{μ} is the energy flux vector ($u^{\mu} q_{\mu} = 0$) in the comoving frame.

- Constraints for u^{μ}

Geodesic flow:

$$u^{\lambda}\nabla_{\lambda}u^{\mu}=0 \tag{3}$$

Irrotational flow:

$$\nabla_{\mu}u_{\nu}-\nabla_{\nu}u_{\mu}=0 \tag{4}$$

The covariant derivative ∇_{μ} is given in terms of a torsion-free connection (Christoffel symbols).

- Covariant conservation of the EMT

$$\nabla_{\mu}T^{\mu\nu}=0 \tag{5}$$

which is a consequence of (1) (Bianchi identities).

3. Einsteins Equations and the Matter Equations in Non-Comoving, Static and Spherically Symmetric Coordinates

Our fluid will be assumed to move with radial velocity v relative to the energy frame (EF). Such a choice of relative motion may be related to the observed motion of e.g. a galaxy relative to the microwave background [6].

For static spherically symmetric coordinates in the EF we use Schwarzschild (canonical) coordinates [7]

$$\mathrm{d}s^2=-\mathrm{e}^{2\phi(r)}\mathrm{d}t^2+\mathrm{e}^{2\lambda(r)}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2 \tag{6}$$

Then u^{μ} and q^{μ} are given by $\left(v=v(r),q=q(r)\right)$

$$u^{\mu}=\beta\left(n^{\mu}+vs^{\mu}\right),\ q^{\mu}=q\beta\left(vn^{\mu}+s^{\mu}\right) \tag{7}$$

with $\beta=\left(1-v^2\right)^{-1/2}$. The time-like and space-like unit vectors n^{μ} and s^{μ} are defined by

$$n^{\mu}=\left(\mathrm{e}^{-\phi},0\right),\ s^{\mu}=\left(0,\mathrm{e}^{-\lambda}\right). \tag{8}$$

Sometimes it is convenient to use instead of $\lambda(r)$ the mass function $M(r)$ related to each other by

$$\mathrm{e}^{2\lambda}=\left(1-\frac{2M}{r}\right)^{-1} \tag{9}$$

The EMT (2) reads in the energy frame (decomposition of $T^{\mu\nu}$ w.r.t. n^{μ} and s^{μ}) [8]

$$T^{\mu\nu}=\rho^{*}n^{\mu}n^{\nu}+p_r^{*}s^{\mu}s^{\nu} \tag{10}$$

where the energy density ρ^{*} and the radial pressure p_r^{*} in the EF are related to the corresponding kinematic quantities in the comoving frame by

$$\rho^{*}=\frac{\rho}{1+v^2},\quad p_r^{*}=-\rho^{*}v^2 \tag{11}$$

In (11) we have used the relation

$$q=-\rho^{*}v \tag{12}$$

which follows from the requirement of the vanishing energy flux in the EF [8].

With the metric (6) and the EMT (10) we obtain for the Einstein Equation (1) [7]

$$\kappa\rho^* = \frac{2M'}{r^2} \tag{13}$$

$$\kappa p_r^* = \frac{1}{r^2}\left(-1 + \left(1 - \frac{2M}{r}\right)(1 + 2r\phi')\right) \tag{14}$$

$$0 = \left(\phi'' + \phi'^2 + \frac{\phi'}{r}\right)\left(1 - \frac{2M}{r}\right) - \left(\phi' + \frac{1}{r}\right)\left(\frac{M}{r}\right)' \tag{15}$$

Here and in what follows a prime denotes differentiation of a function w.r.t. to its argument.

The matter equations consist of two parts:

- The generalized Tolman-Oppenheimer-Volkoff (TOV) equation, following from the space-like part of (5) (or, more directly, from (13)-(15))

$$\left(\rho^* + p_r^*\right)\phi' + \frac{1}{r^2}\left(r^2 p_r^*\right)' = 0 \tag{16}$$

- The geodesic flow constraint (3) [8] (cp. also [9])

$$vv'\beta^2 + \phi' + 0 \tag{17}$$

The irrotational flow constraint (4) is automatically satisfied in spherically symmetric coordinates.

4. Integration of the Einstein and Matter Equations

To integrate the set of independent equations (14)-(17) we will proceed in three steps:

- Equation (17) can easily be integrated

$$v^2 = 1 - \mathrm{e}^{2\phi} \tag{18}$$

- Insertion of $\rho^* = -p_r^*/v^2$ from (11) and (18) into the TOV-Equation (16) leads to

$$\left(r^2 p_r^*\right)\frac{\mathrm{e}^{2\phi}\phi'}{\mathrm{e}^{2\phi} - 1} + \left(r^2 p_r^*\right)' = 0 \tag{19}$$

which again can easily be integrated

$$p_r^* = \frac{\alpha}{r^2}\left(1 - \mathrm{e}^{2\phi}\right)^{-1/2} \tag{20}$$

where α is an integration constant.

For p_r^* to be real valued we have to require

$$\phi(r) \le 0 \tag{21}$$

- Next we insert (20) into the 2nd Einstein Equation (14) and solve for M/r

$$1 - \frac{2M}{r} = \frac{\kappa\alpha\left(1 - \mathrm{e}^{2\phi}\right)^{-1/2} + 1}{1 + 2r\phi'} \tag{22}$$

Insertion of (22) into the 3rd Einstein Equation (15) leads, after some straightforward manipulations, to a nonlinear ordinary differential equation (ODE) for the gravitational potential

$$\left(\left(r^2\phi'\right)' + 2r^2\phi'^2\right)\left(\left(1-e^{2\phi}\right)^{3/2} + \kappa\alpha\left(1-e^{2\phi}\right)\right) + \frac{\kappa\alpha}{2}\left(1+r\phi'\right)\left(1+2r\phi'\right)e^{2\phi} = 0 \quad (23)$$

Comment: The weak field limit ($\phi = 0(\varepsilon)$) of (23) yields (cp. [8], subsection 6.5)

$$\left(r^2\phi'\right)' = \frac{\gamma}{2}\left(-2\phi\right)^{-3/2} \quad (24)$$

where we have put $\gamma = -\kappa\alpha, \left(\gamma = 0\left(\varepsilon^{5/2}\right)\right)$.

The result (24) turns out to be equal to the corresponding equation obtained for the stationary solution of the nonrelativistic darkon fluid model in [5]. In this limit we get from the Poisson equation and (24) the following relation between the energy density ρ and the potential ϕ

$$\kappa\rho = \frac{\gamma}{r^2}\left(-2\phi\right)^{-3/2} \quad (25)$$

Then positivity of ρ requires

$$\gamma > 0 \quad (26)$$

Note that a positive energy density yields a negative radial pressure according to (11).

5. Reformulation of the ODE (23) as a Lie-Nard Differential Equation or as an Abel Differential Equation[1]

With the transformation

$$r \to x = \log r, \quad \phi(r) \to \varphi(x) = -2\phi(r) \quad (27)$$

the ODE (23) becomes the autonomous ODE

$$\left(\varphi'^2 - \varphi'' - \varphi'\right)\left(\left(1-e^{-\varphi}\right)^{3/2} - \gamma\left(1-e^{-\varphi}\right)\right) = \gamma e^{-\varphi}\left(1-\varphi'\right)\left(1-\varphi'/2\right). \quad (28)$$

The further transformation

$$\varphi \to f = 1 - e^{-\varphi} \quad (29)$$

leads to the mixed Lienard differential equation

$$f'' + g_1(f)f' + g_2(f)f'^2 + g_0(f) = 0 \quad (30)$$

with

$$g_0(f) = \gamma\left(1-f\right)^2\left(f^{3/2} - \gamma f\right)^{-1}$$

$$g_1(f) = \left(f^{3/2} + \gamma\frac{f}{2} - \frac{3}{2}\gamma\right)\left(f^{3/2} - \gamma f\right)^{-1}$$

$$g_2(f) = \frac{\gamma}{2}\left(f^{3/2} - \gamma f\right)^{-1} \quad (31)$$

To transform (30) into an Abel differential equation we proceed as usual [10]: With $f(x) \to y(f)$,

[1]For convenience, by "Abel differential equation", we mean Abel differential equation of the first kind.

$$y(f)=\left(1-\gamma f^{-1/2}\right)^{-1}\left(f'(x)\right)^{-1} \tag{32}$$

we obtain from (30)

$$y'(f)=h_2(f)y^2(f)+h_3(f)y^3(f) \tag{33}$$

with

$$h_2(f)=1+\frac{\gamma}{2}f^{-1/2}-\frac{3}{2}\gamma f^{-3/2}$$
$$h_3(f)=\gamma f^{-3/2}(1-f)^2\left(1-\gamma f^{-1/2}\right). \tag{34}$$

By the further transformation

$$f\to x=f^{1/2},\ \ u(x)=y(f) \tag{35}$$

we obtain from (33), (34) a rational Abel differential equation

$$u'(x)=H_2(x)u^2+H_3(x)u^3 \tag{36}$$

with

$$H_2=\frac{2x^3+\gamma x^2-3\gamma}{x^2}$$
$$H_3(x)=\frac{2\gamma(x-\gamma)\left(1-x^2\right)^2}{x^3} \tag{37}$$

Unfortunately (36) does not belong to the known integrable cases of rational Abel differential equations. But, as shown in [11] (see also [12] and [13]), all integrable rational Abel differential equations consist of classes whose members are related to each other by the equivalence transformation

$$x=F(z),\ u(x)=P(z)w(z)+Q(z) \tag{38}$$

where *F*, *P* and *Q* are arbitrary functions of *z* satisfying $F'\neq 0$ and $P\neq 0$.

A computer algebra routine has been presented in [11] which allow us to decide whether a given Abel differential equation belongs to one of the known integrable classes.

6. Open Problems

From the results of Section 5 follow immediately the following open mathematical problems:

- For which values of γ does the Lienard Equation (30) has positive solutions f(x) with $0\leq f(x)\leq 1$
- Check by means of the computer program presented in [11] whether the Abel Equation (36) belongs to one of the known integrable classes.
- If the answer is no, elaborate numerical solutions for Equation (36).
- Stability of the stationary solutions.

7. Physics

Analytic or numerical results for the gravitational potential $\phi(r)$ from solutions of either the nonlinear ODE (23) or any of its equivalent forms given in Section 5 will be suitable for the description of either galactic halos or of dark energy stars.

7.1. Galactic Halos

A star in circular motion in a gravitational potential $\phi(r)$ possesses the tangential velocity $v_{tg}(r)$ given by the relationship

$$v_{tg}^2 = r\phi' \tag{39}$$

which holds also in the general relativistic case (see [14]). Keeping in mind that in our model ϕ is sourced not only by stellar matter but also by the so called dark sector contributions, we may use a solution for ϕ in (39) for modeling of galactic rotation curves (RCs).

For the description of galactic halos, we need gravitational potentials ϕ which vanish for $r \to \infty$. But in the nonrelativistic case, described by (24), solutions vanish already for a finite but very large distance as shown in [5] by a theorem due to Taliaferro [15] as well as by numerical results.

Problem 1: Will admissible solutions (see Section 6) of the Lienard Equation (30) extend up to $x \to +\infty$ or will they end at finite *x*?

The numerical results for the RCs in the nonrelativistic case as shown in ([5] Figure 6) for our model are in good agreement with the observed nearly flat RCs at large radii for "dark matter dominated" galaxies (for a very recent review on the dark matter issues, see [16]).

We do not expect any essential modifications of the weak field limit at large radii for the relativistic model presented in this paper.

On the other hand, at small scales, our nonrelativistic model seems to show numerically a vanishing RC already at a very small but nonzero radius [5]. It was not possible to give a definite answer to this point in [5] because the numerical solutions have shown a discontinuous behavior around the critical value b_c for $b = \varphi'(0)$.

Problem 2: Behavior of the gravitational potential $\phi(r)$ for $r \to 0$ in the weak field limit as well as for the case of strong fields.

7.2. Dark Energy Stars

As has been already stated, our model shows a negative radial pressure for a positive energy density. Therefore, it is predestinated for the description of anisotropic dark energy (DE) stars. To do that, we have to take a solution of our model as interior solution which has to be matched with the exterior Schwarzschild solution.

Recent treatments of anisotropic DE stars are to be found in [17] [18] and [19]. In all these cases, analytic solutions are given by supplying some functions and constants as external input. In [17], proportionalities between energy densities and DE radial pressure as well as analytic expressions for the two metric functions ϕ and λ are assumed. In [18], two equations of state and an analytic expression for the DE energy density are given. An analytic expression for the mass function and a DE-equation of state for the radial pressure are provided in [19].

In our model, no such external inputs are needed. But in order to proceed, we have to succeed in finding (approximate) analytic expressions or numerical results for the gravitational potential.

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