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Solution of Dirac Equation with the Time-Dependent Linear Potential in Non-Commutative Phase Space*

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ABSTRACT

In this paper, the time-dependent invariant of the Dirac equation with time-dependent linear potential has been constructed in non-commutative phase space. The corresponding analytical solution of the Dirac equation is presented by Lewis-Riesenfield invariant method.

Keywords: Non-Commutative; Dirac Equation; Time-Dependent Linear Potentials; Exact Wave Function

1. Introduction

It is well known that the quadratic Hamiltonian system includes not only conservation system but also time-dependent system [1,2]. The study of time-dependent quantum-mechanical system has attracted considerable interest over the past decades because their quantum correspondence provides fundamental structure of basic physics and interpretation of new physics in different areas of physics, such as, gravitation [3], quantum optic [4,5], the Paul trap [6-8] and spintronics [9]. In the past few decades, an extensive effort has been made to obtain exact solution of the Schrödinger equation with time-dependent harmonic oscillator [10-14] making use of different methods, for example path integral, second quantization and dynamical invariant. Besides the time-dependent harmonic oscillator, the time-dependent linear potential model has also been employed to study several problems in physics. For instance, Cocke and Reichl have employed this model to study the effects of static field on the stochastic layer of microwave-driven hydrogen [15] and on nonlinear quantum resonances and the ionization spectrum of a simple bound particle [16] and compute the spectrum of emitted radiation for a particle in a triangular potential well and driven by an electromagnetic field [17]. Recently, Schrödinger equation [18-21] and Dirac equation [22-24] of linear potential model have also been investigated widely and the corresponding exact solutions have been given, in one dimension.

In spite of a large variety of papers that has been published concerning time-dependent Schrödinger equation and Dirac equation in commutative space, no one has reported the study of the time-dependent quantum problems in non-commutative space. While, there has been much interest in the study of physics in non-commutative spaces (NCS) in recent year [25,26], not only because the NCS is necessary when one studies the low energy effective of the D-brane with B field background, but also because in the very tiny string scale or at very high energy situation, the effects of non-commutativity of space may appear. Generally, the theory and methods of researching non-commutative problems are mainly from quantum field theory. But, it is fascinating to speculate whether there might be some low-energy effects of the fundamental quantum field theory. These effects might arise as a non-commutative version of quantum mechanics [27-29]. Under the frame of quantum mechanics, the Quantum Hall effect [30], Aharonov-Casher effect [31], gravitational quantum well [32] and the two-dimensional anharmonic oscillator [33], have been studied extensively in non-commutative space.

In this paper we are interested in non-perturbation investigation of Dirac equation with time-dependent linear potential in non-commutative phase space [32,34,35]. The main purpose of the paper is to construct the time-dependent invariant of the Dirac equation with time-dependent linear potential and to obtain, through the Lewis-

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Riesenfield invariant method [36], a corresponding analytical solution to the Dirac equation in non-commutative phase space.

2. Solution of the Dirac Equation with Time-Dependent Linear Potential in Non-Commutative Phase Space

It is well known that in the two-dimensional commutative space, the coordinates x_i and momenta p_i satisfy the usual canonical commutation relations:

$$\left[x_i, p_j\right] = \mathrm{i}\hbar\delta_{ij}, \left[x_i, x_j\right] = 0, \quad \left[p_i, p_j\right] = 0, (i, j = 1, 2) \tag{1}$$

However, the recent study results on the NCS show that at very tiny scales, say string scale, the space may not commute anymore. Let us denote the operators of coordinates and momenta in non-commutative space as $\hat{x}_i$ and $\hat{p}_i$ ($x_1 \equiv x, x_2 \equiv y, p_1 \equiv p_x$ and $p_2 \equiv p_y$) respectively, then in the two-dimensional non-commutative phase space [32,34,35], the $\hat{x}_i$ and $\hat{p}_i$ satisfy the following commutation relations:

$$\begin{aligned} &[\hat{x}, \hat{y}] = \mathrm{i}\theta, \left[\hat{p}_x, \hat{p}_y\right] = \mathrm{i}\eta, \\ &\left[\hat{x}_i, \hat{p}_j\right] = \mathrm{i}\hbar_{eff}\delta_{ij}, i, j = 1, 2 \\ &\hbar_{eff} = \hbar(1+\xi), \xi = \frac{\theta\eta}{4\hbar^2}. \end{aligned} \tag{2}$$

where θ and η are non-commutative parameters of the non-commutative phase space. Particularly, when $\eta = 0, \theta = 0$ the non-commutative phase space reduces to commutative space. One possible way of implementing algebra Equation (2) is to construct the non-commutative variables $\{\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y\}$ from the commutative variables $\{x, y, p_x, p_y\}$ by the following means of linear transformations:

$$\begin{aligned} &\hat{x} = x - \frac{1}{2\hbar}\theta p_y, \hat{y} = y + \frac{1}{2\hbar}\theta p_x, \\ &\hat{p}_x = p_x + \frac{1}{2\hbar}\eta y, \hat{p}_y = p_y - \frac{1}{2\hbar}\eta x. \end{aligned} \tag{3}$$

Now let us consider a Driac particle moving in the two-dimensional non-commutative phase space where the time-dependent linear potential:
$V(\hat{x}, \hat{y}, t) = f_1(t)\hat{x} + f_2(t)\hat{y}$ ($f_1(t), f_2(t)$ are arbitrary function of time). Then Hamiltonian of the system is given by:

$$\begin{aligned} \hat{H}(t) = {}& c\alpha_x\hat{p}_x + c\alpha_y\hat{p}_y + \beta c^2 m \\ &+ f_1(t)\hat{x} + f_2(t)\hat{y} \end{aligned} \tag{4}$$

with the Pauli matrices:

$$\alpha_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \alpha_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To find the solution of the Dirac equation of system considered here, two approaches are possible. One can directly work in the non-commutative space variables or use the phase space transformations to reduce the problem on the usual commutative space. In following discussion, we work in the second approach.

Substituting Equation (3) into (4), we have corresponding Hamiltonian expressed by the commutative variables

$$\begin{aligned} &\{x, y, p_x, p_y\}(\hbar = c = 1) \\ &H(t) = \alpha_x\left(p_x + \frac{\eta}{2}y\right) + \alpha_y\left(p_y - \frac{\eta}{2}x\right) \\ &\quad + f_1(t)\left(x - \frac{\theta}{2}p_y\right) + f_2(t)\left(y + \frac{\theta}{2}p_x\right) + \beta m \\ &\quad = \left(\alpha_x + \frac{f_2\theta}{2}\right)p_x + \left(\alpha_y - \frac{f_1\theta}{2}\right)p_y \\ &\quad + \left(f_1(t) - \frac{\eta\alpha_y}{2}\right)x + \left(f_2(t) + \frac{\eta\alpha_x}{2}\right)y + \beta m. \end{aligned}$$

In commutative space the time-dependent Dirac equation is written as:

$$i\frac{\partial\psi}{\partial t} = H(t)\psi, (\hbar = 1). \tag{5}$$

So the present task is to solve Equation (5). There are also several methods to study time-dependent Dirac equation [22,23] in commutative space. In this paper, we used the so-called Lewis-Riesenfield invariant method. Now let us suppose that there exists a quantum-mechanical invariant $I(t)$ that satisfies the equation:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = \frac{1}{i}\left[I(t), H(t)\right] + \frac{\partial I}{\partial t} = 0, (\hbar = 1). \tag{6}$$

By applying Equation (6) on ψ and after some minor algebra, we obtain

$$\mathrm{i}\frac{\partial(I\psi)}{\partial t} = H(t)(I\psi) \tag{7}$$

which implies that the action of the invariant operator on a Dirac wave function produces another solution of the Dirac equation. This result is valid for any invariant if the latter involves the operation of time differentiation. For the model considered here let the linear invariant $I(t)$ to be

$$\begin{aligned} I(t) = {}& A_1(t)p_x + B_1(t)x + A_2(t)p_y \\ &+ B_2(t)y + C(t) \end{aligned} \tag{8}$$

where $A_1(t), B_1(t), A_2(t), B_2(t)$, and $C(t)$ are matrices. The substitution of Equations (8) and (4) into Equation (6) we have

$$
\begin{aligned}
&[A_1,\alpha_x]p_x^2+[A_2,\alpha_y]p_y^2+\left[B_1,-\frac{\eta\alpha_y}{2}\right]x^2\\
&+\left[B_2,\frac{\eta\alpha_x}{2}\right]y^2+\left\{[C,\alpha_x]+[A_1,\beta]m+\mathrm{i}\dot{A}_1\right\}p_x\\
&+\left\{[C,\alpha_y]+[A_2,\beta]m+\mathrm{i}\dot{A}_2\right\}p_y\\
&+\left\{\left[C,-\frac{\eta\alpha_y}{2}\right]+[B_1,\beta]m+\mathrm{i}\dot{B}_1\right\}x\\
&+\left\{\left[C,\frac{\eta\alpha_x}{2}\right]+[B_2,\beta]m+\mathrm{i}\dot{B}_2\right\}y\\
&+\left\{[A_2,\alpha_x]+[A_1,\alpha_y]\right\}p_xp_y\\
&+\left\{[B_2,\alpha_x]+\left[A_1,\frac{\eta\alpha_x}{2}\right]\right\}yp_x\\
&+\left\{[B_1,\alpha_y]+\left[A_2,-\frac{\eta\alpha_y}{2}\right]\right\}xp_y\\
&+\left\{\left[B_2,\left(-\frac{\eta\alpha_y}{2}\right)\right]+\left[B_1,\frac{\eta\alpha_x}{2}\right]\right\}xy\\
&+\left\{[B_1,\alpha_x]+\left[A_1,-\frac{\eta\alpha_y}{2}\right]\right\}xp_x\\
&+\left\{[B_2,\alpha_y]+\left[A_2,\frac{\eta\alpha_x}{2}\right]\right\}yp_y\\
&+\mathrm{i}B_1\left(\alpha_x+\frac{f_2\theta}{2}\right)+\mathrm{i}B_2\left(\alpha_y-\frac{f_1\theta}{2}\right)\\
&-\mathrm{i}A_1\left(f_1-\frac{\eta\alpha_y}{2}\right)-\mathrm{i}A_2\left(f_2+\frac{\eta\alpha_x}{2}\right)\\
&-\mathrm{i}[B_1,\alpha_x]-\mathrm{i}[B_2,\alpha_y]+\mathrm{i}\dot{C}+[C,\beta]m=0.
\end{aligned}
\tag{9}
$$

A solution of the above relation is obtained by

$$[A_1,\alpha_x]=0, \tag{10a}$$

$$[A_2,\alpha_y]=0, \tag{10b}$$

$$\left[B_1,\frac{\eta\alpha_y}{2}\right]=0, \tag{10c}$$

$$\left[B_2,\frac{\eta\alpha_x}{2}\right]=0, \tag{10d}$$

$$[C,\alpha_x]+[A_1,\beta]m+\mathrm{i}\dot{A}_1=0, \tag{10e}$$

$$[C,\alpha_y]+[A_2,\beta]m+\mathrm{i}\dot{A}_2=0, \tag{10f}$$

$$\left[C,\left(-\frac{\eta\alpha_y}{2}\right)\right]+[B_1,\beta]m+\mathrm{i}\dot{B}_1=0, \tag{10g}$$

$$\left[C,\frac{\eta\alpha_x}{2}\right]+[B_2,\beta]m+\mathrm{i}\dot{B}_2=0, \tag{10h}$$

$$[A_2,\alpha_x]+[A_1,\alpha_y]=0, \tag{10i}$$

$$[B_2,\alpha_x]+\left[A_1,\frac{\eta\alpha_x}{2}\right]=0, \tag{10j}$$

$$[B_1,\alpha_y]+\left[A_2,\left(-\frac{\eta\alpha_y}{2}\right)\right]=0, \tag{10k}$$

$$\left[B_2,-\frac{\eta\alpha_y}{2}\right]+\left[B_1,\frac{\eta\alpha_x}{2}\right]=0, \tag{10l}$$

$$[B_1,\alpha_x]+\left[A_1,\left(-\frac{\eta\alpha_y}{2}\right)\right]=0, \tag{10m}$$

$$[B_2,\alpha_y]+\left[A_2,\frac{\eta\alpha_x}{2}\right]=0, \tag{10n}$$

$$
\begin{aligned}
&\mathrm{i}B_1\left(\alpha_x+\frac{f_2\theta}{2}\right)+\mathrm{i}B_2\left(\alpha_y-\frac{f_1\theta}{2}\right)-\mathrm{i}A_1\left(f_1-\frac{\eta\alpha_y}{2}\right)\\
&-\mathrm{i}A_2\left(f_2+\frac{\eta\alpha_x}{2}\right)-\mathrm{i}[B_1,\alpha_x]-\mathrm{i}[B_2,\alpha_y]+\mathrm{i}\dot{C}\\
&+[C,\beta]m=0.
\end{aligned}
\tag{10o}
$$

From commutation relations (10a-10d), we have

$$A_1=a_1+a_2\alpha_x, \tag{11a}$$

$$A_2=a_3+a_4\alpha_y, \tag{11b}$$

$$B_1=b_1+b_2\alpha_y, \tag{11c}$$

$$B_2=b_3+b_4\alpha_x, \tag{11d}$$

respectively, where a_i and $b_i\,(i=1,2,3,4)$ are time-dependent arbitrary functions. Next, let us assume that the form of writing C in terms of α_x and α_y is given by

$$C=c_1+c_2\alpha_x+c_3\alpha_y \tag{12}$$

where c_1,c_2 and c_3 are arbitrary functions of time.

Substituting Equations (12) and (11a) into Equation (10e); Equations (12) and (11b) into Equation (10f) respectively, we find

$$\dot{a}_1=0,a_2=0,c_3=0,\dot{a}_3=0,a_4=0,c_2=0, \tag{13}$$

where the dot on variables denotes derivative with respect to time.

Noting $c_3=c_2=0$, then C in Equation (12) is changed to

$$C=c_1. \tag{14}$$

Substituting Equations (14) and (11c) into Equation (10g), Equations (14) and (11d) into Equation (10h) respectively, we get

$$b_2=0,\dot{b}_1=0,b_4=0,\dot{b}_3=0. \tag{15}$$

Making use of Equations (13) and (15), the Equation

(11) can be rewritten in the following form:

$$A_1 = a_1, A_2 = a_3, B_1 = b_1, B_2 = b_3. \tag{16}$$

Substituting Equation (16) into Equation (10o) leads to

$$b_1 = \frac{\eta}{2} a_3, \tag{17a}$$

$$b_3 = -\frac{\eta}{2} a_1, \tag{17b}$$

$$c_1 = \int \left(1 - \frac{\eta\theta}{4}\right)\left(a_1 f_1 + a_3 f_2\right) \mathrm{d}t. \tag{17c}$$

Finally, the invariant of the Dirac equation for time-dependent linear potential can be written as

$$I = a_1 p_x + a_3 p_y + \frac{\eta}{2}\left(a_3 x - a_1 y\right) + \int \left(1 - \frac{\eta\theta}{4}\right)\left(a_1 f_1 + a_3 f_2\right) \mathrm{d}t. \tag{18}$$

It is easy to see that the eigenfunction of $I(t)$ is of the form:

$$\phi_\lambda(x, y, t) \sim \exp\left[\mu_1(t) x + \mu_2(t) y + \mu_3(t) x^2 + \mu_4(t) y^2\right] \tag{19}$$

where $\mu_1(t), \mu_2(t), \mu_3(t)$ and $\mu_4(t)$ are arbitrary time-dependent functions.

From Equation (7), it is obvious that if ψ is a solution of the time-dependent Dirac equation, then the any function defined by $\phi = I\psi$ will also be. In this way, if we take $\phi = \lambda\psi$, we have that ψ is an eigenfunction of I. This suggests that the solutions of the time-dependent Dirac equation have the form of trial function

$$\psi(x, y, t) = \chi(t)\phi_\lambda(x, y, t) \tag{20}$$

where $\chi(t)$ is a 2×1 time-dependent matrix:

$$\chi(t) = \begin{pmatrix} \chi_1(t) \\ \chi_2(t) \end{pmatrix}.$$

The substituting of Equation (20) into Equation (7) gives

$$\dot{\mu}_1(t) = -\mathrm{i} f_1(t) - \theta\mu_3(t) f_2(t) - \sqrt{\left(2\mu_3(t) + \frac{\eta}{2}\right)\left(2\mu_3(t) - \frac{\eta}{2}\right)}, \tag{21a}$$

$$\dot{\mu}_2(t) = -\mathrm{i} f_2(t) + \theta\mu_4(t) f_1(t) - \frac{\mathrm{i}\eta}{2}\left(\sqrt{\frac{\frac{\eta}{2} + 2\mu_4(t)}{\frac{\eta}{2} - 2\mu_4(t)}} + \sqrt{\frac{\frac{\eta}{2} - 2\mu_4(t)}{\frac{\eta}{2} + 2\mu_4(t)}}\right), \tag{21b}$$

$$\mathrm{i}\dot{\chi}(t) = -\mathrm{i}\left(\alpha_x + \frac{f_2(t)\theta}{2}\right)\mu_1(t)\chi(t), -\mathrm{i}\left(\alpha_y - \frac{f_1(t)\theta}{2}\right)\mu_2(t)\chi(t) + \beta m\chi(t), \tag{21c}$$

$$\dot{\mu}_3(t) = 0, \tag{21d}$$

$$\dot{\mu}_4(t) = 0. \tag{21e}$$

Then, if $f_1(t)$ and $f_2(t)$ are specified, we can easily solve Equation (21), and obtain the analytical wave function for the Dirac equation with time-dependent linear potentials in non-commutative phase space.

In particular, when momentum-momentum and space-space are all commutative (namely, $\eta = 0, \theta = 0$), then the solution (21) returns to that of general quantum mechanics.

$$\dot{\mu}_1(t) = -\mathrm{i} f_1(t) - 2\mu_3(t), \tag{22a}$$

$$\dot{\mu}_2(t) = -\mathrm{i} f_2(t), \tag{22b}$$

$$\dot{\chi}(t) = -\alpha_x \mu_1(t)\chi(t), -\alpha_y \mu_2(t)\chi(t) - \mathrm{i}\beta m\chi(t), \tag{22c}$$

$$\dot{\mu}_3(t) = 0, \tag{22d}$$

$$\dot{\mu}_4(t) = 0. \tag{22e}$$

3. Conclusion

In conclusion, the Dirac equation with time-dependent linear potentials has been studied in non-commutative phase space. By Lewis-Riesenfield invariant method, the invariant of system has been constructed and the exact wave function of the system has been obtained. In addition, under the condition that space-space and momentum-momentum are all commutative (namely, η = 0, θ = 0) the results return to that of usual quantum mechanic.

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