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Mathematical Derivation of Angular Momenta in Quantum Physics

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ABSTRACT

For a two-dimensional complex vector space, the spin matrices can be calculated directly from the angular momentum commutator definition. The 3 Pauli matrices are retrieved and 23 other triplet solutions are found. In the three-dimensional space, we show that no matrix fulfills the spin equations and preserves the norm of the vectors. By using a Clifford geometric algebra it is possible in the four-dimensional spacetime (STA) to retrieve the 24 different spins 1/2. In this framework, spins 1/2 are rotations characterized by multivectors composed of 3 vectors and 3 bivectors. Spins 1 can be defined as rotations characterized by 4 vectors, 6 bivectors and 4 trivectors which result in unit multivectors which preserve the norm. Let us note that this simple derivation retrieves the main spin properties of particle physics.

Keywords: Quantum Systems; Spin 1/2; Spin 1; Particle Physics, Spacetime Algebra

1. Introduction

The spin of particles was discovered by Wolfgang Pauli toward the end of 1924 (see Fröhlich's paper for an up-to-date discussion of spins [1]). In quantum mechanics textbooks, the matrix representation of spin is obtained from the commutator general definition of the angular momentum:

$$\left[J_x, J_y\right] = i\hbar J_z \tag{1}$$

$$\left[J_y, J_z\right] = i\hbar J_x \tag{2}$$

$$\left[J_z, J_x\right] = i\hbar J_y \tag{3}$$

These 3 equations are called the spin equations.

The general theory of quantum angular momenta [2] uses the operator $J^2 = J_x^2 + J_y^2 + J_z^2$, the two shift operators $J_+ = J_x + iJ_-, J_- = J_x - iJ_-$ and the positive eigenvalue $j(j+1)\hbar$ of J^2. The values for j are $0, 1/2, 1, 3/2, 2, \cdots$. For $j = 0$, the spin space is of dimension 1 and reduced to a scalar equal to zero. For $j = 1/2$, the spin space is of dimension 2 and the matrix representation is given by

$$J_x = \frac{1}{2}\hbar\sigma_x, J_y = \frac{1}{2}\hbar\sigma_y, J_z = \frac{1}{2}\hbar\sigma_z \tag{4}$$

where σ_i are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

These three J matrices describe the so called spin 1/2 quantum system.

The aim of this work is to address two questions: can the Pauli matrices or more generally any spin system be retrieved directly from the angular momentum definition and are the 3 Pauli matrices the only 2-dimensional quantum angular momenta?

2. Properties of the Minimum-Dimension Quantum Spin System

The smallest quantum angular momentum is of dimension 3 because there are 3 elements J_x, J_y, J_z. As quantum mechanic acts on complex vector spaces, the minimum dimension is a two-dimensional complex space or a four-dimensional real space with two elements multiplied by $i = \sqrt{-1}$ which is different from a true four-dimensional real space.

2.1. Matrix Description

The two components of the vectors of such a space can be written as $(x_1 + ix_2, x_3 + ix_4)$ or in matrix notation:

$$\begin{pmatrix} x_1 + \mathrm{i}x_2 \\ x_3 + \mathrm{i}x_4 \end{pmatrix} = \begin{pmatrix} r_1 \mathrm{e}^{i\varphi_1} \\ r_2 \mathrm{e}^{i\varphi_2} \end{pmatrix} \tag{6}$$

which corresponds to an ordered set of four real elements.

An operator in such a vector space is given by $M = \left[a_{ij}\right]$ a real 2 × 2 matrix multiplied by a complex number α. It can be noticed that the two parts of α provide the information about the position of i in the four real number set a_{ij}.

Quantum mechanics only studies operators which preserve the norm, and in two dimensions this is written as:

$$\alpha\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 + iv_2 \\ v_3 + iv_4 \end{pmatrix} = \begin{pmatrix} v_1' + iv_2' \\ v_3' + iv_4' \end{pmatrix} \quad (7)$$

with

$$v_1'^2 + v_2'^2 + v_3'^2 + v_4'^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2 \quad (8)$$

The solution is:

$$|\alpha|^2\left(a_{11}^2 + a_{21}^2\right) = 1 \quad (9)$$

$$|\alpha|^2\left(a_{12}^2 + a_{212}^2\right) = 1 \quad (10)$$

$$a_{11}a_{12} + a_{21}a_{22} = 0 \quad (11)$$

which corresponds to

$$(\alpha M)^\dagger(\alpha M) = [1] \quad (12)$$

This result can be easily generalized to quantum vector spaces of any dimension. We see that αM is not a unitary matrix because it fulfills only one of the two conditions of a unitary matrix $\left(U^\dagger U = UU^\dagger = [1]\right)$. This implies that quantum operators need not be unitary in order to preserve the norm in the vector space contrary to frequently held views.

The problem studied was to find the 12 real elements of the three following matrices:

$$J_k = \alpha_k\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \quad (13)$$

with the three α_k complex numbers for $k = x, y, z$. The subscripts have presently nothing to do with the space coordinates.

From Equation (1), it follows that the matrix J_z is:

$$J_z = \frac{\alpha_x\alpha_y}{i\hbar}\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \quad (14)$$

with

$$z_{11} = x_{12}y_{21} - x_{21}y_{12}$$

$$z_{12} = y_{12}\left(x_{11} - x_{22}\right) + x_{12}\left(y_{22} - y_{11}\right)$$

$$z_{21} = y_{21}\left(x_{22} - x_{11}\right) + x_{21}\left(y_{11} - y_{22}\right)$$

$$z_{22} = x_{21}y_{12} - x_{12}y_{21}$$

which shows that $z_{22} = -z_{11}$. Mor generally it is well known that the commutator of two matrices results in a matrix with a null trace. As J_x and J_y are defined by a commutator, we also have $x_{22} = -x_{11}$ and $y_{22} = -y_{11}$.

From Equation (2) it follows that the matrix J_x is:

$$J_x = \frac{\alpha_y\alpha_z}{i\hbar}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad (15)$$

with

$$x_{11} = \frac{-2y_{12}y_{21}x_{11} + 2y_{11}\left(x_{21}y_{12} + x_{12}y_{21}\right)}{2y_{12}y_{21} - \hbar^2}$$

$$x_{12} = \frac{2y_{12}^2x_{21} + 4y_{12}y_{11}x_{11}}{4y_{11}^2 + 2y_{12}y_{21} - \hbar^2}$$

$$x_{21} = \frac{2y_{21}^2x_{12} + 4y_{21}y_{11}x_{11}}{4y_{11}^2 + 2y_{12}y_{21} - \hbar^2}$$

$$x_{22} = \frac{2y_{12}y_{21}x_{11} - 2y_{11}\left(x_{12}y_{21} + x_{21}y_{12}\right)}{2y_{12}y_{21} - \hbar^2}$$

From Equation (3) it follows that the matrix J_y has the same expression as J_x just by exchanging y and x. Calculating $x_{21}y_{12} + x_{12}y_{21}$ by eliminating x_{11} gives a relation between the J_y elements:

$$4y_{11}^2\left(2y_{12}y_{21} - \hbar^2\right) - 4y_{12}y_{21}\hbar^2 + \hbar^4 = 0 \quad (16)$$

As J_y is a quantum operator we have the general constraints given by Equations (9)-(11) on the J_y elements:

$$y_{11}^2 + y_{21}^2 = \frac{1}{|\alpha_x|^2} = K \quad (17)$$

$$y_{12}^2 + y_{11}^2 = \frac{1}{|\alpha_x|^2} = K \quad (18)$$

$$y_{11}\left(y_{12} - y_{21}\right) = 0 \quad (19)$$

where K is a positive real constant.

Equation (19) implies that $y_{11} = 0$ or $y_{12} - y_{21} = 0$.

For $y_{11} = 0$, Equations (17) and (18) imply that $y_{21}^2 = y_{12}^2 = K$, therefore $y_{21} = \pm y_{12}$.

For $y_{11} = 0$ and $y_{21} = -y_{12}$, Equation (16) gives $y_{12} = \pm i\hbar/2$ and we obtain the second Pauli matrix

$$J_{y1} = J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (20)$$

or its opposite $J_{y2} = -J_y$.

For $y_{11} = 0$ and $y_{21} = y_{12}$ Equation (16) gives $y_{12} = \pm i\hbar/2$ and we obtain the first Pauli matrix:

$$J_{y3} = J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (21)$$

or its opposite $J_{y4} = -J_{y3}$

The case $y_{11} = 0$ permits to calculate the constant $= \hbar^2/4$.

For $y_{11} \neq 0$ and $y_{21} = y_{12}$, Equation (16) gives $4y_{11}^2\left(2y_{12}^2 - \hbar^2\right) - 4y_{12}^2\hbar^2 + \hbar^4 = 0$ and using Equation (18) $y_{11}^2 + y_{12}^2 = \hbar^2/4$ we obtain $y_{12}^2 = 0$ and $y_{11} = \pm\hbar/2$ which gives the third Pauli matrix:

$$J_{y5} = J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

or its opposite $J_{y6} = -J_{y5}$. The 6 J_{yk} matrices are J_y, $-J_y$, J_x, $-J_x$, J_z, $-J_z$.

The norm preservation and the null trace constraints applied to J_x give the 6 different J_{xk} matrices J_x, $-J_x$, iJ_y, $-iJ_y$, J_z, $-J_z$ which fulfill the angular momentum criteria. Let us note that all are real matrices.

All the matrix triplets which fulfill the angular momentum definition in the two-dimensional complex vector space are obtained with Equation (14). For each J_{xk} there are 6 J_{yk} which gives 36 possibilities for J_z which fulfill the 3 Equations (1)-(3). Out of these 36 possibilities only 24 give a nonzero matrix. Obtaining the 24 solutions is straightforward and only two examples will be given explicitly. The first one is the well-known Pauli spin 1/2 matrices:

$$J_{1x} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J_{1y} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, J_{1z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{23}$$

and the second example is less conventional:

$$J_{13x} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, J_{13y} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J_{13z} = \frac{\hbar}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{24}$$

It can easily be verified that these 3 matrices fulfill the 3 angular momentum definitions given by Equations (1)-(3). The 24 solutions show that the elements of the spin 1/2 matrices can take only four values (−1, −i, i, +1) times $\hbar$/2 and the corresponding real matrices:

$$J_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J_Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, J_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{25}$$

Let us note that:

$$J_X^2 = [1],\ J_Y^2 = [1],\ J_Z^2 = [1] \tag{26}$$

and

$$J_X^\dagger J_x = [1],\ J_Y^\dagger J_Y = [1],\ J_Z^\dagger J_Z = [1] \tag{27}$$

which proves that all the solutions correspond to an operator which preserves the norm of the vectors in the two-dimensional space.

The 24 solutions of the general definition of the angular momenta for spin 1/2 are given in **Table 1**.

The general treatment of angular momenta uses the operator $J^2 = J_x^2 + J_y^2 + J_z^2$. If we calculate this operator for the 24 solutions we find $J^2 = \frac{3}{4}\hbar^2 I_1$ in 16 cases corresponding to the first two and last two rows of **Table 1**. This result is the well-known value obtained with Pauli matrices. In the other 8 cases (rows 3 and 4 of **Table 1**) we obtain $J^2 = -\frac{1}{4}\hbar^2 I_1$.

Table 1. The 24 solutions for spin 1/2 using matrices J_X, J_Y, J_Z, in $\hbar$/2 units.

$J_1(J_X,iJ_Y,J_Z)$	$J_2(J_X,-iJ_Y,-J_Z)$	$J_3(J_X,J_Z,-iJ_Y)$	$J_4(J_X,-J_Z,iJ_Y)$
$J_5(-J_X,iJ_Y,-J_Z)$	$J_6(-J_X,-iJ_Y,J_Z)$	$J_7(-J_X,J_Z,-iJ_Y)$	$J_8(-J_X,-J_Z,iJ_Y)$
$J_9(J_Y,J_X,iJ_Z)$	$J_{10}(J_Y,-J_X,-iJ_Z)$	$J_{11}(J_Y,J_Z,-iJ_X)$	$J_{12}(J_Y,-J_Z,iJ_X)$
$J_{13}(-J_Y,J_X,-iJ_Z)$	$J_{14}(-J_Y,-J_X,-iJ_Z)$	$J_{15}(-J_Y,J_Z,iJ_X)$	$J_{16}(-J_Y,-J_Z,-iJ_X)$
$J_{17}(J_Z,iJ_Y,-J_X)$	$J_{18}(J_Z,-iJ_Y,J_X)$	$J_{19}(J_Z,J_X,iJ_Y)$	$J_{20}(J_Z,-J_X,-iJ_Y)$
$J_{21}(-J_Z,iJ_Y,J_X)$	$J_{22}(-J_Z,-iJ_Y,-J_X)$	$J_{23}(-J_Z,J_X,-iJ_Y)$	$J_{24}(-J_Z,-J_X,iJ_Y)$

We can associate the spin 1/2 with components J_{kx}, J_{ky} and J_{kz} to the space coordinates in the laboratory frame (O,u_x,u_y,u_z) to define the spin vector of a particle:

$$\boldsymbol{J}_k = J_{kx}\boldsymbol{u}_x + J_{ky}\boldsymbol{u}_y + J_{kz}\boldsymbol{u}_z \tag{28}$$

2.2. Dirac Equation

It is well known [2] that the spin of a particle can be retrieved using an equation which satisfies the special theory of relativity and quantum mechanics postulates, such as the Dirac equation. In Dirac's book [3], the Pauli matrices are obtained from the commutation definition with the additional constraint:

$$J_x^2 = J_y^2 = J_z^2 = \frac{\hbar^2}{4} \tag{29}$$

Owing to the fact that each observable has only two eigenvalues $\hbar$/2 and −$\hbar$/2. He obtained

$$\sigma_x = \begin{pmatrix} 0 & \mathrm{e}^{ia} \\ \mathrm{e}^{-ia} & 0 \end{pmatrix} \tag{30}$$

and stated that the phase α could be adequately chosen so as to obtain the three Pauli matrices. It is clear that Dirac did not consider non observable spins and he only took the 3 Pauli matrices as components of the spin vector.

In his relativistic theory of the electron, he found that in order to obtain a linear wave equation it was necessary to introduce four 4 × 4 matrices solving the following equations:

$$\begin{aligned} \alpha_x^2 &= 1, & \alpha_x\alpha_y + \alpha_y\alpha_x &= 0 \\ \beta^2 &= m^2c^2, & \alpha_x\beta + \beta\alpha_x &= 0 \end{aligned} \tag{31}$$

and the same relations obtained by permuting x,y and z.

A solution given by Dirac to these four matrices was:

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix},\quad \beta = \begin{pmatrix} I_1 & \sigma_k \\ \sigma_k & -I_1 \end{pmatrix} \tag{32}$$

with k = x, y, z, which nowadays is referenced as the standard form. But it is well known that any set of matrices obtained with:

$$\alpha' = U\alpha U^{\dagger} \quad \text{and} \quad \beta' = U\beta U^{\dagger} \tag{33}$$

where U is any unitary matrix $\left(U^{\dagger}U = UU^{\dagger} = 1\right)$, is also a solution to Equations (31). It can be shown that for a homogeneous magnetic field, the energy of the interaction of the intrinsic momentum with the magnetic field is

$$E_s = \frac{\hbar q}{2m_0}\sigma B \tag{34}$$

and the spin definition can also be retrieved as

$$\boldsymbol{S} = \frac{1}{2}\hbar\boldsymbol{\sigma} \tag{35}$$

where $\boldsymbol{\sigma}$ is the spin vector defined by the 3 Pauli matrices as in the non relativistic case. Dirac's equation has to be discussed more precisely only when time evolution is introduced.

An important question is why does the general treatment of quantum angular momentum give only one solution? This comes from the definition of $J^2 = J_x^2 + J_y^2 + J_z^2$ which is a dot product and not the matrix product of two angular momentum components. This definition introduces some constraints which permit to eliminate the solutions where $J^2 < 0$, but by taking only the positive values of j, also eliminates the negative components in the combination of the Pauli matrices. Therefore only the solution $J_1\left(\sigma_x, \sigma_y, \sigma_z\right)$ is obtained.

3. Angular Momenta for 3-Dimensional Quantum System

If we consider the extension to three dimensions, a vector has now 3 components and any quantum operator is described by a real 3 × 3 matrix multiplied by a complex number α'. We can choose a basis for the vector space where one angular momentum is diagonal. If we take the z direction for this, the angular momentum is:

$$J_z = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{36}$$

where the λ_k are three complex numbers.

As the angular momenta are defined by Equations (1)-(3) which correspond to non commuting matrices the trace of the 3 angular momentum matrices J_x, J_y, J_z is null. If we suppose that in the same basis and for the most general case $J_y = \left[\beta_{ij}\right]$ where β_{ij} are complex numbers, we obtain according to Equation (2):

$$J_x = \frac{1}{i\hbar}\begin{pmatrix} 0 & \beta_{12}\left(\lambda_2 - \lambda_1\right) & \beta_{13}\left(\lambda_3 - \lambda_1\right) \\ -\beta_{21}\left(\lambda_2 - \lambda_1\right) & 0 & \beta_{23}\left(\lambda_3 - \lambda_2\right) \\ -\beta_{31}\left(\lambda_3 - \lambda_1\right) & -\beta_{32}\left(\lambda_3 - \lambda_2\right) & 0 \end{pmatrix} \tag{37}$$

and Equation (3) gives:

$$J_y = \frac{-1}{\hbar^2} \cdot \begin{pmatrix} 0 & -\beta_{12}\left(\lambda_2 - \lambda_1\right)^2 & -\beta_{13}\left(\lambda_3 - \lambda_1\right)^2 \\ -\beta_{21}\left(\lambda_2 - \lambda_1\right)^2 & 0 & -\beta_{23}\left(\lambda_3 - \lambda_2\right)^2 \\ -\beta_{31}\left(\lambda_3 - \lambda_1\right)^2 & -\beta_{32}\left(\lambda_3 - \lambda_2\right)^2 & 0 \end{pmatrix} \tag{38}$$

and Equation (1):

$$J_z = \frac{1}{\hbar^4}\left[\alpha_{ij}\right] \tag{39}$$

$$\begin{aligned}
\alpha_{11} &= 2\left(\beta_{12}\beta_{21}\left(\lambda_1 - \lambda_2\right)^3 + \beta_{13}\beta_{31}\left(\lambda_1 - \lambda_3\right)^3\right) \\
\alpha_{22} &= 2\left(\beta_{12}\beta_{21}\left(\lambda_2 - \lambda_1\right)^3 + \beta_{23}\beta_{32}\left(\lambda_2 - \lambda_3\right)^3\right) \\
\alpha_{33} &= 2\left(\beta_{13}\beta_{31}\left(\lambda_3 - \lambda_1\right)^3 + \beta_{23}\beta_{32}\left(\lambda_3 - \lambda_2\right)^3\right) \\
\alpha_{12} &= \beta_{13}\beta_{32}\left(\lambda_3 - \lambda_1\right)\left(\lambda_2 - \lambda_3\right)\left(\lambda_1 + \lambda_2 - 2\lambda_3\right) \\
\alpha_{13} &= \beta_{12}\beta_{23}\left(\lambda_2 - \lambda_1\right)\left(\lambda_3 - \lambda_2\right)\left(\lambda_2 + \lambda_3 - 2\lambda_2\right) \\
\alpha_{23} &= \beta_{21}\beta_{13}\left(\lambda_1 - \lambda_2\right)\left(\lambda_3 - \lambda_1\right)\left(\lambda_2 + \lambda_3 - 2\lambda_1\right) \\
\alpha_{21} &= \beta_{23}\beta_{31}\left(\lambda_3 - \lambda_2\right)\left(\lambda_1 - \lambda_3\right)\left(\lambda_1 + \lambda_2 - 2\lambda_3\right) \\
\alpha_{31} &= \beta_{21}\beta_{32}\left(\lambda_1 - \lambda_2\right)\left(\lambda_2 - \lambda_{31}\right)\left(\lambda_1 + \lambda_3 - 2\lambda_2\right) \\
\alpha_{32} &= \beta_{31}\beta_{12}\left(\lambda_1 - \lambda_3\right)\left(\lambda_2 - \lambda_1\right)\left(\lambda_2 + \lambda_3 - 2\lambda_1\right)
\end{aligned} \tag{40}$$

To find the angular momenta, Equation (39) must be equal to Equation (36). There are several solutions obtained from the non diagonal J_z matrix elements equal to zero.

If $\beta_{13} = \beta_{31} = \lambda_2 = 0$, then $\lambda_1 = \pm\hbar$. For $\lambda_1 = \hbar$ we obtain a set of three matrix solutions to the spin equation:

$$J_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, J_x = \begin{pmatrix} 0 & \mathrm{i}\beta_{12} & 0 \\ \frac{-\mathrm{i}\hbar^2}{2\beta_{12}} & 0 & \mathrm{i}\beta_{23} \\ 0 & \frac{-\mathrm{i}\hbar^2}{2\beta_{23}} & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & \beta_{12} & 0 \\ \frac{\hbar^2}{2\beta_{12}} & 0 & \beta_{23} \\ 0 & \frac{\hbar^2}{2\beta_{23}} & 0 \end{pmatrix} \tag{41}$$

For $\lambda_1 = -\hbar$ the sign of J_z and J_x is changed but J_y is unchanged. If $\beta_{12} = \beta_{23} = -i\hbar/\sqrt{2}$ we retrieve the conventional matrices for a spin 1:

$$J_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}, J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix} \tag{42}$$

Another solution is obtained with $\beta_{23} = \beta_{32} = \lambda_1 = 0$, then $\lambda_2 = \pm\hbar$ and for $\lambda_2 = \hbar$ the angular momenta are:

$$J_z = \hbar\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, J_x = \begin{pmatrix}0 & -i\beta_{12} & -i\beta_{13}\\ \frac{i\hbar^2}{2\beta_{12}} & 0 & 0\\ \frac{-i\hbar^2}{2\beta_{13}} & 0 & 0\end{pmatrix}, \quad J_y = \begin{pmatrix}1 & \beta_{12} & \beta_{13}\\ \frac{\hbar^2}{2\beta_{12}} & 0 & 0\\ \frac{\hbar^2}{2\beta_{13}} & 0 & 0\end{pmatrix} \tag{43}$$

The third set of solutions is obtained with $\beta_{12} = \beta_{21} = \lambda_3 = 0$, then $\lambda_1 = \pm\hbar$ and for $\lambda_1 = \hbar$ the angular momenta are:

$$J_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, J_x = \begin{pmatrix}0 & 0 & i\beta_{13}\\ 0 & 0 & -i\beta_{23}\\ \frac{-i\hbar^2}{2\beta_{13}} & \frac{i\hbar^2}{2\beta_{23}} & 0\end{pmatrix}, \quad J_y = \begin{pmatrix}0 & 0 & \beta_{13}\\ 0 & 0 & \beta_{23}\\ \frac{\hbar^2}{2\beta_{13}} & \frac{\hbar^2}{2\beta_{23}} & 0\end{pmatrix} \tag{44}$$

The choice of the basis of the 3-dimensional vector space can also be made so as to have a diagonal angular operator in the *x* or *y* directions, then the equivalent sets of solutions will be obtained by permuting the *x*, *y* and *z* subscripts. The important result of this derivation is that, contrary to the 2-dimensional case, there is no solution where the derived angular momentum operator preserves the norm of the vectors in the 3-dimensional vector space. Indeed, $J_k^\dagger J_k$ is never equal to 1.

Therefore, the spin of any boson in particle physics requires a description other than the matrix representation of the angular momentum.

An interesting way to define the spin is to use the framework of a Clifford algebra defined in the quantum vectorial space.

4. Clifford Algebra and Spins

In his paper, in order to define the spin, Fröhlich [1] used a Clifford algebra involving $2^k \times 2^k$ matrices over complex numbers, which has to be compared to the geometric algebra of spacetime initiated by Hestenes [4-6] and also developed by the Cambridge astrophysical group [7-9].

Grassmann and Clifford's geometric algebra is based on the definition of the *geometric product uv* for vectors *u*, *v*, *w* obeying the following rules:

$$(uv)w = u(vw) \tag{45}$$

$$u(v+w) = uv + uw \tag{46}$$

$$(v+w)u = vu + wu \tag{47}$$

$$v^2 = \varepsilon_v |v|^2 \tag{48}$$

where ε_v is the *signature* of *v* and the *magnitude* $|v|$ is a real positive scalar. The geometric product can be decomposed into a symmetric *inner product*:

$$u \bullet v = v \bullet u \tag{49}$$

and an antisymmetric *outer product*:

$$u \wedge v = -v \wedge u \tag{50}$$

such that

$$uv = u \bullet v + u \wedge v. \tag{51}$$

Orthonormal vectors are defined by:

$$\begin{matrix} u_1 \bullet u_1 = 1 & u_1 \wedge u_1 = 0 \\ u_2 \bullet u_2 = 1 & u_2 \wedge u_2 = 0 \end{matrix} \tag{52}$$

and

$$u_1 \bullet u_2 = 0 \tag{53}$$

The signature of orthonormal vectors u_k are $u_k^2 = \pm 1$. The fact that real numbers can give negative squares is a little surprising but it is well illustrated by matrix J_Y of Equation (25).

4.1. Basis of Geometric Algebra in Two and Three Dimensions

In two dimensions we have the geometric algebra of the plane with four elements:

$$\begin{matrix} 1, & u_1, & u_2, & u_1u_2 = i. \\ \text{scalar,} & \text{vector,} & \text{vector,} & \text{bivector.} \end{matrix} \tag{54}$$

The bivector u_1u_2 has a property such that its square $(u_1u_2)^2 = u_1u_2u_1u_2 = -u_1^2u_{12}^2$. If the signatures $u_1^2 = u_2^2 = 1$

or $u_1^2 = u_2^2 = -1, (u_1u_2)^2 = -1$. It can be shown that $u_1u_2 = i$ forms a natural subalgebra equivalent to complex numbers $z = x + yu_1u_2$. The bivector i is called a pseudoscalar which anticommutes with vectors u_1 and u_2.

An arbitrary linear sum over the four basis elements is called *a* itmultivector:

$$A \equiv a_0 + a_1u_1 + a_2u_2 + a_3i \tag{55}$$

with components $a_i \in \Re$.

The sum of two multivectors is obtained by adding each component and the geometric product is obtained from the multiplication table (**Table 2**).

We can notice that the same algebraic properties are obtained by using the {1, u_1, u_2, $\tilde{i} = u_2u_1$} basis vectors (where ~ indicates a reversion of vector products). Again $\tilde{i}^2 = -u_1^2u_2^2$ is equal to −1 if the signatures are $u_1^2 = u_2^2 = 1$ or $u_1^2 = u_2^2 = -1$.

The product of two multivectors AB is explicitly given by:

$$AB = \left(a_0b_0 + a_1b_1u_1^2 + a_2b_2u_2^2 - a_3b_3\right)1 \tag{56}$$

$$\left(a_0b_1 + a_1b_0 - a_2b_3u_2^2 + a_3b_2u_2^2\right)u_1$$

$$\left(a_0b_2 + a_1b_3u_1^2 + a_2b_0 - a_3b_1u_1^2\right)u_2$$

$$\left(a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0\right)i$$

and the square of a multivector is

$$\begin{aligned} A^2 &= \left(a_0^2 + a_1^2u_1^2 + a_2^2u_2^2 - a_3^2\right)1 \\ &\quad + 2\left(a_0a_1u_1 + a_0a_2u_2 - a_0a_3i\right) \end{aligned} \tag{57}$$

Unit multivectors are characterized by:

$$\begin{aligned} A^2 = 1 \Leftrightarrow &\left(a_0 = \pm 1, a_1 = a_2 = a_3 = 0\right) \forall u_1^2, u_2^2 \\ &\text{or } \left(a_0 = 0, a_1^2u_1^2 + a_2^2u_2^2 - a_3^2 = 1\right) \end{aligned} \tag{58}$$

Rotation in geometric algebra is based on a theorem by Hamilton: given any unit vector n (n^2 = 1), we can resolve an arbitrary vector x into parts parallel and perpendicular to n: $x = x_\perp + x_\parallel$. These components are identified algebraically through their commutation properties:

$$nx_\parallel = x_\parallel n \tag{59}$$

Table 2. Multiplication table for geometric algebra in two dimensions.

1	u_1	u_2	i
u_1	$u_1^2 1$	i	$u_1^2 u_2$
u_2	$-i$	$u_2^2 1$	$-u_2^2 u_1$
i	$-u_1^2 u_2$	$u_2^2 u_1$	-1

$$nx_\perp = -x_\perp n \tag{60}$$

The vector $x_\parallel - x_\perp$ can therefore be written $-nxn$. Geometrically, the transformation $x \to -nxn$ represents a *reflection* in the plane x, n. To make a rotation we need two such reflections:

$$x \to mn\ x\ nm \tag{61}$$

with $mn\ nm$ = 1 [7]. The multivector $R \equiv mn$ is called a *rotor* and $\tilde{R} = nm$ is called the "reverse" of R. R satisfying $R\tilde{R} = \tilde{R}R = 1$ is defined as "unimodular".

The bilinear transformation of vectors is a very general way of handling rotations which can easily be generalized to vector spaces of any dimension [7].

In three dimensions we have the geometric algebra of space with 8 elements:

$$\begin{matrix} 1, & \{u_1, u_2, u_3\}, & \{u_1u_2, u_3u_1, u_2u_3\}, & u_1u_2u_3 = i \\ \text{1 scalar,} & \text{3 vectors,} & \text{3 bivectors,} & \text{1 trivector} \end{matrix} \tag{62}$$

Again the pseudoscalar $i = u_1u_2u_3$ is defined by its property such that $i^2 = u_1u_2u_3u_1u_2u_3 = -1$, therefore the signature of the orthonormal vectors must be one of the four possibilities: $\left(u_1^2 = u_2^2 = u_3^2 = 1\right)$ or

$\left(u_1^2 = u_2^2 = -1 \text{ and } u_3^2 = 1\right)$ or

$\left(u_1^2 = u_3^2 = -1 \text{ and } u_2^2 = 1\right)$ or

$\left(u_2^2 = u_3^2 = -1 \text{ and } u_1^2 = 1\right)$.

Contrary to the 2-dimensional case, varying the order of bivectors does not give the same algebraic properties. By choosing the bivectors:

$$\left\{\rho_1 = u_1u_2, \quad \rho_2 = u_3u_1, \quad \rho_3 = u_2u_3\right\} \tag{63}$$

for the basis of the 3-dimensional vector space, we have the relation defining the quaternion algebra:

$$\rho_1^2 = \rho_2^2 = \rho_3^2 = \rho_1\rho_2\rho_3 = -1 \tag{64}$$

which is only valid with the signatures:

$$u_1^2 = u_2^2 = u_3^2 = 1 \tag{65}$$

The important property given by Equation (64) is that the trivector i commutes not only with vectors but also with bivectors.

The multiplication table for 3-dimensional geometric algebra with the signatures, in order to have the general case, is given in **Table 3**.

The product of two multivectors is given explicitly by:

$$\begin{aligned} R = AB = R_0 1 + R_1u_1 + R_2u_2 + R_3u_3 + R_4\rho_1 \\ + R_5\rho_2 + R_6\rho_3 + R_7i \end{aligned} \tag{66}$$

with

Table 3. Multiplication table for geometric algebra in three dimensions.

1	u_1	u_2	u_3	u_1u_2	u_3u_1	u_2u_3	i
u_1	$u_1^2 1$	u_1u_2	$-u_3u_1$	$u_1^2u_2$	$-u_1^2u_3$	i	$u_1^2u_2u_3$
u_2	$-u_1u_2$	$u_2^2 1$	u_2u_3	$-u_2^2u_1$	i	$u_2^2u_3$	$u_2^2u_3u_1$
u_3	u_3u_1	$-u_2u_3$	$u_3^2 1$	i	$u_3^2u_1$	$-u_3^2u_2$	$u_3^2u_1u_2$
u_1u_2	$-u_1^2u_2$	$u_2^2u_1$	i	$-u_1^2u_2^2 1$	$u_1^2u_2u_3$	$-u_2^2u_3u_1$	$-u_1^2u_2^2u_3$
u_3u_1	$u_1^2u_3$	i	$-u_3^2u_1$	$-u_1^2u_2u_3$	$-u_1^2u_3^2 1$	$u_3^2u_1u_2$	$-u_1^2u_3^2u_2$
u_2u_3	i	$-u_2^2u_3$	$u_3^2u_2$	$u_2^2u_3u_1$	$-u_3^2u_1u_2$	$-u_2^2u_3^2 1$	$-u_2^2u_3^2u_1$
i	$u_1^2u_2u_3$	$u_2^2u_3u_1$	$u_3^2u_1u_2$	$-u_1^2u_2^2u_3$	$-u_1^2u_3^2u_2$	$-u_2^2u_3^2u_1$	-1

$$
\begin{aligned}
R_0 &= a_0b_0 + a_1b_1u_1^2 + a_2b_2u_2^2 + a_3b_3u_3^2 \\
&\quad - a_4b_4u_1^2u_2^2 - a_5b_5u_1^2u_3^2 - a_6b_6u_2^2u_3^2 - a_7b_7 \\
R_1 &= a_0b_1 + a_1b_0 - a_2b_4u_2^2 + a_3b_5u_3^2 \\
&\quad + a_4b_2u_2^2 - a_5b_3u_3^2 - a_6b_7u_2^2u_3^2 - a_7b_6u_2^2u_3^2 \\
R_2 &= a_0b_2 + a_1b_4u_1^2 + a_2b_0u_2^2 - a_3b_6u_3^2 \\
&\quad - a_4b_1u_1^2 - a_5b_7u_1^2u_3^2 + a_6b_3u_3^2 - a_7b_5u_1^2u_3^2 \\
R_3 &= a_0b_3 - a_1b_5u_1^2 + a_2b_6u_2^2 + a_3b_0 \\
&\quad - a_4b_7u_1^2u_2^2 + a_5b_1u_1^2 - a_6b_2u_2^2 - a_7b_4u_1^2u_2^2 \\
R_4 &= a_0b_4 - a_1b_2 - a_2b_1u_2^2 + a_3b_7u_3^2 \\
&\quad + a_4b_0 + a_5b_6u_3^2 - a_6b_5u_3^2 + a_7b_3u_3^2 \\
R_5 &= a_0b_5 - a_1b_3 + a_2b_7u_2^2 + a_3b_1 \\
&\quad - a_4b_6u_2^2 - a_5b_0 + a_6b_4u_2^2 + a_7b_2u_2^2 \\
R_6 &= a_0b_6 + a_1b_7u_1^2 + a_2b_3 - a_3b_2 \\
&\quad + a_4b_5u_1^2 - a_5b_4u_1^2 + a_6b_0 + a_7b_1u_1^2 \\
R_7 &= a_0b_7 + a_1b_6 + a_2b_5 + a_3b_4 \\
&\quad + a_4b_3 + a_5b_2 + a_6b_1 + a_7b_0
\end{aligned}
$$

According to the previous mentioned Hamilton's theorem, any multivector A can be resolved in a commuting (perpendicular) part: C_a and anticommuting (parallel) part: J_a, therefore:

$$A = C_a + J_a \tag{67}$$

with

$$C_a = a_0 1 + a_7 i$$

$$J_a = a_1u_1 + a_2u_2 + a_3u_3 + a_4\sigma_1 + a_5\sigma_2 + a_6\sigma_3$$

because 1 and i commute with all the elements of a multivector.

If we look at the spin definition in natural units ($\hbar = 1$) given by the spin equations in a geometric algebra framework, J_xJ_y can be considered as anticommuting multivectors defining a rotation R_Z in the x, y plane.

To prove the validity of this unconventional definition of spin 1/2, we consider the first spin equation, $J_xJ_y - J_yJ_x = i\hbar J_z$ which can be easily calculated with Equation (66). It appears that the scalar and the pseudoscalar components of J_z are equal to zero. Since this relation applies to the other two spin equations, every J_x, J_y and J_z have no scalar and no pseudoscalar components. The first spin equation gives the following 6 equations:

$$
\begin{aligned}
2\left(x_3y_5u_3^2 - x_5y_3u_3^2 + x_4y_2u_2^2 - x_2y_4u_2^2\right) &= -z_6u_2^2u_3^2 \\
2\left(x_1y_4u_1^2 - x_4y_1u_1^2 + x_6y_3u_3^2 - x_3y_6u_3^2\right) &= -z_5u_1^2u_3^2 \\
2\left(x_2y_6u_2^2 - x_6y_2u_2^2 + x_5y_1u_1^2 - x_1y_5u_1^2\right) &= -z_4u_1^2u_2^2 \\
2\left(x_1y_2 - x_2y_1 + x_5y_6u_3^2 - x_6y_5u_3^2\right) &= z_3u_3^2 \\
2\left(x_3y_1 - x_1y_3 + x_6y_4u_2^2 - x_4y_6u_2^2\right) &= z_2u_2^2 \\
2\left(x_2y_3 - x_3y_2 + x_4y_5u_1^2 - x_5y_4u_1^2\right) &= z_1u_1^2
\end{aligned}
\tag{68}
$$

Calculating the components of the product of two multivectors $R_z = J_xJ_y$ allows to retrieve the 6 previous r_{zk} factors and two new factors r_{z0} and r_{z7} for the scalar and the pseudoscalar components of the product:

$$
\begin{aligned}
R_z &= r_{z0}u_1 + r_{z1}u_1 + r_{z2}u_2 + r_{z3}u_3 + r_{z4}\rho_1 \\
&\quad + r_{z5}\rho_2 + r_{z6}\rho_3 + r_{z7}i
\end{aligned}
\tag{69}
$$

with

$$
\begin{aligned}
r_{z0} &= x_1y_1u_1^2 + x_2y_2u_2^2 + x_3y_3u_3^2 - x_4y_4u_1^2u_2^2 \\
&\quad - x_5y_5u_1^2u_3^2 - x_6y_6u_2^2u_3^2 \\
r_{z7} &= x_1y_6 + x_2y_5 + x_3y_4 + x_4y_3 + x_5y_2 + x_6y_1
\end{aligned}
$$

therefore

$$
\begin{aligned}
R_z = J_xJ_y &= r_{z0}1 + \frac{1}{2}\left(J_xJ_y - J_yJ_x\right) + r_{z7}i \\
&= r_{z0}1 + r_{z7}i + \frac{1}{2}iJ_z
\end{aligned}
\tag{70}
$$

An analogous derivation gives:

$$
\begin{aligned}
\widetilde{R_z} = J_yJ_x &= r_{z0}1 + \frac{1}{2}\left(J_yJ_x - J_xJ_y\right) + r_{z7}i \\
&= r_{z0}1 + r_{z7}i - \frac{1}{2}iJ_z
\end{aligned}
\tag{71}
$$

Therefore

$$R_z\widetilde{R_z} = \widetilde{R_z}R_z = \left(r_{z0}1 + r_{z7}i\right)^2 - \frac{1}{4}i^2J_z^2 \tag{72}$$

where J_k^2 (with $k = x$, y, z) have only a scalar and a pseudoscalar component. The important consequence is that the J_k^2 commute.

Hamilton's theorem proves that there exists a unit vector which resolves any vector into a perpendicular and a parrallel part and u_3 is a good candidate to play this role for R_z because u_3 is perpendicular to any multivector J_x and any multivector J_y. The permutation of x,y,z shows that there is no incompatibility in doing so for J_yJ_z and J_zJ_x. We retrieve the relation (65):

$$u_1^2 = u_2^2 = u_3^2 = 1 \tag{73}$$

and find the 3 relations for $R_k\widetilde{R_k} = 1$:

$$\begin{aligned} R_z\widetilde{R_z} &= \left(r_{z0}1 + r_{z7}i\right)^2 + \frac{1}{4}J_z^2 = 1 \\ R_y\widetilde{R_y} &= \left(r_{y0}1 + r_{y7}i\right)^2 + \frac{1}{4}J_y^2 = 1 \\ R_x\widetilde{R_x} &= \left(r_{x0}1 + r_{x7}i\right)^2 + \frac{1}{4}J_x^2 = 1 \end{aligned} \tag{74}$$

Therefore a spin 1/2 ($S1/2$) can be defined in geometric algebra as a set of 3 rotors:

$$R_z = J_xJ_y,\ R_y = J_zJ_x,\ R_x = J_yJ_z \tag{75}$$

As R_k are unimodular vectors, they preserve the magnitude of multivectors. Equations (68) and (69) show that the solution for the component values of the set of the 3 rotors depends on the signatures of the orthonormal vectors.

In summary, in geometric algebra a spin 1/2 is a set of 3 rotations obtained from the product of two multivectors which are the linear summation of 3 orthonormal vectors u_1, u_2, u_3 and 3 bivectors ρ_1, ρ_2 , ρ_3 according to:

$$\begin{aligned} &J_k = k_1u_1 + k_2u_2 + k_3u_3 + k_4\rho_1 + k_5\rho_2 + k_6\rho_3 \\ &k = \{x, y, z\} \\ &\text{with } \rho_1^2 = \rho_2^2 = \rho_2^2 = \rho_1\rho_2\rho_3 = -1 \end{aligned} \tag{76}$$

which fulfill the spin equations:

$$\left[J_xJ_y\right] = u_1u_2u_3\hbar J_z \tag{77}$$

and the permutation of (x,y,z).

It can be noticed that the only solution if we try to solve the spin equations in the two-dimensional case, is $J_x = J_y = J_z = 0$. This result is not surprising because in two dimensions, rotations commute and $J_xJ_y - J_yJ_x = 0$.

4.2. Spacetime Algebra

According to Hestenes [5], the standard model for spacetime is a real 4D *Minkowski spacetime* with vector addition and scalar multiplication where we can impose the geometric product defined by equations 45 to 51 in order to generate a geometric algebra called *spacetime algebra* (STA).

A basis for STA can be generated by a frame $\{u_1, u_2, u_3, u_4\}$ of orthonormal vectors which determines the pseudoscalar:

$$i = u_1u_2u_3u_4 \tag{78}$$

In order for the pseudoscalar i to keep the same properties as in the two- and three-dimensional cases, i^2 must be equal to −1, which imposes two signature combinations which are called the metric of STA. One possibility to fulfill $i^2 = -1$ is that three signatures should be equal to 1 and the fourth equal to −1, the second possibility is that three signatures should be equal to −1 and the fourth equal to 1. The metrics are defined by $\{1,1,1,-1\}$ and $\{-1,-1,-1,1\}$. The 3 orthonormal vectors with identical signature will be associated with the space components x,y,z and the other with the time component t so that:

$$i^2 = u_x^2u_y^2u_z^2u_t^2 = -1 \tag{79}$$

By forming all distinct products of the four u_k we obtain a complete basis for STA consisting of $2^4 = 16$ linearly independent elements.

There are 12 bivectors obtained from the arrangement of 2 out of the 4 orthonormal vectors, but there are only 6 with different magnitudes:

$$\begin{aligned} u_xu_y &= -u_yu_x = \rho_1 \qquad & u_xu_t &= -u_tu_x = \sigma_1 \\ u_zu_x &= -u_xu_z = \rho_2 \qquad & u_yu_t &= -u_tu_y = \sigma_2 \\ u_yu_z &= -u_zu_y = \rho_3 \qquad & u_zu_t &= -u_tu_z = \sigma_3 \end{aligned} \tag{80}$$

As regards trivectors, there are 24 products to be obtained from the arrangement of 3 out of the 4 orthonormal vectors. There are 8 distinct products and only fourhaving different magnitudes:

$$\begin{aligned} u_xu_yu_z &= -u_xu_zu_y = \frac{iu_t}{u_t^2} = \xi_0 \\ u_yu_tu_z &= -u_yu_zu_t = \frac{iu_x}{u_x^2} = \xi_1 \\ u_xu_tu_z &= -u_xu_zu_t = -\frac{iu_y}{u_y^2} = \xi_2 \\ u_xu_tu_y &= -u_xu_yu_t = \frac{iu_z}{u_z^2} = \xi_3 \end{aligned} \tag{81}$$

We can verify that i anticommutes with vectors and trivectors but commutes with bivectors.

A complete basis for STA is:

$$\begin{aligned} &1\ \text{scalar}: 1 \\ &4\ \text{vectors}: \{u_1, u_2, u_3, u_4\} \\ &6\ \text{trivectors}: \{3\rho, 3\sigma\} \\ &4\ \text{pseudovectors}: \{\xi\} \\ &1\ \text{pseudoscalar}: i \end{aligned} \tag{82}$$

According to the 3-dimensional geometric algebra case we have the quaternion relation (64) for the ρ_κ bivectors. By adequately choosing the bivectors of the vector space basis, the 3 new bivectors σ_k fulfill the following quaternion relation:

$$(i\sigma_1)^2 = (i\sigma_2)^2 = (i\sigma_3)^2 = (i\sigma_1)(i\sigma_2)(i\sigma_3) = -1 \quad (83)$$

The four pseudovectors are: ξ_0 the pseudoscalar of dimension 3 and the three other ξ_k which, again, fulfill a quaternion relation between the geometric product of the time orthonormal vector and ξ_k:

$$\begin{aligned}(u_t\xi_1)^2 &= (u_t\xi_2)^2 = (u_t\xi_3)^2 \\ &= (u_t\xi_1)(u_t\xi_2)(u_t\xi_3) = -1\end{aligned} \quad (84)$$

The 3 quaternion relations are verified with the metric $\{1,1,1,-1\}$ and the choice of the basis defined by Equations (80) and (81).

For the other metric, Equation (84) is still valid, but to keep the quaternion relations (63) and (83) for bivectors, we have to consider another basis where $\widetilde{\rho_2} = -\rho_2 = u_x u_z$, which corresponds to reverse rotations in the x,z plane.

If we consider the geometric algebra definition of spin 1/2 given by Equations (76) to (77) we retrieve the spin 1/2 characterized by the 6-element multivector:

$$S_{x,y,z,1}\{u_x,u_y,u_z,\rho_1,\rho_2,\rho_{3z}\} \quad (85)$$

but in the 4-dimensional vector space there are two other spins 1/2:

$$\begin{aligned}&S_{x,y,z,2}\{u_x,u_y,u_z,\mathrm{i}\sigma_1,\mathrm{i}\sigma_2,\mathrm{i}\sigma_3\} \\ &= \{u_x,u_y,u_z,-u_z^2u_t^2\rho_1,u_y^2u_t^2\rho_2,-u_x^2u_t^2\rho_3\}\end{aligned} \quad (86)$$

$$\begin{aligned}&S_{x,y,z,3}\{u_x,u_y,u_z,u_t\xi_1,u_t\xi_2,u_t\xi_3\} \\ &= \{u_x,u_y,u_z,u_t^2\rho_1,-u_t^2\rho_2,u_t^2\rho_3\}\end{aligned} \quad (87)$$

for the metric $\{1,1,1,-1\}$. For the other metric with the other basis there are three new spins 1/2 obtained from the 3 vectors u_x, u_y and u_z:

$$S_{x,y,z,4}\{u_x,u_y,u_z,\rho_1,\widetilde{\rho_2},\rho_3\} \quad (88)$$

$$S_{x,y,z,5}\{u_x,u_y,u_z,-u_z^2u_t^2\rho_1,u_y^2u_t^2\widetilde{\rho_2},-u_x^2u_t^2\rho_3\} \quad (89)$$

$$S_{x,y,z,6}\{u_x,u_y,u_z,u_t^2\rho_1,-u_t^2\widetilde{\rho_2},u_t^2\rho_3\} \quad (90)$$

There are 18 other spins 1/2 obtained with the different vectors of the 4-dimensional case and the same bivectors:

$$S_{x,y,t,1..6}, S_{x,z,t,1..6}, S_{y,z,t,1..6} \quad (91)$$

Therefore the 24 spins 1/2 found with the matrix derivation are retrieved in the STA description but here with a clear distinction between spins 1/2 which fall in two metric-dependent categories. If the 12 spins 1/2 of each category were sorted as a function of bivectors we would find 3 families of 4 spins 1/2 as in elementary particle physics.

As the definition of rotation given by Equation (61) applies whatever the dimension of the vector space, in four dimensions the spin 1 can be defined by set of 3 rotors:

$$\begin{aligned}&R_{z4} = J_{x4}J_{y4},\quad R_{y4} = J_{z4}J_{y4},\quad R_{x4} = J_{y4}J_{z4} \\ &\text{with}\quad J_k = k_1u_1 + k_2u_2 + k_3u_3 + k_4u_4 + k_5\rho_1 \\ &+k_6\rho_2 + k_7\rho_3 + k_8\sigma_1 + k_9\sigma_2 + k_{10}\sigma_3 + k_{11}\xi_0 \\ &+k_{12}\xi_1 + k_{13}\xi_2 + k_{14}\xi_3 \qquad k = \{x,y,z\}\end{aligned} \quad (92)$$

where the J_k fulfill the spin equations. From the generalization to 4 dimensions of the vector product given by Equation (66), it is easy to see that the scalar component is null because real components commute and the pseudo scalar component is null because its parts are independent of the signatures. As the R_k are unimodular vectors, they induce rotations which preserve the norm of any vector. Again the solution to the equations which determine the components of the multivectors J_k is dependent on the signature of the orthonormal vectors. Therefore there are 4 different solutions with space-like signatures $u_x^2 = u_y^2 = u_z^2 = \pm 1$ and time-like signature $u_t^2 = \pm 1$ and 12 different solutions if we consider each signature independently.

5. Conclusion

We have shown that for the two-dimensional complex vector space, the spin matrices can be calculated directly from the angular momentum commutator definition. We have retrieved the 3 Pauli matrices and found 23 other triplet solutions. When extended to the three-dimensional space, we have shown that there is no matrix which preserves the norm of the vectors and fulfills the spin equations. By using a geometric algebra with a vector product which combines a commuting product and an anticommuting product it has been possible in four-dimensional spacetime to retrieve the 24 different spins 1/2 defined as 12 clockwise and 12 counter-clockwise rotations. These rotations are characterized by anticommuting parts composed of 3 vectors and 3 bivectors which fulfill the spin equations. Spin 1 can be defined as 3 rotations characterized by 4 vectors, 6 bivectors and 4 trivectors which fulfill the spin equations. These unimodular spin 1 rotations preserve the magnitude of multivectors. There are 12 different spins 1 depending on the signature of the 4 orthonormal vectors of the four-dimensional vector space. The correspondence between this derivation and particle physics is perhaps fortuitous but the use of STA offers the advantage of formulating conventional relativistic physics in invariant form without reference to a coordinate system [5] and it seems promising to analyze how time evolution and spin interaction can be used in order to predict the gyromagnetic ratio of the proton and the neutron.

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