Journal of Modern Physics, 2012, 3, 1849-1855
http://dx.doi.org/10.4236/jmp.2012.312232 Published Online December 2012 (http://www.SciRP.org/journal/jmp)

Solutions of Schrödinger Equation with Generalized Inverted Hyperbolic Potential

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Received May 10, 2012; revised October 3, 2012; accepted October 18, 2012

ABSTRACT

The bound state solutions of the Schrödinger equation with generalized inverted hyperbolic potential using the Nikiforov-Uvarov method are reported. We obtain the energy spectrum and the wave functions with this potential for arbitrary l-state. It is shown that the results of this potential reduced to the standard potentials—Rosen-Morse, Poschl-Teller and Scarf potential as special cases. We also discussed the energy equation and the wave function for these special cases.

Keywords: Schrodinger Equation; Inverted Hyperbolic Potential; Nikiforov-Uvarov Method

1. Introduction

The analytical and numerical solutions of the wave equations for both relativistic and non-relativistic cases have taken a great deal of interest recently. In many cases different attempts have been developed to solve the energy eigenvalues from the wave equations exactly or numerically for non-zero angular momentum quantum number $(l \neq 0)$ for a given potential [1-16]. It is well known that these solutions play an essential role in the relativistic and non-relativistic quantum mechanics for some physical potentials of interest, [1,2,12,17-19].

In this paper, we aim to solve the radial Schrödinger equation for quantum mechanical system with inverted generalized hyperbolic potential and show the results for this potential using Nikiforov-Uvarov method (NU), [20].

The present paper is an attempt to carry out the analytical solutions of the Schrodinger equation with the generalized inverted hyperbolic potential using the NU method.

The hyperbolic potentials under investigations are commonly used to model inter-atomic and intermolecular forces [10,21]. Among such potentials are Poschl-Teller, Rosen-Morse and Scarf potential, which have been studied extensively in the literatures, [5-8,22-25]. However, some of these hyperbolic potentials are exactly solvable or quasi-exactly solvable and their bound state solutions have been reported, [3,4,11,13,26-28]. We seek to present and study a generalized hyperbolic potential which other potentials can be deduced as special cases within the framework of Schrödinger equation with mass m and potential V.

The paper is organized as follows: Section 2 is devoted to the review of the Nikiforov-Uvarov method. In Section 3 we present the exact solution of the Schrodinger equation. Discussion and results are presented in Section 4. Finally we give a brief conclusion in Section 5.

2. Review of Nikiforov-Uvarov Method

The Nikiforov-Uvarov (NU) method, [20] was proposed and applied to reduce the second order differential equation to the hypergeometric—type equation by an appropriate co-ordinate transformation $S = S(r)$ as, [15,20].

$$\psi''(s)+\frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \tag{1}$$

where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials at most in the second order, and $\bar{\tau}$ is a first order polynomial. In order to find a particular solution of Equation (1) we use the separation of variables with the transformation

$$\psi(s)=\varphi(s)\chi(s) \tag{2}$$

It reduces Equation (1) to an equation as hypergeometric type

$$\sigma(s)\chi''(s)+\tau(s)\chi'(s)+\lambda\chi(s)=0 \tag{3}$$

and $\varphi(s)$ is defined as a logarithmic derivative in the following form and its solution can be obtained from

$$\frac{\varphi'(s)}{\varphi(s)}=\frac{\pi(s)}{\sigma(s)} \tag{4}$$

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The other part of the wave formation $\chi(s)$ is the hypergeometric type function whose polynomial solutions are given by Rodriques relations.

$$\chi_n(s)=\frac{B_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[\sigma^n(s)\rho(s)\right] \tag{5}$$

where B_n is a normalization constant, and the weight function $\rho(s)$ must satisfy the condition.

$$\frac{\mathrm{d}}{\mathrm{d}s}(\sigma\rho)=\tau(s)\rho(s) \tag{6}$$

with

$$\tau(s)=\bar{\tau}(s)+2\pi(s) \tag{7}$$

The function $\pi(s)$ and the parameter λ requires for the NU method are defined as follows:

$$\pi(s)=\frac{\sigma'-\bar{\tau}}{2}\pm\sqrt{\left(\frac{\sigma'-\bar{\tau}}{2}\right)^2-\bar{\sigma}+k\sigma} \tag{8}$$

$$\lambda=k+\pi'(s) \tag{9}$$

On the other hand in order to find the value of k, the expression under the square root of Equation (8) must be square of polynomial. Thus, a new eigenvalue for the second order equation becomes

$$\lambda=\lambda_n=-n\frac{\mathrm{d}\tau}{\mathrm{d}s}-\frac{n(n-1)\sigma''}{2} \tag{10}$$

where the derivative $\frac{\mathrm{d}\tau(s)}{\mathrm{d}s}$ is negative. By comparing Equations (9) and (10), we obtained the energy eigenvalues.

3. Bound State Solutions of the Schrödinger Equation

The Schrödinger equation with mass m and potential $V(r)$ takes the following form, [15]

$$\Psi''(r)+\frac{2m}{\hbar^2}\left[E-V(r)\right]\Psi(r)=0 \tag{11}$$

where the generalized hyperbolic potential $V(r)$ under investigation is defined as

$$\begin{aligned}V_{a,b,c,d}(r)=&-aV_0\coth(\alpha r)+bV_1\coth^2(\alpha r)\\&-cV_2\operatorname{cosech}^2(\alpha r)+d\end{aligned} \tag{12}$$

Here V_0, V_1 and V_2 are the depth of the potential and a, b, c and d are real numbers. The generalized hyperbolic potential $V(r)$ of Equation (12) has the following special cases:

(i) $V_{-a,0,c,0}(r)=aV_0\coth(\alpha r)-cV_2\operatorname{cosech}^2(\alpha r)$ (13)

(ii) $V_{0,0,c,0}(r)=cV_2\operatorname{cosech}^2(\alpha r)$ (14)

(iii) $V_{0,b,0,0}(r)=b\coth^2(\alpha r)$ (15)

The potentials (13)-(15) are the Rosen-Morse potential, Poschl-Teller potential and Scarf potential respectively.

We now perform the transformation [6,7,15]

$$\Psi(r)=\frac{R(r)}{r} \tag{16}$$

on Equation (1) and obtain

$$R''(r)+\frac{2m}{h^2}\left[E+aV_0\coth(\alpha r)-bV_1\coth^2(\alpha r)+cV_2\operatorname{cosech}^2(\alpha r)-d\right]R(r)=0 \tag{17}$$

where the prime indicates differentiation both respect to r.

Now using a new ansaltz for the wave function in the form [3,4,11]

$$R(r)=\mathrm{e}^{-\frac{\beta r}{2}}F(r) \tag{18}$$

and including the centrifugal term, reduces Equation (17) into the following differential equation,

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}-\beta\frac{\mathrm{d}F}{\mathrm{d}r}+\frac{2m}{h^2}\left[E+aV_0\coth(\alpha r)-bV_1\coth^2(\alpha r)+cV_2\operatorname{cosech}^2(\alpha r)-d-\frac{(l+1)}{r^2}+\left(\frac{\beta}{2}\right)^2\right]F(r)=0 \tag{19}$$

Because of the centrifugal term in Equation (19), this equation cannot be solved analytically when the angular momentum quantum number $\ell\neq 0$. Therefore, in order to find the approximate analytical solution of Equation (19) with $\ell\neq 0$, we must make an approximation for the centrifugal term. Thus, when $\alpha r\ll 1$ we use the approximation scheme [9,29] for the centrifugal term,

$$\frac{1}{r^2}\approx\alpha^2\operatorname{cosech}^2(\alpha r) \tag{20}$$

Substituting Equation (20) into Equation (19), we get

$$\begin{aligned}&\frac{\mathrm{d}^2F(r)}{\mathrm{d}r^2}-\frac{\beta\mathrm{d}F(r)}{\mathrm{d}r}\\&+\frac{2m}{h^2}\left[E+\left(\frac{\beta}{\alpha}\right)^2+aV_0\coth(\alpha r)-bV_1\coth^2(\alpha r)+cV_2\operatorname{cosech}^2(\alpha r)-\alpha^2l(l+1)\operatorname{cosech}^2(\alpha r)-d\right]F(r)=0\end{aligned} \tag{21}$$

Now making the change of variable

$$s = \coth(\alpha r) \tag{22}$$

we obtain

$$\begin{aligned}&\left(1+s^2\right)^2 \frac{\mathrm{d}^2F}{\mathrm{d}s^2} + \left(1+s^2\right)(\beta+2s)\frac{\mathrm{d}F}{\mathrm{d}s} \\ &+ \frac{2m}{\alpha^2\hbar^2}\left[E + \left(\frac{\beta}{2}\right)^2 + aV_0 s - bV_1 s^2 + cV_2\left(1+s^2\right) - \alpha^2 l(l+1)\left(1+s^2\right) - d\right]F(s) = 0\end{aligned} \tag{23}$$

Simplifying Equation (23), we have

$$\frac{\mathrm{d}^2F}{\mathrm{d}s^2} + \frac{\beta+2s}{\left(1+s^2\right)}\frac{\mathrm{d}F}{\mathrm{d}s} + \frac{1}{\left(1+s^2\right)^2}\left[-\varepsilon^2 + \beta^2 s + \gamma^2 s^2\right]F(s) = 0 \tag{24}$$

where the following dimensionless parameters have been employed:

$$\begin{aligned}\varepsilon^2 &= -\frac{2m}{\alpha^2\hbar^2}\left[E + \left(\frac{\beta}{2}\right) + cV_2 - \alpha^2\ell\left(\ell+1+d\right)\right], \\ \beta^2 &= \frac{2maV_0}{\hbar^2\alpha^2} \\ \gamma^2 &= \frac{2m}{\hbar^2\alpha^2}\left(cV_2 - bV_1 - \alpha^2\ell(\ell+1)\right)\end{aligned} \tag{25}$$

Comparing Equations (1) and (24), we obtain the following polynomials,

$$\begin{aligned}&\bar{\tau}(s) = (\beta+2s), \sigma(s) = \left(1+s^2\right), \\ &\bar{\sigma}(s) = -\varepsilon^2 + \beta^2 s + \gamma^2 s^2.\end{aligned} \tag{26}$$

Substituting these polynomials into Equation (8), we obtain the $\pi(s)$ function as

$$\begin{aligned}\pi(s) &= -\frac{\beta}{2} \\ &\pm\frac{1}{2}\sqrt{\left(4k-4\gamma^2\right)s^2 - 4\beta^2 s + \beta^2 + 4\varepsilon^2 + 4k}\end{aligned} \tag{27}$$

The expression in the square root of Equation (27) must be square of polynomial in respect of the NU method. Therefore, we determine the $\pi(s)$-values as

$$\begin{aligned}&\sqrt{(u+v)s} + \sqrt{(u-v)} \\ &\pi(s) = -\frac{\beta}{2} \pm \frac{1}{2}\begin{cases}\sqrt{(u+v)}s + \sqrt{(u-v)}, \text{for } k = \gamma^2 - \varepsilon^2 - \left(\frac{\beta}{2}\right)^2 + \sqrt{u^2-v^2} \\ \sqrt{(u+v)}s - \sqrt{(u-v)}, \text{for } k = \gamma^2 - \varepsilon^2 - \left(\frac{\beta}{2}\right)^2 - \sqrt{u^2-v^2}\end{cases}\end{aligned} \tag{28}$$

where $u = \left(\varepsilon^2\sqrt{1+\frac{\beta^2}{2\varepsilon^2}} + \gamma^2\right)$ and $V = i\beta\sqrt{\gamma^2 + \frac{5}{2}\beta^2}$.

For the polynomial of $\tau = \bar{\tau} + 2\pi$ which has a negative derivative, we get

$$k = \gamma^2 - \varepsilon^2 - \left(\frac{\beta}{2}\right)^2 + \sqrt{u^2-v^2} \tag{29}$$

$$\pi(s) = -\frac{\beta}{2} - \frac{1}{2}\left[\sqrt{(u+v)}s - \sqrt{u-v}\right] \tag{30}$$

Now using $\lambda = k + \pi'(s)$, we obtain $\tau(s)$ and λ valued as

$$\tau(s) = 2s - \sqrt{(u+v)}s + \sqrt{u-v} \tag{31}$$

$$\lambda(s) = \gamma^2 - \varepsilon^2 - \left(\frac{\beta}{2}\right)^2 - \sqrt{u^2-v^2} - \frac{1}{2}\sqrt{u+v} \tag{32}$$

Another definition of λ_n is as given in Equation (10), thus using values of $\tau(s)$ and $\sigma(s)$, we get,

$$\lambda = \lambda_n = u\sqrt{u+v} - u(u+1) \tag{33}$$

Comparing Equations (32) and (33), we obtain the energy eigenvalue equation as

$$\begin{aligned}&\left[\frac{(n+1)}{8\sqrt{2}\beta\gamma}\right] + i\left(\frac{\gamma}{8\sqrt{2}\beta} - \frac{1}{2v}\right)\varepsilon^4 - \left[1 + \frac{i\beta^2}{4}\left(1+\frac{1}{v}\right)\right]\varepsilon^2 \\ &- \left[\Sigma - \frac{(n+1)}{2}\sqrt{v} + iv + \frac{\beta\gamma}{2\sqrt{2}}\left((n+1) + i\gamma^2\right) - \frac{i\gamma^4}{2}\right] = 0\end{aligned} \tag{34}$$

where $\Sigma = \left(\frac{\beta}{2}\right)^2 - \gamma^2 - n(n+1)$.

Solving the energy eiginvalue equation explicitly, we obtain the energy eiginvalues as

$$\varepsilon^2 = \frac{\left[1+\frac{i\beta^2}{4}\left(1+\frac{1}{v}\right)\right]}{2\left[\frac{(n+1)}{8\sqrt{2}\beta r}+i\left(\frac{\gamma}{8\sqrt{2}\beta}-\frac{1}{2v}\right)\right]} \pm \frac{1}{2\left[\frac{(n+1)}{8\sqrt{2}\beta\gamma}+i\left(\frac{\gamma}{8\sqrt{2}\beta}-\frac{1}{2v}\right)\right]}\sqrt{\left(1+\frac{i\beta^2}{4}\left(1+\frac{1}{v}\right)\right)^2+4\left[\frac{(n+1)}{8\sqrt{2}\beta\gamma}+i\left(\frac{\gamma}{8\sqrt{2}\beta}-\frac{1}{2v}\right)\right]} \times\left[\sum-\frac{(n+1)\sqrt{v}}{2}+iV+\frac{\beta\gamma}{2\sqrt{2}}\left((n+1)+i\gamma^2\right)-\frac{i\gamma^4}{2}\right] \tag{35}$$

Now using these quantities of Equation (20) and the definition for $\sum$ and V given as

$$\sum = \frac{2m}{\hbar^2\alpha^2}\left[\frac{av_0}{2}-cV_2+bV_1+\alpha^2 l(\ell+1)+\frac{\hbar^2\alpha^2 n(n+1)}{2m}\right] \tag{36}$$

$$V = i\left(\frac{2m}{\hbar^2\alpha^2}\right)\sqrt{aV_0+cV_2-bV_1-\alpha^2\ell(\ell+1)} \tag{37}$$

We obtain the energy spectrum of the Equation (35) for the Schrödinger equation with the generalized inverted hyperbolic potential as

$$E_{nl} = \frac{-\frac{\hbar^2\alpha^2}{2m}\left[1+\frac{i\beta^2}{4}(1+v)\right]}{2\left[\frac{(n+1)}{8\sqrt{2}\beta r}+i\left(\frac{r}{8\sqrt{2}\beta}-\frac{1}{2v}\right)\right]} \pm \frac{\frac{\hbar^2\alpha^2}{2m}}{2\left[\frac{(n+1)}{8\sqrt{2}\beta r}+i\left(\frac{r}{8\sqrt{2}\beta}-\frac{1}{2v}\right)\right]}\sqrt{\left(1+\frac{i\beta^2}{4}\left(1+\frac{1}{v}\right)\right)^2+4\left[\frac{(n+1)}{8\sqrt{2}\beta r}+i\left(\frac{r}{8\sqrt{2}\beta}-\frac{1}{v}\right)\right]} \sqrt{x\left[\sum-\frac{(n+1)}{2}\sqrt{v}+iv+\frac{\beta r}{2\sqrt{2}}\left(n+1+ir^2\right)-\frac{ir^4}{2}\right]} -\left(\frac{\beta}{2}\right)^2-cV_2+\alpha^2\ell(\ell+1)-d \tag{38}$$

We now find the corresponding eigenfunctions. The polynomial solutions of the hyperbolic function $\chi_n(s)$ depend on the determination of weight function $\rho(s)$. Thus we determine the $\rho(s)$ function in Equation (6) as

$$\rho(s) = \left(1+s^2\right)^{-2}\left(\frac{1+is}{1-is}\right)^{\mu+iv} \tag{39}$$

where $\mu = 2-\sqrt{u+v}$, $\upsilon = \sqrt{u-v}$ and substituting Equation (39) in to the Rodriques relation of Equation (5), we have

$$\chi_n(s) = B_n\left(1+is\right)^{2+\mu+2\upsilon}\left(l-is\right)^{2-\mu-i\upsilon} \times\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left[\left(1+is\right)^{N+\mu+i\upsilon-2}\left(1-is\right)^{n-\upsilon-iv-2}\right] \tag{40}$$

The polynomial solution of $\chi_n(s)$ can be expressed in terms of Jacobi polynomials which is one of the orthogonal polynomials, which is $\rho_n^{2+A,2-A}(x)$, where $A = \mu+i\upsilon$, and $x = is$.

The other part of the wave function is obtain from Equation (4) as,

$$\varphi(s) = \left(1+is\right)^{\frac{\mu+B}{2}}\left(1-is\right)^{\frac{\mu-B}{2}} \tag{41}$$

where $B = \dfrac{\mu+\beta}{2i}$.

Combining the Jacobi polynomials of Equations (40) and (41), we obtain the redial wave function of the Schrödinger equation with inverted generalized hyperbolic potential as

$$F_{nl}(S) = N_n\left(1+x\right)^{\frac{\mu+B}{2}}\left(1-x\right)^{\frac{\mu-B}{2}}P_n^{2+A,2-A}(x) \tag{43}$$

where N_n is a new normalization constant and obeys the condition

$$\int_{-\infty}^{\infty} R_n^2(s)\mathrm{d}s = 1.$$

The total radial wave function is obtain using Equations (18) and (22) as

$$R_{nl}(r) = N_n\left[1+\mathrm{i}\coth(\alpha r)\right]^{-\frac{\mu+B}{2}} \cdot\left(1+\mathrm{i}\coth(\alpha r)\right)^{\frac{\mu-B}{2}}P_n^{2+A,2-A}\left(\mathrm{i}\coth(\alpha r)\right) \tag{43}$$

4. Results and Discussion

The well-known potentials are obtained from the generalized inverted hyperbolic potential if we make appropri-

ate choose for the values of the parameters in the generalized inverted potentials as stated in Section 3. We plotted the variation of the generalized inverted hyperbolic potential as a function of r for $a = 1$, $b = 0.01$, $V_0 = 1$ MeV, $V_1 = 0.5$ MeV, $C_2 = 2$, $V_2 = 0.02$ MeV, $d = 2$ MeV at different parameters of $\alpha = 1, 2, 3$ and 4 as display in **Figure 1**.

Rosen-Morse Potential: For $b = d = 0$, the Rosen-Morse Potential is obtain as given in Equation (13). We plotted the variation of Rosen-Morse $V(r)$ with r for $a = -1$, $V_0 = 1$ MeV, $c = 2$ and $V_2 = 0.02$ MeV with different α parameters of $\alpha = 1, 2, 3$ and 4 in **Figure 2**. Substituting $b = d = 0$ in Equations (38) and (43), we obtain the energy spectrum and the wave function of the Rosen-Morse potential respectively.

Poschl-Teller Potential: Poschl-Teller Potential is obtain from the generalized inverted hyperbolic potential by setting $a = b = d = 0$ and $c = -c$ as given in Equation (14). The Poschl-Teller potential is plotted as a function of r for $c = -2$ and $V_2 = 0.02$ MeV in **Figure 3**. Substituting these parameters in the energy equation of Equation (38) and wave function (43), we obtain the desired energy spectrum and the wave function of the Poschl-Teller potential.

Scarf Potential: We can deduce the Scarf potential from the generalized inverted hyperbolic potential by setting $a = c = d = 0$. We display in **Figure 4** the plot of Scarf potential as a function of r for $b = 0.05$, $V_1 = 0.5$ MeV with various parameter of $\alpha = 1, 2, 3$ and 4 . Setting the above limiting values in Equations (38) and (43) we obtain the energy eigen-values and wave function for the Scarf potential respectively.

5. Conclusion

The bound state solutions of the Schrödinger equation with a generalized inverted hyperbolic potential have been investigated within the framework of the NU method. Three well-known potential have been deduced from this potential. We discussed the energy spectrum and the wave function of the SE with this potential for an arbitrary l-state. We also discussed the special cases of the generalized inverted hyperbolic potential: Rosen Morse, Poschl-Teller and Scarf potentials. Finally, we plotted the effective potential as a function of r for different $l = 1$, 2, 3 and 4 as shown in **Figure 5**.

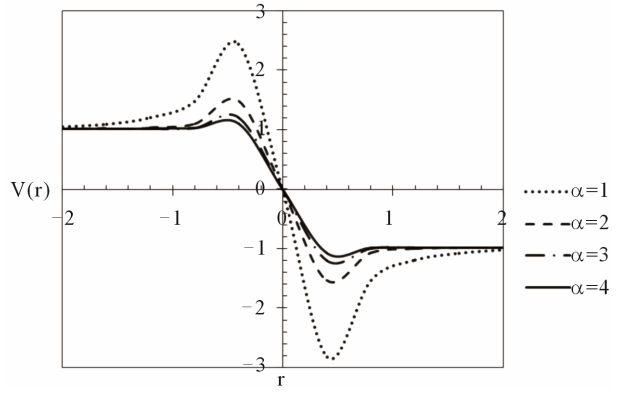

Figure 1. A plot of inverted generalized hyperbolic potenrial with r for $a = 1$, $b = 0.01$, $c = 2$, $d = 2$ MeV, $V_0 = 1$ MeV, $V_1 = 0.5$ MeV, $V_2 = 0.02$ MeV and $\alpha = 1, 2, 3$, and 4.

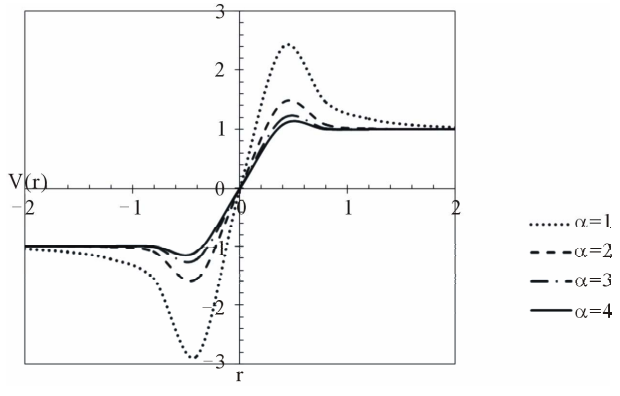

Figure 2. Variation of Rosen-Morse potential with r for $a = -1$, $b = 0$, $c = 2$, $d = 0$, $V_0 = 1$ MeV, $V_2 = 0.02$ MeV with various parameter of $\alpha = 1, 2, 3$, and 4.

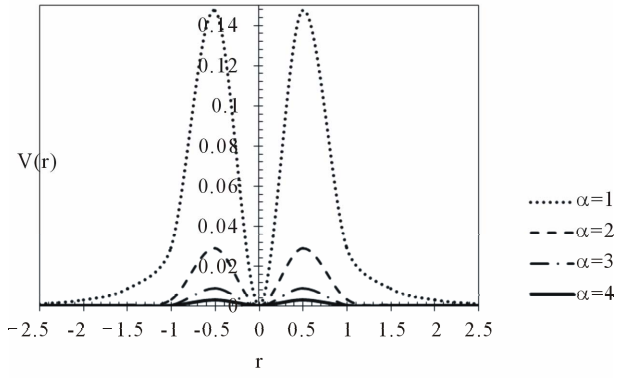

Figure 3. A plot of Poschl-Teller potenrial with r for 0.01, $c = -2$, $V_2 = 0.02$ MeV with various parameter of $\alpha = 1, 2, 3$, and 4.

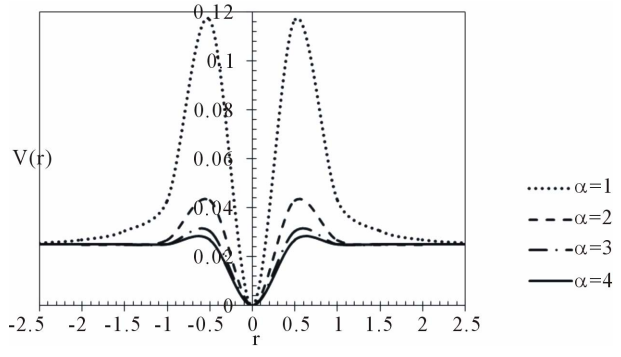

Figure 4. A plot of Scarf potenrial with r for $a = 0$, $b = 0.05$, $c = 0$, $d = 0$, $V_1 = 0.5$ MeV with various parameter of $\alpha = 1, 2, 3$, and 4.

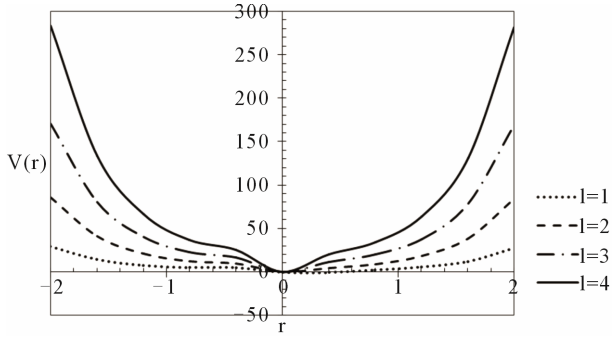

Figure 5. A variation of the effective potential as a function of r for $l = 1, 2, 3$ and 4 with $\alpha = 1$.

6. Acknowledgements

This work is partially supported by the Nandy Research Grant No. 64-01-2271.

REFERENCES

[1] O. M. Al-Dossary, "Morse Potential Eigen-Energies through the Asymptotic Iteration Method," *International Journal of Quantum Chemistry*, Vol. 107, No. 10, 2007, pp. 2040-2046. doi:10.1002/qua.21335

[2] Y. F. Cheng and T. Q. Dai, "Exact Solutions of the Klein-Gordon Equation with a Ring-Shaped Modified Kratzer Potential," *Chinese Journal of Physics*, Vol. 45, No. 5, 2007, p. 480.

[3] C. B. Compean and M. Kirchbach, "Trigonometric Quark Confinement Potential of QCD Traits," *The European Physical Journal A*, Vol. 33, No. 1, 2007, pp. 1-4. doi:10.1140/epja/i2007-10444-0

[4] C. B. Compean and M. Kirchbach, "The Trigonometric Rosen-Morse Potential in the Supersymmetric Quantum Mechanics and Its Exact Solutions," *Journal of Physics A*, Vol. 39, No. 3, 2006, p. 547. doi:10.1088/0305-4470/39/3/007

[5] A. Contreras-Astorga and D. J. Fernandez, "Supersymmetric Partners of the Trigonometric Poschl-Teller Potentials," *Journal of Physics A*, Vol. 41, No. 47, 2008, Article ID: 475303.

[6] F. Cooper, A. Khare and U. Sukhature, "Supersymmetry and Quantum Mechanics," *Physics Reports*, Vol. 251, No. 5-6, 1990, pp. 267-385. doi:10.1016/0370-1573(94)00080-M

[7] J. J. Diaz, J. Negro, I. M. Nieto and O. Rosas, "The Supersymmetric Modified Poschl-Teller and Delta-Well Potentials," *Journal of Physics A*, Vol. 32, No. 48, 1999, p. 8447. doi:10.1088/0305-4470/32/48/308

[8] D. J. Fernandez, "Supersymmetric Quantum Mechanics," arxiv: 0910.0192v1, 2009.

[9] R. I. Greene and C. Aldrich, "Variational Wave Function for a Screened Coulomb Potential," *Journal of Physics A*, Vol. 14, No. 6, 1976, pp. 2363-2366.

[10] A. S. Halberg, "Quasi-Exact Solvability of a Hyperbolic Intermolecular Potential Induced by an Effective Mass Step," *International Journal Mathematics and Mathematical Sciences*, Vol. 2011, Article ID: 358198.

[11] S. D. Hernandez and D. J. Fernandez, "Rosen-Morse Potentials and Its Supersymmetric Partners," *International Journal of Theoretical Physics*, Vol. 50, 2011, pp. 1993-2001.

[12] S. Ikhdair and R. Sever, "Polynomial Solutions of Non-Central Potentials," *International Journal of Theoretical Physics*, Vol. 46, No. 10, 2002, pp. 2384-2395. doi:10.1007/s10773-007-9356-8

[13] A. N. Ikot and L. E. Akpabio, "Approximate Solution of Schrodinger Equation with Rosen-Morse Potential Including a Centrifugal Term," *Applied Physics Research*, Vol. 2, No. 2, 2010, p. 202.

[14] K. J. Oyewumi, E. O. Akinpelu and A. D. Agboola, "Exactly Complete Solutions of the Pseudoharmonic Potential in N-Dimensions," *International Journal of Theoretical Physics*, Vol. 47, No. 4, 2008, pp. 1039-1057. doi:10.1007/s10773-007-9532-x

[15] R. Sever, C. Tezcan, O. Yesiltas and M. Bucurget, "Exact Solution of Effective Mass Schrodinger Equation for the Hulthen Potential," *International Journal of Theoretical Physics*, Vol. 47, No. 9, 2008, pp. 2243-2248. doi:10.1007/s10773-008-9656-7

[16] G. F. Wei, X. Y. Liu and W. L. Chen, "The Relativistic Scattering States of the Hulthen Potential with an Improved New Approximation Scheme to the Centrifugal Term," *International Journal of Theoretical Physics*, Vol. 48, No. 6, 2009, pp. 1649-1658. doi:10.1007/s10773-009-9937-9

[17] H. X. Quan, L. Guang, W. Z. Min, N. L. Bin and M. Yan, "Solving Dirac Equation with New Ring-Shaped Non-Spherical Harmonic Oscillator," *Communications in Theoretical Physics*, Vol. 53, No. 2, 2010, p. 242. doi:10.1088/0253-6102/53/2/07

[18] A. N. Ikot, A. D. Antia, L. E. Akpabio and J. A. Obu, "Analytic Solutions of Schrodinger Equation with Two-Dimensional Harmonic Potential in Cartesian and Polar Coordinates via Nikiforov-Uvarov Method," *Journal of Vectorial Relativity*, Vol. 6, No. 2, 2011, pp. 65-76.

[19] A. N. Ikot, L. E. Akpabio and E. J. Uwah, "Bound State Solution of the Klein-Gordon Equation with Hulthen Potential," *Electronic Journal of Theoretical Physics*, Vol. 8, No. 25, 2011, pp. 225-232.

[20] A. F. Nikiforov and U. B. Uvarov, "Special Functions of Mathematical Physics," Birkhausa, Basel, 1988.

[21] A. N. Ikot, L. E. Akpabio and J. A. Obu, "Exact Solutions of Schrodinger Equation with Five-Parameter Potential," *Journal of Vectorial Relativity*, Vol. 6, No. 1, 2011, p. 1.

[22] S. Meyur and S. Debnath, "Lie Algebra Approach to Non-Hermitian Hamiltonians," *Bulgarian Journal of Physics*, Vol. 36, No. 2, 2009, pp. 77-87.

[23] S. Meyur, "Algebraic Aspect for Two Solvable Potentials," *Electronic Journal of Theoretical Physics*, Vol. 8, No. 25, 2011, pp. 217-224.

[24] S. G. Roy, J. Choudhury, N. K. Sarkar, S. R. Karumuri and R. Bhattacharjee, "A Lie Algebra Approach to the Schrodinger Equation for Bound States of Poschl-Teller

Potential," *Electronic Journal of Theoretical Physics*, Vol. 7, No. 24, 2010, pp. 235-240.

[25] C. S. Jia, T. Chen and L. G. Cui, "Approximate Analytic Solutions of the Dirac Equation with the Generalized Poschl-Teller Potential Including the Pseudo-Spin-Centrifugal Term," *Physics Letters A*, Vol. 373, 2009, pp. 1621-1626. doi:10.1016/j.physleta.2009.03.006

[26] K. J. Oyewumi and C. O. Akoshile, "Bound State Solutions of the Dirac-Rosen-Morse Potential with Spin and Pseudospin Symmetry," *The European Physical Journal A*, Vol. 45, 2010, pp. 311-318. doi:10.1140/epja/i2010-11007-0

[27] K. J. Oyewumi, "Analytic Solution of the Kratzer-Feus Potential in an Arbitrary Number of Dimensions," *Foundations of Physics Letters*, Vol. 18, No. 75, 2005.

[28] W. A. Yahya, K. J. Oyewumi, C. O. Akoshile and T. T. Ibrahim, "Bound State Solutions of Relativistic Dirac Equation with Equal Scalar and Vector Eckart Potentials Using the Nikivorov-Uvarov Method," *Journal of Vectorial Relativity*, Vol. 5, No. 3, 2010, pp. 1-8.

[29] Y. Xu, S. He and C. S. Jia, "Approximate Analytical Solutions of the Klein-Gordon Equation with Poschl-Teller Potential Including the Centrifugal Term," *Physica Scripta*, Vol. 81, No. 4, 2010, Article ID: 045001. doi:10.1088/0031-8949/81/04/045001

