

# Application of Homotopy Perturbation Method and Parameter Expanding Method to Fractional Van der Pol Damped Nonlinear Oscillator

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## ABSTRACT

In this study, homotopy perturbation method and parameter expanding method are applied to the motion equations of two nonlinear oscillators. Our results show that both the (HPM) and (PEM) yield the same results for the nonlinear problems. In comparison with the exact solution, the results show that these methods are very convenient for solving nonlinear equations and also can be used for strong nonlinear oscillators.

**Keywords:** Nonlinear Oscillators; Homotopy Perturbation Method; Parameter Expanding Method; Fractional Van der Pol Damped Nonlinear Oscillator

## 1. Introduction

The study of nonlinear oscillators has been important in the development of the theory of dynamical systems. The Van der Pol oscillator can be regarded as describing a mass-spring-damper system with a nonlinear position-dependent damping coefficient or, equivalently, an RLC electrical circuit with a negative nonlinear resistor, and has been used to develop models in many applications, such as electronics, biology or acoustics. It represents a nonlinear system with an interesting behavior that arises naturally in several applications. Very recently, various kinds of analytical and numerical methods have been used to solve the problems of nonlinear oscillators, such as Frequency-amplitude formulation [1-3], Energy balance method [4-6], Variational iteration method [7,8], Homotopy perturbation method [9-12] and Parameter expanding method [13-15].

In recent years, the application of the homotopy perturbation method in nonlinear problems has been developed by scientists and engineers, because this method continuously deforms the difficult problem under study into a simple problem which is easy to solve. The homotopy perturbation method was proposed first by He in 1998 and was further developed and improved by him

[10,13,16]. Homotopy is an important part of differential topology. The homotopy perturbation method is, in fact, a coupling of the traditional perturbation method and homotopy in topology [17]. The essential idea of this method is to introduce a homotopy parameter, say  $p$ , which takes the values from 0 to 1. When  $p = 0$ , the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. At  $p = 1$ , the problem takes the original form and the final stage of deformation gives the desired solution. Since most phenomena in our world are essentially nonlinear and are described by nonlinear equations, it is very difficult to solve nonlinear problems, and in general, it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem.

Parameter expanding method proposed by He [18] is the most effective and convenient method for handling nonlinear problems [13]. In this method, the solution and unknown frequency of oscillation are expanding in a series by a bookkeeping parameter.

This paper applies (HPM) and (PEM) to fractional Van der Pol damped nonlinear oscillator. Comparison of the period of oscillation and the exact solution shows that both methods are very effective and convenient and quite accurate to nonlinear engineering problems.

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## 2. Applications

In order to show the accuracy of homotopy perturbation method (HPM) and parameter expanding method (PEM) for solving nonlinear equations and to compare it with exact solutions, we will consider the following examples.

### 2.1. Example 1: The Classical Fractional Van der Pol Oscillator

The classical fractional Van der Pol damped nonlinear oscillator can be represented by the following nonlinear equation [19-21].

$$\ddot{x} + x^{1/3} + \varepsilon(1 - x^2)\dot{x} = 0, \tag{1}$$

with the initial conditions:

$$x(0) = a, \quad \dot{x}(0) = 0. \tag{2}$$

#### 2.1.1. Application of Homotopy Perturbation Method (HPM)

The following homotopy can be constructed

$$\ddot{x} + \omega^2 x + p[x^{1/3} - \omega^2 x + \varepsilon(1 - x^2)\dot{x}] = 0, \quad p \in [0, 1]. \tag{3}$$

As in He's homotopy perturbation method [9,10,16], it is obvious that when  $p = 0$  Equation (3) becomes the linearized equation,  $\ddot{x}_0 + \omega^2 x_0 = 0$ , for  $p = 1$  Equation (3) then becomes the original problem. Assume that the periodic solution to Equation (3) may be written as a power series in  $p$ :

$$x = x_0 + px_1 + p^2 x_2 + p^3 x_3 + \dots, \tag{4}$$

Setting  $p = 1$ , leads to the approximate solution of the problem:

$$x = \lim_{p \rightarrow 1} x = x_0 + x_1 + x_2 + x_3 + \dots \tag{5}$$

Substituting Equation (4) into Equation (3) and equating the terms with the identical powers of  $p$ ,

$$p^0 : \ddot{x}_0 + \omega^2 x_0 = 0, \quad x(0) = a, \quad \dot{x}(0) = 0, \tag{6}$$

$$p^1 : \ddot{x}_1 + \omega^2 x_1 + x_0^{1/3} - \omega^2 x_0 + \varepsilon(1 - x_0^2)\dot{x}_0 = 0, \tag{7}$$

$$x_1(0) = 0, \dot{x}_1(0) = 0.$$

Solving Equation (6), we have

$$x_0 = a \cos \omega t. \tag{8}$$

The Fourier series for  $(\cos \omega t)^{1/3}$  has been calculated [22] and is given by.

$$(\cos \omega t)^{1/3} = b_1 \cos \omega t + b_3 \cos 3\omega t + \dots, \tag{9}$$

where  $b_1 = 1.15960$ ,  $b_3 = -0.231919$ ,

Substituting Equation (8) into Equation (7) leads to

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 + \frac{1}{4}(4b_1 a^{1/3} - 4a\omega^2) \cos \omega t + b_3 a^{1/3} \cos 3\omega t \\ + \frac{1}{4}(\varepsilon a^3 \omega - 4\varepsilon a \omega) \sin \omega t + \frac{1}{4} \varepsilon a^3 \omega \sin 3\omega t = 0. \end{aligned} \tag{10}$$

Eliminating the secular term, we have

$$4b_1 a^{1/3} - 4a\omega^2 = 0, \quad a^2 - 4 = 0. \tag{11}$$

From the above equation, we can easily find that

$$a = 2, \quad \omega = 0.8547. \tag{12}$$

which is same with that obtained by iteration procedure (see Equation (40) in Ref [19] and Equation (23) in Ref [20]). Hence, the approximate periodic solution can be readily obtained:

$$x_{app}(t) = 2 \cos(0.8547t). \tag{13}$$

#### 2.1.2. Application of Parameter Expanding Method (PEM)

We rewrite Equation (1) in the form

$$\ddot{x} + 0 \cdot x + 1 \cdot x^{1/3} + \varepsilon(1 - x^2)\dot{x} = 0. \tag{14}$$

To solve Equation (1) by parameter expanding method we expand the solution of the problem and coefficients 0 and 1 in the left side of Equation (14) in series of  $\varepsilon$  as follows

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots, \tag{15}$$

$$0 = \omega^2 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots, \tag{16}$$

$$1 = \varepsilon c_1 + \varepsilon^2 c_2 + \dots \tag{17}$$

Substituting Equations (15)-(17) into Equation (14), and processing as the standard perturbation method, we have

$$\ddot{x}_0 + \omega^2 x_0 = 0, \quad x(0) = a, \quad \dot{x}(0) = 0, \tag{18}$$

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 + \omega_1 x_0 + c_1 x_0^{1/3} + \dot{x}_0 - \dot{x}_0 x_0^2 = 0, \\ x(0) = a, \dot{x}(0) = 0. \end{aligned} \tag{19}$$

The solution of Equation (18) can be easily obtained

$$x_0 = a \cos \omega t. \tag{20}$$

Substituting the result into Equation (19) yields

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 + \omega_1 a \cos \omega t + c_1 a^{1/3} (\cos \omega t)^{1/3} \\ - a\omega \sin \omega t + \frac{a^3 \omega}{4} (\sin \omega t + \sin 3\omega t) = 0. \end{aligned} \tag{21}$$

Using the relation (9) Equation (21) can be rewritten as

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 + (\omega_1 a + c_1 a^{1/3} b_1) \cos \omega t + c_1 a^{1/3} b_3 \cos 3\omega t \\ + a\omega \left( \frac{a^2}{4} - 1 \right) \sin \omega t + \frac{a^3 \omega}{4} \sin 3\omega t = 0. \end{aligned} \tag{22}$$

No secular terms requires

$$\left(\frac{a^2}{4} - 1\right) = 0, \quad \omega_1 a + c_1 a^{1/3} b_1 = 0. \quad (23)$$

If the first order approximation is enough, then setting  $\varepsilon = 1$  in Equations (15)-(17) yields

$$x = x_0 + x_1 \quad 0 = \omega^2 + \omega_1, \quad 1 = c_1. \quad (24)$$

From relation (23) and (24) we have

$$a = 2, \quad \omega = 0.8547 \quad (25)$$

which is the same with that obtained by iteration procedure in [19,20] (see Equation (40) in Ref [19] and Equation (23) in Ref [20])

Hence, the approximate periodic solution can be readily obtained:

$$x_{app}(t) = 2 \cos(0.8547t). \quad (26)$$

To illustrate and verify the accuracy of this method, we may compare the approximate periodic solution  $x_{app}(t)$  and the exact periodic solution. For reference, the exact solution  $x_{ex}$  for the classical fractional Van der Pol damped nonlinear oscillator is as follows [23]:

$$x_{ex} = \frac{2a}{\sqrt{a^2 + (4 - a^2)e^{-\varepsilon t}}} \cos(\omega_{ex} t), \quad (27)$$

where

$$\omega_{ex} = \frac{\sqrt{\pi} \Gamma(5/4)}{\sqrt{(8/3)a^{1/3} \Gamma(7/4)}}. \quad (28)$$

In **Figure 1**, we presented the variety of comparisons between approximate and exact solution for Equation (1). For  $a = 2, \varepsilon = 1$ . It can be observed that Equation (26) provides excellent approximation to the exact periodic solution in Equation (27).

### 2.2. Example 2: The Special Case of the Fractional Van der Pol Equation or Rayleigh Equation

The special case of the fractional Van der Pol damped nonlinear oscillator or the Rayleigh equation can be represented by [21,24,25].

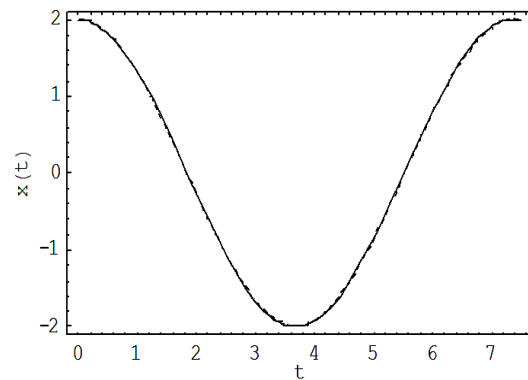
$$\ddot{x} + x^3 = \varepsilon \left( \dot{x} - \frac{1}{3} \dot{x}^3 \right), \quad (29)$$

with the initial conditions:

$$x(0) = a, \quad \dot{x}(0) = 0. \quad (30)$$

#### 2.2.1. Application of Homotopy Perturbation Method (HPM)

We can establish the following homotopy



**Figure 1** Comparison of the analytical solutions (–) with the exact solution (···) for  $a = 2; \omega = 0.8547$ .

$$\ddot{x} + \omega^2 x + p \left[ x^3 - \omega^2 x - \varepsilon \left( \dot{x} - \frac{1}{3} \dot{x}^3 \right) \right] = 0, \quad p \in [0, 1]. \quad (31)$$

Similarly, when  $p = 0$ , Equation (31) becomes a linear equation; for  $p = 1$ , Equation (31) then becomes the original problem. Applying the perturbation technique, the solution of Equation (31) can be expressed as a power series in  $p$ :

$$x = x_0 + px_1 + p^2 x_2 + p^3 x_3 + \dots, \quad (32)$$

Setting  $p = 1$ , leads to the approximate solution of the problem:

$$x = \lim_{p \rightarrow 1} x = x_0 + x_1 + x_2 + x_3 + \dots \quad (33)$$

Substituting Equation (32) into Equation (31) and equating the terms with the identical powers of  $p$ ,

$$p^0 : \ddot{x}_0 + \omega^2 x_0 = 0, \quad x(0) = a, \quad \dot{x}(0) = 0, \quad (34)$$

$$p^1 : \ddot{x}_1 + \omega^2 x_1 + x_0^3 - \omega^2 x_0 - \varepsilon \left( \dot{x}_0 - \frac{1}{3} \dot{x}_0^3 \right) = 0, \quad (35)$$

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0.$$

The solution for  $x_0$  is

$$x_0 = a \cos \omega t. \quad (36)$$

Substituting Equation (36) into Equation (35) leads to

$$\ddot{x}_1 + \omega^2 x_1 = \left( a\omega^2 - \frac{3}{4} a^3 \right) \cos \omega t - \frac{1}{4} a^3 \omega^3 \cos 3\omega t + \left( \frac{1}{4} a^3 \omega^3 - a\omega \right) \sin \omega t - \frac{1}{12} \varepsilon a^3 \omega^3 \sin 3\omega t = 0. \quad (37)$$

No secular terms requires

$$\left( \frac{3}{4} a^2 - \omega^2 \right) = 0, \quad \left( \frac{1}{4} a^2 \omega^2 - 1 \right) = 0. \quad (38)$$

From the above equation, we can easily find that

$$a = 1.51967, \quad \omega = 1.31607. \quad (39)$$

Hence, the approximate periodic solution can be readily obtained:

$$x_{app}(t) = 1.51967 \cos(1.31607t). \quad (40)$$

**2.2.2. Application of Parameter Expanding Method (PEM)**

We rewrite Equation (29) in the form

$$\ddot{x} + 0 \cdot x + 1 \cdot x^3 = \varepsilon \left( \dot{x} - \frac{1}{3} \dot{x}^3 \right). \quad (41)$$

Substituting Equations (15)-(17) into Equation (41), collecting the same power of  $\varepsilon$ , and equating each coefficient of  $\varepsilon$  to zero, we obtain

$$\ddot{x}_0 + \omega^2 x_0 = 0, \quad x(0) = a, \quad \dot{x}(0) = 0, \quad (42)$$

$$\ddot{x}_1 + \omega^2 x_1 + \omega_1 x_0 + c_1 x_0^3 = \dot{x}_0 - \frac{1}{3} \dot{x}_0^3, \quad x_1(0) = a, \quad \dot{x}_1(0) = 0. \quad (43)$$

Solving Equation (42), we have

$$x_0 = a \cos \omega t. \quad (44)$$

Substituting  $x_0$  into Equation (43) results in

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 + a \left( -\frac{3}{4} c_1 a^2 - \omega_1 \right) \cos \omega t \\ - \frac{1}{4} c_1 a^3 \cos 3\omega t + \omega a \left( \frac{1}{4} a^2 \omega^2 - 1 \right) \sin \omega t \\ - \frac{1}{12} a^3 \omega^3 \sin 3\omega t = 0. \end{aligned} \quad (45)$$

Eliminating the secular terms needs

$$\left( \frac{3}{4} c_1 a^2 + \omega_1 \right) = 0, \quad \left( \frac{1}{4} a^2 \omega^2 - 1 \right) = 0. \quad (46)$$

If the first order approximation is enough, then setting  $\varepsilon = 1$  in Equations (15)-(17) yields

$$x = x_0 + x_1, \quad 0 = \omega^2 + \omega_1, \quad 1 = c_1. \quad (47)$$

From relation (46) and (47) we have

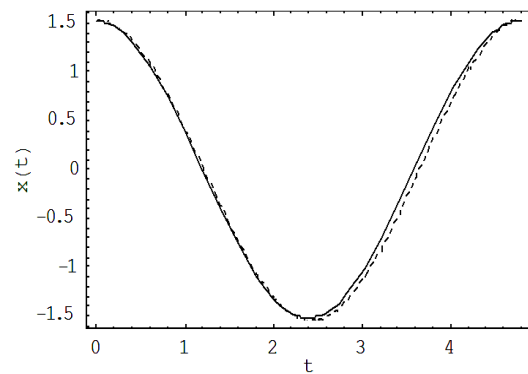
$$a = 1.51967, \quad \omega = 1.31607. \quad (48)$$

Hence, the approximate periodic solution can be readily obtained:

$$x_{app}(t) = 1.51967 \cos(1.31607t). \quad (49)$$

To write the exact periodic solution  $x_{ex}(t)$ , we used its values from [23], and for approximate values we used Equation (49). In fact the exact periodic solution it can be obtained from the following complicated relation that is given in [23].

$$x_{ex} = \frac{2a}{\sqrt{a^2 \omega_{ex}^2 - (a^2 \omega_{ex}^2 - 4)e^{-\varepsilon t}}} \cos(\omega_{ex} t), \quad (50)$$



**Figure 2. Comparison of the analytical solutions (–) with the exact solution (---) for  $a = 1.51967$ ;  $\omega = 1.31607$ .**

where

$$\omega_{ex} = \frac{\sqrt{\pi} \Gamma(3/4)}{\sqrt{8} \Gamma(5/4)} a. \quad (51)$$

To compare with the exact period, **Figure 2** shows the comparison between the approximate periodic solution  $x_{app}(t)$  obtained from formula (49) and the exact periodic solution  $x_{ex}(t)$  obtained from formula (50) see [23]. It can be seen from **Figure 2** that the approximate periodic solution is nearly identical with that given by the exact periodic solution.

**3. Conclusion**

In this paper, we give a comparative study between Homotopy perturbation method (HPM) and Parameter expanding method (PEM) to obtain the approximate periodic solutions to fractional Van der Pol damped nonlinear oscillators. Illustrative examples reveal that these methods are very effective and convenient for solving nonlinear differential equations. Comparisons are also made between the exact solution and the results of the Homotopy perturbation method and Parameter expanding method in order to prove the precision of the results obtained from both methods mentioned.

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