

The $AdS_5 \times S^5$ Fermionic Model

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Received 25 October 2013; revised 22 November 2013; accepted 14 December 2013

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Abstract

We consider the $AdS_5 \times S^5$ integrable model. As it turns out, relying on well known arguments, we claim that the conformally invariant fermionic model is solvable, the resulting solution given in terms of two current algebras realizations.

Keywords

Component, Formatting, Style, Styling, Insert

1. Introduction

Integrable models have a long and successful history [1]. In particular, models defined on a symmetric space are generally integrable [2]-[4]. This means that an infinite number of local conservation laws exist [2], or at least one nonlocal conservation law [1] [5]. In general, such integrable models display a non vanishing mass gap, useful for describing the exact S-matrix in terms of rapidities [3] [6]. In such a line, a large number of models have been solved and their exactness on shell solution was obtained [7]-[10].

There is also at least one model where no mass gap exists, but comprising non trivial conservation laws. It is the case of the chiral Gross-Neveu model [11]. Supposing the existence of a mass gap, the model has been solved on shell [12] [13]. However, it is known that there is a non trivial fix point such that the theory allows for a conformally invariant solution as well, for a given value of the coupling constant [14].

This means that an integrable model can also contain a conformally invariant solution. This is a quite non trivial fact that we wish to explore in case of integrable models relevant for string theory, where conformal invariance is a very desirable property.

In the framework of string theory, it is possible to gather information about the Yang-Mills theory at intermediate coupling. Obtaining a strongly coupled field theory underlying the QCD string actually provides an integrable model in the world sheet, and the low dimensionality of the problem may imply exact solvability [15].

In that case, the symmetry of the integrable model is $PSU(2,2|4)/(SO(4,1) \times SO(5))$.

The bosonic part of such a coset is $AdS_5 \times S^5$, which will be our main concern. Most of the literature is related, in this case, to integrable models and their nonlocal conservation laws [16] [17]. Currents for the pure spinor superstring in $AdS_5 \times S^5$ have subsequently been constructed [18]. While the role of AdS_5 is largely discussed in relation to string solutions [19] [20], integrable structures are related to the underlying string spectrum [21].

Later, the non local charges have also been related to a BRST cohomology [22] ensuring κ -symmetry. One thus conjectured that conformal invariance should be related to the integrable models relevant to string theory.

On the other hand, in string theory, a lot has been done concerning integrability of the underlying symmetry of strings in certain backgrounds. In Maldacena's conjecture, four-dimensional $N = 4$ super Yang-Mills theory is dual to super strings in $AdS_5 \times S^5$ background [23]. But

$$AdS_5 \times S^5 \equiv \frac{SO(5,1)}{O(4,1)} \times \frac{SO(6)}{O(5)} \quad (1)$$

This means that the model is defined on a symmetric space, thus implying a non trivial (and non local) conservation law [2]. Moreover, since the symmetric space is a direct product of symmetric spaces with simple gauge groups, the sigma model defined on that space is also integrable at the quantum level [4]. On the other hand, conformal invariance is very useful in string theory and the question is whether these models display conformal invariance, at least in some form. The answer is positive, as we show.

We shall consider a fermionic model defined upon the space (1). Following old and well established arguments we see that at a well defined value of the coupling constant the theory is conformally invariant.

2. Conserved Currents

The above mentioned fermionic model is defined by the lagrangian density

$$L = \bar{\psi}_{ia} i\gamma^\mu \partial_\mu \psi_{ia} + g_1 J_{\mu ab} J_{ba}^\mu + g_2 J_{\mu ij} J_{ji}^\mu \quad (2)$$

where we define the currents are given by $J_{\mu ab} = \bar{\psi}_{ia} \gamma^\mu \psi_{ib}$ and $J_{\mu ij} = \bar{\psi}_{ia} \gamma^\mu \psi_{ja}$. They are related to the first or second factors defining the underlying symmetry group, that is, we identify the labels a, b, \dots as being in $SO(5,1)$ and i, j, \dots in $SO(6)$. Here, g_1 and g_2 are, up to now, arbitrary coupling constants.

The field equation for ψ_{ia} is

$$i\gamma^\mu \partial_\mu \psi_{ia} = -2g_1 \gamma^\mu \psi_{ic} J_{\mu ca} - 2g_2 \gamma^\mu \psi_{ka} J_{\mu ki} \quad (3)$$

while

$$i\partial_\mu \bar{\psi}_{ia} \gamma^\mu = 2g_1 J_{\mu ac} \bar{\psi}_{ic} \gamma^\mu + 2g_2 J_{\mu ik} \bar{\psi}_{ka} \gamma^\mu \quad (4)$$

is the field equation for $\bar{\psi}_{ia}$.

The Noether currents related to the symmetries $SO(5,1)$ and $SO(6)$, respectively, obey the conservation equations $\partial_\mu J_{ab}^\mu = 0$ and $\partial_\mu J_{ij}^\mu = 0$.

Let us now consider the axial currents (non) conservation laws. Using the relations for the γ_{μ} matrices we have

$$\varepsilon^{\mu\nu} \gamma_\nu = \gamma^5 \gamma^\mu, \varepsilon^{01} = 1$$

$$(\gamma^5)^2 = 1, (\gamma^0)^2 = 1, (\gamma^1)^2 = -1 \quad (5)$$

$$\gamma^1 = -\gamma_1, \gamma^5 = \gamma^0 \gamma^1, \gamma^5 \gamma^0 = -\gamma_1 \quad (6)$$

We can compute the divergence of the axial current,

$$\begin{aligned} & \varepsilon^{\mu\nu} \partial_\mu J_{\nu ab}, \\ & \varepsilon^{\mu\nu} i\partial_\mu (\bar{\psi}_{ia} \gamma_\nu \psi_{ib}) = i\partial_\mu (\bar{\psi}_{ia} \gamma^5 \gamma^\mu \psi_{ib}) = -[i\partial_\mu \bar{\psi}_{ia} \gamma^\mu] \gamma^5 \psi_{ib} + \bar{\psi}_{ia} \gamma^5 [i\partial_\mu \gamma^\mu \psi_{ib}] \end{aligned} \quad (7)$$

Taking into account the field equations we get

$$\begin{aligned} \varepsilon^{\mu\nu} i \partial_\mu (\bar{\psi}_{ia} \gamma_\nu \psi_{ib}) = & -2g_1 \left[\bar{\psi}_{ia} \gamma^5 \gamma^\mu \psi_{ic} \right] J_{\mu cb} - 2g_2 \left[\bar{\psi}_{ia} \gamma^5 \gamma^\mu \psi_{jb} \right] J_{\mu ji} \\ & + 2g_1 J_{\mu ac} \left[\bar{\psi}_{ic} \gamma^\mu \gamma^5 \psi_{ib} \right] - 2g_2 J_{\mu ij} \left[\bar{\psi}_{ja} \gamma^\mu \gamma^5 \psi_{ib} \right] \end{aligned} \quad (8)$$

Here we note that the terms with the g_1 coefficient are products of two currents while the terms with g_2 coefficient are cancelled, that is,

$$\begin{aligned} \varepsilon^{\mu\nu} i \partial_\mu (\bar{\psi}_{ia} \gamma_\nu \psi_{ib}) = & -2g_1 \varepsilon^{\mu\nu} \left[J_{vac} J_{\mu cb} - J_{\mu ac} J_{vcb} \right] \\ & - 2g_2 \varepsilon^{\mu\nu} \left[\bar{\psi}_{ia} \gamma_\nu \psi_{jb} J_{\mu ji} - J_{\mu ij} \bar{\psi}_{ja} \gamma_\nu \psi_{ib} \right] \end{aligned} \quad (9)$$

Using the identity

$$\varepsilon^{\mu\nu} (\gamma_\mu)_{\alpha\beta} (\gamma_\nu)_{\gamma\delta} = \delta_{\alpha\beta} (\gamma_5)_{\beta\gamma} - (\gamma_5)_{\alpha\delta} \delta_{\beta\gamma} \quad (10)$$

the final result is

$$\varepsilon^{\mu\nu} \partial_\mu (J_\nu)_{ab} - 4ig_1 \varepsilon^{\mu\nu} (J_\mu J_\nu)_{ab} = 0 \quad (11)$$

Therefore, the axial current $J_{vac}^{(5)} = \varepsilon^{\mu\nu} J_{vac}$ fails to be conserved classically. A similar result follows for the axial current $J_{vij}^{(5)} = \varepsilon^{\mu\nu} J_{vij}$,

$$\varepsilon^{\mu\nu} \partial_\mu (J_\nu)_{ij} - 4ig_2 \varepsilon^{\mu\nu} (J_\mu J_\nu)_{ij} = 0 \quad (12)$$

We consider now the axial anomaly contribution to the field equations. We introduce the gauge field $(A_\mu)_{ab} = 2g_1 (J_\mu)_{ab}$ in order to identify the anomaly term

$$\frac{N}{2\pi} \varepsilon^{\mu\nu} F_{\mu\nu} = \frac{N}{\pi} \varepsilon^{\mu\nu} (\partial_\mu A_\nu - iA_\mu A_\nu) = \frac{N}{\pi} \left[2g_1 \varepsilon^{\mu\nu} \partial_\mu J_\nu - i(2g_1)^2 J_\mu J_\nu \right] \quad (13)$$

to be added to the divergence equation for $J_{vac}^{(5)}$,

$$\varepsilon^{\mu\nu} \partial_\mu (J_\nu)_{ab} - 4ig_1 \varepsilon^{\mu\nu} (J_\mu J_\nu)_{ab} = \frac{N}{2\pi} \varepsilon^{\mu\nu} (F_{\mu\nu})_{ab} \quad (14)$$

where N is the number of species, in this case equal to 6. We are thus led to

$$\varepsilon^{\mu\nu} \partial_\mu (J_\nu)_{ab} = 4ig_1 \frac{\left(1 - g_1 \frac{N}{\pi}\right)}{\left(1 + g_1 \frac{N}{\pi}\right)} \varepsilon^{\mu\nu} (J_\mu J_\nu)_{ab} \quad (15)$$

Therefore, the choice $g_1 = (\pi/N)$ implies that the axial current is also conserved,

$$\varepsilon^{\mu\nu} \partial_\mu (J_\nu)_{ab} = 0 \quad (16)$$

This means conformal invariance in the coset $SO(5,1)/O(4,1)$. Notice that, mutatis mutandis we get similar a result for the $SO(6)/O(5)$ factor, as well as conformal invariance for all spaces of the kind $AdS_p \times S^p$ in case we carefully choose the coupling. Thus, at the point $g_2 = (2\pi)/6$ the second axial current is conserved

$$\varepsilon^{\mu\nu} \partial_\mu (J_\nu)_{ij} = 0 \quad (17)$$

and the fermionic theory in the coset $SO(6)/O(5)$ is conformally invariant.

An alternative and equivalent proof of conformal invariance at a given coupling can be obtained by arguments already known in [14]. Thus, for these values of g_1 and g_2 we get the conformal field ψ_{ia} with $SO(5,1)/SO(4,1) \times SO(6)/SO(5)$ conformal invariance.

3. Currente Algebra

We can write the equal-time commutation rules

$$\begin{aligned} [J_{0ab}(t, x), J_{0cd}(t, y)] &= if_{abcd}^{ef} J_{0ef}(t, x) \delta(x - y) \\ [J_{0ab}(t, x), J_{1cd}(t, y)] &= if_{abcd}^{ef} J_{0ef}(t, x) \delta(x - y) + iC_1 \delta_{ac} \delta_{bd} \delta'(x - y) + iC_2 \delta_{ad} \delta_{bc} \delta'(x - y) \\ [J_{1ab}(t, x), J_{1cd}(t, y)] &= if_{abcd}^{ef} J_{0ef}(t, x) \delta(x - y) \end{aligned} \quad (18)$$

where C_1 and C_2 ($= 0$ or $-C_1$) are c -number Schwinger terms. In addition, we also have

$$\begin{aligned} [J_{0ij}(t, x), J_{0kl}(t, y)] &= if_{ijkl}^{pq} J_{0pq}(t, x) \delta(x - y) \\ [J_{0ij}(t, x), J_{1kl}(t, y)] &= if_{ijkl}^{pq} J_{0pq}(t, x) \delta(x - y) + iD_1 \delta_{ik} \delta_{jl} \delta'(x - y) + iD_2 \delta_{il} \delta_{jk} \delta'(x - y) \\ [J_{1ij}(t, x), J_{1kl}(t, y)] &= if_{ijkl}^{pq} J_{0ef}(t, x) \delta(x - y) \end{aligned} \quad (19)$$

where D_1 and D_2 ($= 0$ or $-D_1$) are also c -number Schwinger terms.

Here we note the structure constants f_{abcd}^{ef} of the factor group $SO(5,1)/O(4,1)$ and f_{ijkl}^{pq} of $SO(6)/O(5)$.

Using

$$\begin{aligned} x_+ &= t + x, \quad x_- = t - x, \\ J_{\pm ab} &= J_{0ab} \pm J_{1ab}, \quad J_{\pm ab} = J_{\pm ab}(x_{\pm}) \end{aligned} \quad (20)$$

we can deduce from the equal-time commutation relations the commutation rules for any space-time point,

$$\begin{aligned} [J_{\pm ab}(x_{\pm}), J_{\pm cd}(y_{\pm})] &= 2if_{abcd}^{ef} J_{\pm ef}(x_{\pm}) \delta(x_{\pm} - y_{\pm}) + 2iC_1 \delta_{ac} \delta_{bd} \delta'(x_{\pm} - y_{\pm}) \\ &\quad + 2iC_2 \delta_{ad} \delta_{bc} \delta'(x_{\pm} - y_{\pm}) \end{aligned} \quad (21)$$

We can now decompose also the currents $J_{\pm ab}$ into creation and annihilation parts, each one of massless excitations. We have

$$J_{\pm ab}(x_{\pm}) = J_{\pm ab}^{(+)}(x_{\pm}) + J_{\pm ab}^{(-)}(x_{\pm}) \quad (22)$$

where (+) is the creation part and (−) the annihilation part. Note that here two creation or two annihilation operators of different $SO(5,1)/O(4,1)$ indices do not commute.

One finds also

$$[J_{\pm ab}(x_{\pm}), \psi_{ic}(x'_+, x'_-)] = -\sigma \delta_{ab} (1 \pm \delta \gamma_5) \psi_{ic}(x'_+, x'_-) \delta(x_{\pm} - x'_\pm) \quad (23)$$

where, due to Jacobi identity $\sigma = 1$ and $\delta^2 = 1$. A similar construction with the current $J_{\mu ij}$ can be trivially obtained.

Correlation functions are now immediately obtained from the methods of two-dimensional conformally invariant Quantum Field Theory [1].

The by now rather expected results displayed above mean that integrable models can have a conformally invariant counterpart. The fact that in string theory one needs conformal invariance as a building block forces us into the above solution at least for the fermionic models in question.

The rather important unanswered question is about what happens in case of a purely bosonic theory, or also, maybe even more important, to the model defined on a graded manifold. In the last case, in view of the unbroken supersymmetry, we are led to a conjecture concerning such sigma models, namely we conjecture that such models have a conformal fix point where the correlators are exactly solvable and present the previous symmetry.

Acknowledgements

This work has been supported by FAPESP and CNPq, Brazil.

References

- [1] Abdalla, E., Abdalla, M.C.B. and Rothe, K.D. (2001) *Non Perturbative Methods in 2-Dimensional Quantum Field Theory*, World Scientific.
- [2] Eichenherr, H. and Forger, M. (1981) *Communications in Mathematical Physics*, **82**, 227.
- [3] Abdalla, E., Abdalla, M.C.B. and Forger, M. (1988) *Nuclear Physics B*, **297**, 374.
Abdalla, E. and Forger, M. (1986) *Communications in Mathematical Physics*, **104**, 123.
Abdalla, E. and Abdalla, M.C.B. (1981) *M. Gomes Physical Review D*, **23**, 1800.
<http://dx.doi.org/10.1103/PhysRevD.27.825>
- [4] Abdalla, E., Forger, M. and Gomes, M. (1982) *Nuclear Physics*, **B210**, 181.
[http://dx.doi.org/10.1016/0550-3213\(82\)90238-3](http://dx.doi.org/10.1016/0550-3213(82)90238-3)
- [5] Luscher, M., Pohlmeier, K. (1978) *Nuclear Physics*, **B137**, 46. [http://dx.doi.org/10.1016/0550-3213\(78\)90049-4](http://dx.doi.org/10.1016/0550-3213(78)90049-4)
- [6] Zamolodchikov, A.B. and Zamolodchikov, A.B. (1979) *Annals of Physics*, **120**, 253-291.
[http://dx.doi.org/10.1016/0003-4916\(79\)90391-9](http://dx.doi.org/10.1016/0003-4916(79)90391-9)
- [7] Abdalla, E., Abdalla, M.C.B., Lima-Santos, A. (1984) *Physical Letters*, **B140**, 71-75.
[http://dx.doi.org/10.1016/0370-2693\(84\)91050-5](http://dx.doi.org/10.1016/0370-2693(84)91050-5)
- [8] Abdalla, E. and Forger, M. (1985) *A. Lima Santos Nuclear Physics*, **B256**, 145.
[http://dx.doi.org/10.1016/0550-3213\(85\)90389-X](http://dx.doi.org/10.1016/0550-3213(85)90389-X)
- [9] Abdalla, E. (1984) *A. Lima-Santos Physical Review*, **D29**, 1851. <http://dx.doi.org/10.1103/PhysRevD.29.1851>
- [10] Lima-Santos, A. and Martins, M.J. (2007) *Nuclear Physics*, **B760**, 184-196.
Lima-Santos, A. (2006) *Journal of Statistical Mechanics*, **0612**, P12018.
Malara, R. and Lima-Santos, A. (2006) *Journal of Statistical Mechanics*, **0609**, P09013.
Kurak, V. (2005) *A. Lima-Santos Journal of Physics*, **A38**, 2359-2374.
- [11] Gross, D.J. (1974) *Andre Neveu Physical Reviews*, **D10**, 3235. <http://dx.doi.org/10.1103/PhysRevD.10.3235>
- [12] Abdalla, E., Berg, B. and Weisz, P. (1979) *Nuclear Physics*, **B157**, 387.
[http://dx.doi.org/10.1016/0550-3213\(79\)90110-X](http://dx.doi.org/10.1016/0550-3213(79)90110-X)
- [13] Koberle, R., Kurak, V., Swieca, J.A. (1979) *Physical Reviews*, **D20**, 897. <http://dx.doi.org/10.1103/PhysRevD.20.897>
- [14] Dashen, R.F. and Frishman, Y. (1975) *Physical Reviews*, **D11**, 2781. <http://dx.doi.org/10.1103/PhysRevD.11.2781>
- [15] Bena, I., Polchinski, J. and Roiban, R. (2004) *Physical Reviews*, **D69**, 046002.
<http://dx.doi.org/10.1103/PhysRevD.69.046002>
- [16] Maldacena, J. and Ooguri, H. (2001) *Journal of Mathematical Physics*, **42**, 2929.
<http://dx.doi.org/10.1063/1.1377273>
- [17] Maldacena, J. and Maoz, L. (2002) 046; I. Bakas and J. Sonnenschein JHEP 0212 (2002) 049.
- [18] Vallilo, B. (2004) JHEP 0403 (2004) 037.
- [19] Mandal, G., Suryanarayana, N.V. and Wadia, S.R. (2002) *Physical Letters*, **B543**, 81.
[http://dx.doi.org/10.1016/S0370-2693\(02\)02424-3](http://dx.doi.org/10.1016/S0370-2693(02)02424-3)
- [20] Berenstein, D., Maldacena, J.M. and Nastase, H. (2002) JHEP 0204 (2002) 013.
- [21] Hatsuda, M. and Yoshida, K. (2005) *Advances in Theoretical and Mathematical Physics*, **9**, 703-728.
- [22] Kluson, J. (2007) 040; Nathan Berkovits JHEP 0502 (2005) 060.
- [23] Maldacena, J.M. (1998) *Advances in Theoretical and Mathematical Physics*, **2**, 231-252.