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Pedestrian Analysis of Harmonic Plane Wave Propagation in 1D-Periodic Media

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Abstract

The propagation of TE, TM harmonic plane waves impinging on a periodic multilayer film made of a stack of slabs with the same thickness but with alternate constant permittivity is analyzed. To tackle this problem, the same analysis is first performed on only one slab for harmonic plane waves, solutions of the wave equation. The results obtained in this case are generalized to the stack, taking into account the boundary conditions generated at both ends of each slab by the jumps of permittivity. Differential electromagnetic forms are used to get the solutions of Maxwell's equations.

Keywords: Periodic Slabs, Multilayer Film, TE, TM Waves, Propagation

1. Introduction

The modern approach to harmonic plane wave propagation in periodic materials such as photonic crystals [1,2] relies on the Floquet-Bloch modes [1,2,3] and on a quantum mecha-ics-like technique. We present here for 1D-periodic media, made of a stack of slabs with al-ternate but constant permittivity, in brief a multilayer film, a less powerfull pedestrian tech-nique but providing the explicit expressions of the electromagnetic TM and TE fields. We start with the analysis of a TM plane wave propagation inside an horizontal x,y-slab of thick-ness a, permittivity ε(z) and afterwards, the results obtained in this case are transposed to the stack of slabs.

Harmonic plane wave propagation in a multilayer film has been known for a long time [4], the traditional approach being to consider the multiple reflections that take place at the inter-faces [4], using for instance the S-matrix propagation technique [5], but because of the per-mittivity periodicity, we proceed differently dwelling on boundary conditions at both ends of each slab where exists a jump of permittivity. A particular attention is given to evanescent waves because of their interest in meta-materials with negative permittivity and permeability.

In adddition, we start this paper with a succinct introduction of electromagnetic differential forms [6,7] more efficient than the conventional formalism to tackle the kind of problems to be discussed here. We only use the strong solutions of Maxwell's equations supplied by the differential-form formulation so that we have no need of a computational tool as required by the weak solutions [8].

2. Differential-form Formulation of Maxwell's Equations

We work with the subscript j,k,l, taking the values 1,2,3 associated respectively to the coordinates x,y,z. The summation convention is used and εjkl is the antisymmetric Levi-Civita tensor.

The 3D differential-form formulation of Maxwell's equations is [6,7] in absence of charge and current with the exterior derivative operator d and $\tau = ct$

$$\begin{aligned} &\mathrm{d}\wedge \boldsymbol{E}+\partial_\tau \boldsymbol{B}=0 \ \ \text{a)} \quad \mathrm{d}\wedge \boldsymbol{B}=0 \ \ \text{b)} \\ &\mathrm{d}\wedge \boldsymbol{H}+\partial_\tau \boldsymbol{D}=0 \ \ \text{a)} \quad \mathrm{d}\wedge \boldsymbol{D}=0 \ \ \text{b)} \end{aligned} \tag{1}$$

In these relations $\mathrm{d}=\mathrm{d}x_i\partial_i$, $\boldsymbol{E}, \boldsymbol{H}$ are the 1-forms

$$(\boldsymbol{E},\boldsymbol{H})=(E_i,H_i)\mathrm{d}x_i \tag{2a}$$

and ($\boldsymbol{B}$, $\boldsymbol{D}$) the 2-forms

$$(\boldsymbol{B},\boldsymbol{D})=\frac{1}{2}\varepsilon_{jkl}(B_i,D_i)(\mathrm{d}x_k\wedge \mathrm{d}x_1) \tag{2b}$$

We consider these equations in a medium with permittivity $\varepsilon(r)$ (r is written for x, y, z) and constant permeability μ.

Then, let $*h$ be the Hodge star operator [6,7]

$$*h\left(\mathrm{d}x_j\right)=\frac{1}{2}\varepsilon_{jkl}\left(\mathrm{d}x_k\wedge \mathrm{d}x_l\right) \tag{3}$$

supplying the permittivity and permeability operators $*\varepsilon$, $*\mu$

$$*\varepsilon=\varepsilon\left(r\right)*h, *\mu=\mu*\mathrm{h} \tag{3a}$$

from which the 2-forms $\boldsymbol{D}$, $\boldsymbol{B}$ become

$$\boldsymbol{D}=*\varepsilon\boldsymbol{E}, \boldsymbol{B}=*\mu\boldsymbol{H} \tag{4}$$

so that the coefficients of the differential terms in (2b) are

$$\left(D_i,B_i\right)=\left[\varepsilon\left(r\right)E_i,\mu H_i\right] \tag{4a}$$

Taking into account (4), the Maxwell Equations (1a) become for harmonic fields exp(iωt)

$$\mathrm{d}\wedge\boldsymbol{E}+i\omega/c*\mu\boldsymbol{H}=0, \mathrm{d}\wedge\boldsymbol{H}+i\omega/c*\varepsilon\left(r\right)\boldsymbol{E}=0 \tag{5}$$

that is according to (2a,b) and (4a)

$$\left(\partial_j E_k-\partial_k E_j+\mathrm{i}\left(\omega\mu/2c\right)\varepsilon_{jkl}H_l\right)\left(\mathrm{d}x_j\wedge\mathrm{d}x_k\right)=0 \tag{6a}$$

$$\left(\partial_j H_k-\partial_k H_j+\mathrm{i}\left(\omega\varepsilon\left(r\right)/2c\right)\varepsilon_{jkl}E_l\right)\left(\mathrm{d}x_j\wedge\mathrm{d}x_k\right)=0 \tag{6b}$$

Finally, a simple calculation gives for the second set (1b) of Maxwell's equations

$$\begin{aligned}&\partial_j H_j\left(\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3\right)=0,\\&\partial_j\left[\varepsilon\left(r\right)E_j\right]\left(\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3\right)=0\end{aligned} \tag{7}$$

Electromagnetic 2-forms supply weak solutions of Maxwell's equations by integration on 2D-manifolds, understood as limit over 2D-small simplexes made of triangular elements as used in numerical electromagnetics when physical regions are approximated by finite elements [8]. Strong solutions, the only ones considered here, are obtained by making null the cofficients of the differential terms, they are solutions of conventional Maxwell's equations but easier to get as shown here below.

2.1. TM Field

The wave equation for the magnetic field is obtained by eliminating E from (5) which gives

$$\mathrm{d}\wedge*\varepsilon^{-1}\mathrm{d}\wedge\boldsymbol{H}-\omega^2c^{-2}*\mu\boldsymbol{H}=0 \tag{8}$$

Now, according to (2a) and (3a), the second term on the right hand side of (8) is

$$\omega^2c^{-2}*\mu\boldsymbol{H}=\frac{1}{2}\omega^2c^{-2}\mu\varepsilon_{jkl}H_j\left(\mathrm{d}x_k\wedge\mathrm{d}x_l\right) \tag{9}$$

while in the first term

$$\mathrm{d}\wedge\boldsymbol{H}=\left(\partial_j H_k-\partial_k H_j\right)\left(\mathrm{d}x_j\wedge\mathrm{d}x_k\right) \tag{10}$$

and, using the inverse Hodge star operator $*\varepsilon^{-1}$

$$*\varepsilon^{-1}\mathrm{d}\wedge\boldsymbol{H}=\varepsilon^{-1}\left(r\right)\varepsilon_{jkl}\partial_k H_l\mathrm{d}x_j \tag{10a}$$

Then, we get in Appendix A

$$\mathrm{d}\wedge*\varepsilon^{-1}\mathrm{d}\wedge\boldsymbol{H}=\frac{1}{2}\varepsilon_{ikl}\Psi_j\left(\mathrm{d}x_k\wedge\mathrm{d}x_l\right) \tag{11}$$

in which with the Laplacian operator $\Delta=\partial_j\partial_j$

$$\Psi_j=\varepsilon^{-1}\left(r\right)\Delta H_j+\varepsilon^{-2}\left(r\right)\partial_k\varepsilon\partial_k H_j-\varepsilon^{-2}\left(r\right)\partial_j\left(\partial_j\varepsilon H_k\right) \tag{11a}$$

Subsituting (9) and (11) into (8) gives the differential form of the wave equation

$$\varepsilon_{jkl}\left(\Psi_j-\omega^2c^{-2}\mu H_j\right)\left(\mathrm{d}x_k\wedge\mathrm{d}x_l\right)=0 \tag{12}$$

Let us now consider the TM field in which $\phi(z)$ is an arbitrary function

$$H_y=H_z=0, H_x=\exp\left(\mathrm{i}k_y y\right)\phi\left(z\right) \tag{13}$$

in a medium with permittivity ε(z) depending only on z.

A simple look to (12) shows that this equation reduces to

$$\left(\Psi_x-\omega^2c^{-2}\mu H_x\right)\left(\mathrm{d}x_y\wedge\mathrm{d}x_z\right)=0 \tag{14}$$

with the strong solution $\Psi_x-\omega^2c^{-2}\mu H_x=0$, that is according to (11a) and (13)

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\varepsilon^{-1}\left(z\right)\mathrm{d}\phi\left(z\right)\right]+\left[\omega^2c^{-2}\mu-k_y^2\varepsilon^{-1}\left(z\right)\right]\phi\left(z\right)=0 \tag{15}$$

Once obtained the solutions of (15) and consequently H_x according to (13), the electric field is provided by the 2-form (6b)

$$\begin{aligned}&E_x\left(\mathrm{d}y\wedge\mathrm{d}z\right)=0,\left[\partial_z H_x-i\omega c^{-1}\mu\varepsilon\left(z\right)E_y\right]\left(\mathrm{d}z\wedge\mathrm{d}x\right)\\&+\left[\partial_y H_x+i\omega c^{-1}\mu\varepsilon\left(z\right)E_z\right]\left(\mathrm{d}x\wedge\mathrm{d}y\right)=0\end{aligned} \tag{16}$$

with the strong solution

$$\begin{aligned}&E_x=0,\\&E_y=\left[ic/\omega\mu\varepsilon\left(z\right)\right]\partial_z H_x,\\&E_z=\left[ic/\omega\mu\varepsilon\left(z\right)\right]\partial_y H_x\end{aligned} \tag{16a}$$

which achieves to determine the TM harmonic plane wave E_y, E_z, H_x. From now on, $\mu=1$.

2.2. TE Field

The wave equation for the electric field is obtained by eliminating H from (5)

$$\mathrm{d}\wedge*\mu^{-1}\mathrm{d}\wedge\boldsymbol{E}-\omega^2c^{-2}*\varepsilon\left(r\right)\boldsymbol{E}=0 \tag{17}$$

According to (2a) and (3a), the second term in (17) is

$$\omega^2 c^{-2} * \varepsilon(r)\boldsymbol{E} = \frac{1}{2}\omega^2 c^{-2}\varepsilon(r)\varepsilon_{jkl}E_j\left(\mathrm{d}x_k \wedge \mathrm{d}y_l\right) \quad (18)$$

while in the first term

$$\mathrm{d}\wedge\boldsymbol{E} = \left(\partial_j E_k - \partial_k E_j\right)\left(\mathrm{d}x_j \wedge \mathrm{d}x_k\right) \quad (19)$$

and using the inverse Hodge star operator $*\mu^{-1}$

$$*\mu^{-1}\mathrm{d}\wedge\boldsymbol{E} = \mu^{-1}\varepsilon_{jkl}\partial_k E_l \mathrm{d}x_j \quad (19a)$$

Then

$$\begin{aligned}\mathrm{d}\wedge *\mu^{-1}\mathrm{d}\wedge\boldsymbol{E} &= \frac{1}{2}\partial_m \mathrm{d}x_j \wedge\left(\mu^{-1}\varepsilon_{jkl}\partial_k E_l\right)\mathrm{d}x_j \\ &= \frac{1}{2}\mu\varepsilon_{jkl}\partial_m\partial_l E_k\left(\mathrm{d}x_m \wedge \mathrm{d}x_j\right)\end{aligned} \quad (20)$$

so that with A_2, A_3 deduced from A_1 by a circular permutation of x,y,z, a simple calculation gives

$$\mathrm{d}\wedge *\mu^{-1}\mathrm{d}\wedge\boldsymbol{E} = A_1(x,y,z) + A_2(y,z,x) + A_3(z,x,y) \quad (21)$$

with

$$A_1(x,y,z) = \Phi_z(x,y,z)\left(\mathrm{d}x \wedge \mathrm{d}y\right) \quad (22)$$

in which

$$\Phi_z(x,y,z) = \mu^{-1}\left[\partial_z\left(\partial_x E_x + \partial_y E_y\right) - \partial_z^2 E_z - \partial_y^2 E_z\right] \quad (22a)$$

and, taking into account the divergence Equation (7) $\varepsilon\partial_j E_j + \partial_j \varepsilon E_j = 0$, this expression be-comes

$$\Phi_z(x,y,z) = -\mu^{-1}\left[\Delta E_z + \partial_z\left(E_j\partial_j \varepsilon/\varepsilon\right)\right] \quad (23)$$

supplying Φ_x, Φ_y by a circular permutation of x, y, z so that according to (21), (22):

$$\mathrm{d}\wedge *\mu^{-1}\mathrm{d}\wedge\boldsymbol{E} = \frac{1}{2}\varepsilon_{ikl}\Phi_j\left(\mathrm{d}x_k \wedge \mathrm{d}x_l\right) \quad (24)$$

Substituting (18) and (24) into (17) gives the different-tial form of the wave equation for the electric field

$$\varepsilon_{ikl}\left[\Phi_j - \omega^2 c^{-2}\varepsilon(r)\varepsilon(r)E_j\right]\left(\mathrm{d}x_k \wedge \mathrm{d}x_l\right) = 0 \quad (25)$$

For the TE wave

$$E_y = E_z = 0,\ E_x = \exp\left(\mathrm{i}k_y y\right)\phi(z) \quad (26)$$

in a medium with permittivity $\varepsilon(z)$ depending only on z, $E_j\partial_j\varepsilon = 0$ in (23): Equation (25) reduces to

$$\left[\Delta E_x + \omega^2 c^{-2}\varepsilon(z)E_x\right]\left(\mathrm{d}y \wedge \mathrm{d}z\right) = 0 \quad (27)$$

and, taking into account (26), this differential form has the strong solution

$$\partial_z^2\phi(z) + \left[\omega^2 c^{-2}\mu\varepsilon(z) - k_y^2\right]\phi(z) = 0 \quad (28)$$

The components H_y, H_z of this TE field are in terms of E_x:

$$\partial_z E_x + i\omega\mu/c\,H_y = 0,\ \partial_y E_x + i\omega\mu/c\,H_z = 0 \quad (29)$$

3. Harmonic Plane Waves in a 1D-Periodic Medium

3.1. TM Plane Wave in a Slab ($0 < z < a$)

Suppose that the plane wave (13), (16a) with $\phi(z) = \exp(\mathrm{i}k_i z)$ impinges of the $z = 0$ face of an horizontal slab of thickness a and constant permittivity ε_1 endowed in a medium with permit-tivity ε_0 for $z < 0$ and ε_2 for $z > a$, that is

$$\varepsilon(z) = \begin{cases}\varepsilon_0 : z < 0; \\ \varepsilon_1 : 0 < z < a \\ \varepsilon_2 : z > a\end{cases} \quad (30)$$

So, according to (13) and (16a), the components of the incident and reflected fieds in the half space $z < 0$ are ($\mathrm{i} = \sqrt{-1}$)

$$\begin{aligned}H_x^i &= A_i\exp\left(\mathrm{i}\tau_i\right), \\ E_y^i &= -\left(ck_i A_i/\omega\varepsilon_0\right)\exp\left(\mathrm{i}\tau_i\right), \\ E_z^i &= -\left(ck_y A_i/\omega\varepsilon_0\right)\exp\left(\mathrm{i}\tau_i\right)\end{aligned} \quad (31)$$

$$\begin{aligned}\tau_i &= k_y y + k_z z, \\ k_y^2 + k_i^2 &= \omega^2 c^{-2}\varepsilon_0,\ z < 0\end{aligned} \quad (31a)$$

and

$$\begin{aligned}H_x^r &= A_r\exp\left(\mathrm{i}\tau_r\right), \\ E_y^r &= -\left(ck_i A_r/\omega\varepsilon_0\right)\exp\left(\mathrm{i}\tau_r\right), \\ E_z^r &= -\left(ck_y A_r/\omega\varepsilon_0\right)\exp\left(\mathrm{i}\tau_r\right)\end{aligned} \quad (32)$$

$$\begin{aligned}\tau_r &= k_y y + k_i z, \\ k_y^2 + k_i^2 &= \omega^2 c^{-2}\varepsilon_0,\ z < 0\end{aligned} \quad (32a)$$

Now, in the other two intervalls (30) where ε_j, j = 1, 2 is constant, Equation (15) reduces to

$$\begin{aligned}\mathrm{d}^2\phi/\mathrm{d}z^2 + \Omega_j^2\phi &= 0, \\ \Omega_j^2 &= \omega^2 c^{-2}\varepsilon_j - k_y^2 \quad j = 1,2\end{aligned} \quad (33)$$

and, we shall consider the three situations

$$\Omega_1^2 > 0, \Omega_2^2 > 0; \Omega_1^2 > 0, \Omega_2^2 < 0; \Omega_1^2 < 0, \Omega_2^2 > 0 \quad (33)$$

1) taking into account (30), the solutions of Equation (33) in the first situation $\Omega_j^2 > 0$, j = 1, 2, are with amplitudes A, B, A_t

$$\begin{aligned}\phi(z) &= A\exp(\mathrm{i}kz) + B\exp(-\mathrm{i}kz), \\ k_y^2 + k^2 &= \omega^2 c^{-2}\varepsilon_1,\ 0 < z < a\end{aligned} \quad (34a)$$

$$\phi_t(z) = A_t \exp(\mathrm{i}k_t z),\quad k_y^2 + k_t^2 = \omega^2 c^{-2}\varepsilon_2,\; z > a \tag{34b}$$

Remark 1. If the region with permittivity ε_2 above the slab is limited at z = 2a, $\phi_t(z)$ is chan-ged into

$$\phi_t(z) = A_t \exp(\mathrm{i}k_t z) + B_t \exp(-\mathrm{i}k_t z),\; a < z < 2a \tag{34c}$$

a remark of interest in the next section. Now, taking into account (34a), the components of the TM wave for $0 < z < a$ are, acccording to (13) and (16a)

$$H_x = A\exp(\mathrm{i}\tau) + B\exp(\mathrm{i}\tau'),\quad E_y^i = -(ck/\omega\varepsilon_1)\left[A\exp(\mathrm{i}\tau_i) - B\exp(\mathrm{i}\tau')\right],\quad E_z = (ck_y/\omega\varepsilon_1)\left[A\exp(\mathrm{i}\tau) + B\exp(\mathrm{i}\tau')\right] \tag{35}$$

$$\tau = k_y y + kz,\quad \tau' = k_y y - kz,\quad k_y^2 + k^2 = \omega^2 c^{-2}\varepsilon_1,\; 0 < z < a \tag{35a}$$

while for $z > a$, taking into account (34b)

$$H_x^t = A_t \exp(\mathrm{i}\tau_t),\quad E_y^t = -(ck_t/\omega\varepsilon_2) A_t \exp(\mathrm{i}\tau_t),\quad E_z^t = -(ck_y/\omega\varepsilon_2) A_t \exp(\mathrm{i}\tau_t), \tag{36}$$

$$\tau_t = k_y y + k_t z,\quad k_y^2 + k_t^2 = \omega^2 c^{-2}\varepsilon_2,\; z > a \tag{36a}$$

We now have to take into account the boundary conditions at $z = 0$ and $z = a$, imposed by the continuity of the Hx, Ey components of the electromagnetic field at permittivity jumps.

Then, according to (31), (32), (35), we get at $z = 0$ the two relations

$$A_i + A_r = A + B,\; k_i\varepsilon_0^{-1}(A_i - A_r) = k\varepsilon_0^{-1}(A - B) \tag{37a}$$

while at $z = a$, taking into account (35), (36) it comes

$$A\exp(\mathrm{i}ka) + B\exp(-\mathrm{i}ka) = A_t \exp(\mathrm{i}k_t a)$$
$$Ak\varepsilon_1^{-1}\exp(\mathrm{i}ka) - Bk\varepsilon_1^{-1}\exp(-\mathrm{i}ka) = A_t k_t \varepsilon_2^{-1}\exp(\mathrm{i}k_t a) \tag{37b}$$

The four relations (24(a,b)) supply in Appendix B the four amplitudes A_r, A, B, A_t in terms of the incident amplitude A_i which achieves to determine the fields (32), (35), (36). These boundary conditions impose no constraint on frequency when all the possible values of k_y are considered.

2) In the second situation, $\Omega_1^2 > 0$, $\Omega_2^2 < 0$, the component k_t of the wave vector is pure imaginary and, according to (36a)

$$\phi_t(z) = A_t \exp\left[-k_t(z - a)\right],\quad z > a \tag{38}$$

Remark 2: similarly to the previous remark, for a upper region bounded at $z = 2a$

$$\phi_t(z) = A_t \exp\left[-k_t(z - a)\right] + B_t \exp\left[-k_t(2a - z)\right],\quad a < z < 2a \tag{38a}$$

So, according to (38), the field above the slab is evanescent, does not propagate and the components (36) H_x^t, E_y^t become

$$H_x^t = A_t \exp(\mathrm{i}k_y y)\exp\left[-k_t(z - a)\right]$$
$$E_y^t = (-ick_t/\omega\varepsilon_2) A_t \exp(\mathrm{i}k_y y)\exp\left[-k_t(z - a)\right],\; z > a \tag{39}$$

and, using (35) in $0 < z < a$, the boundary conditions at $z = a$ imply

$$A_t = A\exp(\mathrm{i}ka) + B\exp(-\mathrm{i}ka),\quad -ik_t\varepsilon_2^{-1}A_t = -k\varepsilon_1^{-1}\left[A\exp(\mathrm{i}ka) - B\exp(-\mathrm{i}ka)\right] \tag{40}$$

from which we get

$$2A_t = (1 - \mathrm{i}k\varepsilon_2/k_t\varepsilon_1)A\exp(\mathrm{i}ka) + (1 + \mathrm{i}k\varepsilon_2/k_t\varepsilon_1)B\exp(-\mathrm{i}ka) \tag{41}$$

$$0 = (1 + \mathrm{i}k\varepsilon_2/k_t\varepsilon_1)A\exp(\mathrm{i}ka) + (1 - \mathrm{i}k\varepsilon_2/k_t\varepsilon_1)B\exp(-\mathrm{i}ka) \tag{42}$$

These relations have to be made complete with the boundary conditons (37a) at $z = 0$ from which A, B are provided in terms of A_i and A_r so that according to (41), A, B, A_r, A_t are ob-tained in terms of A_i. Explicitly, substituting into (41) the relations (B.1) of Apendix B, we get with the γ functions $\gamma_\pm = \frac{1}{2}(1 \pm \varepsilon_1 k_i/\varepsilon_0 k)$

$$(1 + \mathrm{i}k\varepsilon_2/k_t\varepsilon_1)[\gamma_+ A_i + \gamma_- A_r]\exp(\mathrm{i}ka) + (1 - \mathrm{i}k\varepsilon_2/k_t\varepsilon_1)[\gamma_- A_i + \gamma_+ A_r]\exp(-\mathrm{i}ka) = 0 \tag{43}$$

supplying Ar from which the amplitudes A, B, A are obtained.

In this case also, the boundary conditions impose no constraint on ω when k_y takes all the possible values but, the situation is different when $k_y = 0$ for propagation in the z-direction. Then $k_y = k$ and $\omega = ck/\sqrt{\varepsilon_j}$, $j = 1, 2$, with $j = 1$ in $0 < z < a$ and $j = 2$ for $z > a$ so that if $\varepsilon_2 > \varepsilon_1$ there is a frequency band gap in the intervall ($ck/\sqrt{\varepsilon_2}$, $ck/\sqrt{\varepsilon_1}$).

3) Finally in the third situation: $\Omega_1^2 < 0$, $\Omega_2^2 > 0$, the TM plane wave (13), (16a) ge-nerates in the slab an evanescent plane wave with the components H_x, E_y deduced from (35), (35a) by changing k into ik and it comes (for simplification, the coefficient exp(i$k_y y$) pre-

sent in each component is discarded)

$$H_x = A\exp(-kz) + B\exp[-k(a-z)]$$
$$E_y = (-ik/\omega\varepsilon)\{A\exp(-kz) - B\exp[-k(a-z)]\}, \quad (44)$$
$$0 < z < a$$

Then, according to (31), (32), (44), the boundary conditions at $z = 0$ supply the two relations

$$\begin{aligned} &A_i + A_r = A + B\exp(-ka), \\ &k_i\varepsilon_0^{-1}(A_i - A_r) = -\mathrm{i}k\varepsilon_0^{-1}[A - B\exp(-ka)] \end{aligned} \quad (45)$$

from which we get

$$\begin{aligned} &2A_i = (1 - \mathrm{i}k\varepsilon_0/k_i\varepsilon_1)A + (1 + \mathrm{i}k\varepsilon_0/k_i\varepsilon_1)B, \\ &2A_r = (1 + \mathrm{i}k\varepsilon_0/k_i\varepsilon_1)A + (1 - \mathrm{i}k\varepsilon_0/k_i\varepsilon_1)B \end{aligned} \quad (46)$$

Now, at $z = a$, the boundary conditions imply

$$\begin{aligned} &A\exp(-ka) + B = A_t\exp(\mathrm{i}k_t a), \\ &\mathrm{i}k\varepsilon_1^{-1}[A\exp(-ka) - B] = -k_t\varepsilon_2^{-1}A_t\exp(\mathrm{i}k_t a) \end{aligned} \quad (47)$$

which gives

$$A\exp(-ka)(k_t/\varepsilon_2 + \mathrm{i}k/\varepsilon_1) + B(k_t/\varepsilon_2 - \mathrm{i}k/\varepsilon_1) = 0 \quad (48)$$

together with (46), this last relation supplies A, B in terms of A^i which achieves to determine A_r and A_t according to (46), (47).

Of course, if the region with permittivity ε_2 is bounded at $z = 2a$, $\phi_t = A_t\exp(\mathrm{i}k_t z)$ is changed into $\phi_t = A_t\exp(\mathrm{i}k_t z) + B_t\exp(-\mathrm{i}k_t z)$. In this situation also, there is a frequency band gap if $\varepsilon_1 > \varepsilon_2$.

3.2. TM Wave Propagation in a Periodic Multilayer Film

We now consider a stack of slabs with each the same thickness a but with an alternate value $\varepsilon(z)$ constant inside the slabs

$$\varepsilon(z) = \begin{cases} \varepsilon_1 : 2ma < z < (2m+1)a \\ \varepsilon_2 : (2m+1)a < z < (2m+2)a \end{cases} \quad m = 0,1,2,\cdots \quad (49)$$

so that $\varepsilon(z + 2ma) = \varepsilon(z)$.

The TM plane wave (13) (16a) is assumed to impinge on the $z = 0$ face of this stack ($m = 0$) and the following notations are used for the field $\phi(z)$.

$$\phi(z) = \begin{cases} \phi_{2m}(z) : 2ma < z < (2m+1)a \\ \phi_{2m+1}(z) : (2m+1)a < z < (2m+2)a \end{cases} \quad (50)$$

The equation (15) becomes inside the slabs since $\mathrm{d}\varepsilon/\mathrm{d}z = 0$ and $\Omega_1^2 = \omega^2 c^{-2}\varepsilon_j - k_y^2,\ j = 1,2$

$$\begin{aligned} &\mathrm{d}^2\phi_{2m}/\mathrm{d}z^2 + \Omega_1^2\phi_{2m} = 0,\ k_1^2 + k_y^2 = \omega^2 c^{-2}\varepsilon_1, \\ &2ma < z < (2m+1)a \\ &\mathrm{d}^2\phi_{2m+1}/\mathrm{d}z^2 + \Omega_2^2\phi_{2m+1} = 0,\ k_2^2 + k_y^2 = \omega^2 c^{-2}\varepsilon_2, \\ &(2m+1)a < z < (2m+2)a \end{aligned} \quad (51)$$

We consider the three situations (33a),

1) In the first situation $\Omega_1^2 > 0,\ \Omega_2^2 > 0 \quad \Omega_1^2 > 0$, $\Omega_2^2 > 0$, the equations (51) have the solutions with amplitudes (A_{2m}, B_{2m}) and (A_{2m+1}, B_{2m+1})

$$\begin{aligned} &\phi_{2m}(z) = A_{2m}\exp(\mathrm{i}k_1 z) + B_{2m}\exp(\mathrm{i}k_1 z), \\ &2ma < z < (2m+1)a \\ &\phi_{2m+1}(z) = A_{2m+1}\exp(\mathrm{i}k_2 z) + B_{2m+1}\exp(-\mathrm{i}k_2 z), \\ &(2m+1)a < z < (2m+2)a \end{aligned} \quad (52)$$

so that according to (13) and (16a) we get for the $H_x E_y$ components intervening in the boun-dary conditions

$$\begin{aligned} &H_x^{2m} = \exp(\mathrm{i}k_y y)\phi_{2m}(z), \\ &H_x^{2m+1} = \exp(\mathrm{i}k_y y)\phi_{2m+1}(z) \\ &E_y^{2m} = (-ck_1/\omega\varepsilon_1)\exp(\mathrm{i}k_y y)\,\phi_{2m}^{\dagger}(z), \\ &E_y^{2m+1} = (-ck_2/\omega\varepsilon_2)\exp(\mathrm{i}k_y y)\phi_{2m+1}^{\dagger}(z) \end{aligned} \quad (53)$$

in which

$$\begin{aligned} &\phi_{2m}^{\dagger}(z) = A_{2m}\exp(\mathrm{i}k_1 z) - B_{2m}\exp(-\mathrm{i}k_1 z) \\ &\phi_{2m+1}^{\dagger}(z) = A_{2m+1}\exp(\mathrm{i}k_2 z) - B_{2m+1}\exp(-\mathrm{i}k_2 z) \end{aligned} \quad (53a)$$

Now, the intervalls $\{(2m-1)a, 2ma\}$ and $\{2ma, (2m+1)a\}$ have a joint boundary at $z = 2ma$ and the continuity of H_x, E_y at these boundaries implies

$$\begin{aligned} &\phi_{2m-1}(2ma) = \phi_{2m}(2ma), \\ &(k_2/\varepsilon_2)\phi_{2m-1}^{\dagger}(2ma) = (k_1/\varepsilon_1)\phi_{2m}^{\dagger}(2ma) \end{aligned} \quad (54)$$

and these relations made explicit in Appendix C supply (A_{2m}, B_{2m}) in terms of $(A_{2m-1}, B_{2m-1})\, m = 1,2\cdots M$. and changing m into $m+1$ gives a similar result at $z = (2m+1)a$ for (A_{2m+1}, B_{2m+1}) in terms of (A_{2m}, B_{2m}).

To achieve the determination of these amplitudes, we need the boundary conditions at $z = 0$ where the incident field impinges on the multilayer film, and at $z = 2Ma$ output of the stack.

The first and last slab in this stack being assumed to have the permittiviy ε_1, we get at $z = 0$ according to (37a) with an evident change in notations

$$A_i + A_r = A_0 + B_0,\ k_i\varepsilon_0^{-1}(A_i - A_r) = k_1\varepsilon_0^{-1}(A_0 - B_0) \quad (55)$$

from which (A_0, B_0) are obtained in terms of (A_i, B_r) and consequently (A_{2M}, B_{2M}) reas-cending the series

(C,3), (C,6) of Appendix C for $m = 0,1,2,\cdots,M$.

Then, $A_t \exp\left(\mathrm{i}k_y y + \mathrm{i}k_t z\right)$ being the field outside the stack for $z > 2Ma$, the boundary conditions are similarly to (37b), ε_0 being the permittivity outside the stack

$$\begin{aligned} A_{2M} \exp\left(2M\mathrm{i}ka\right) + B_{2M} \exp\left(-2M\mathrm{i}ka\right) &= A_t \exp\left(2M\mathrm{i}k_t a\right) \\ A_{2M} k_1 \varepsilon_1^{-1} \exp\left(2M\mathrm{i}ka\right) - B_{2M} k_1 \varepsilon_1 \exp\left(-2M\mathrm{i}ka\right) &= A_t k_t \varepsilon_0^{-1} \exp\left(2M\mathrm{i}k_t a\right) \end{aligned} \tag{56}$$

from which $\left(A_r, A_t\right)$ are obtained in terms of A_i which achieves to determine the amplitudes $\left(A_{2m}, B_{2m}\right), \left(A_{2m+1}, B_{2m+1}\right) m = 1,2\cdots M-1$

2) In the second situation, $\Omega_1^2 > 0, \Omega_2^2 < 0, k_2$ is pure imaginary and, in the intervalls $\left(2m-1\right)a < z < 2ma,$ $m = 1,2\cdots M$, the function $\phi_{2m-1}\left(z\right)$ becomes according to (38a)

$$\phi_{2m-1}\left(z\right) = A_{2m-1} \exp\left[-k_2 \left\{z - \left(2m-1\right)a\right\}\right] + B_{2m-1} \exp\left[-k_2 \left(2ma - z\right)\right] \tag{57}$$

and the boundary conditions (54) at z = 2ma are changed into

$$\begin{aligned} A_{2m} \exp\left(2m\mathrm{i}ak_1\right) + B_{2m} \exp\left(-2m\mathrm{i}ak_1\right) &= A_{2m-1} \exp\left(-k_2 a\right) + B_{2m-1} \\ \left(k_1/\varepsilon_1\right)\left[A_{2m} \exp\left(2m\mathrm{i}ak_1\right) - B_{2m} \exp\left(-2m\mathrm{i}ak_1\right)\right] &= \left(\mathrm{i}k_2/\varepsilon_2\right)\left[A_{2m-1} \exp\left(-k_2 a\right) - B_{2m-1}\right] \end{aligned} \tag{58}$$

from which we get $\left(A_{2m}, B_{2m}\right)$ in terms of $\left(A_{2m-1}, B_{2m-1}\right)$

$$\begin{aligned} 2A_{2m} \exp\left(2m\mathrm{i}ak_1\right) &= \left(1 + \mathrm{i}k_2\varepsilon_1/k_1\varepsilon_2\right) \exp\left(-k_2 a\right) A_{2m-1} + \left(1 - \mathrm{i}k_2\varepsilon_1/k_1\varepsilon_2\right) B_{2m-1} \\ 2B_{2m} \exp\left(-2m\mathrm{i}ak_1\right) &= \left(1 - \mathrm{i}k_2\varepsilon_1/k_1\varepsilon_2\right) \exp\left(-k_2 a\right) A_{2m-1} + \left(1 + \mathrm{i}k_2\varepsilon_1/k_1\varepsilon_2\right) B_{2m-1} \end{aligned} \tag{59}$$

which takes the place of (C.3) in Appendix C while in (C.6) k_2 has to be changed into $\mathrm{i}k_2$ to get $\left(A_{2m+1}, B_{2m+1}\right)$ in terms of $\left(A_{2m}, B_{2m}\right)$.

The relations (55), (56) achieve to determine the amplitudes $\left(A_{2m-1}, B_{2m-1}\right)$, $\left(A_{2m}, B_{2m}\right)$, $\left(A_{2m+1}, B_{2m+1}\right)$, $m = 1,2,\cdots M$ in terms of the amplitude A_i of the incident field. If $\varepsilon_2 > \varepsilon_1$, there exists a frequency band gap for propagation in the z-direction as discussed in Sec.3.1.

3) In the third situation $\Omega_1^2 < 0, \Omega_2^2 > 0, k_1$ is pure imaginary and in the intervalls $2ma < z < \left(2m+1\right)a$ and $\phi_{2m}\left(z\right)$ becomes

$$\phi_{2m}\left(z\right) = A_{2m} \exp\left[-k_1\left(z - 2ma\right)\right] + B_{2m} \exp\left[-k_1\left\{\left(2m+1\right)a - z\right\}\right] \tag{60}$$

and the boundary conditions at $z = \left(2m+1\right)$ a are

$$A_{2m+1} \exp\left[\left(2m+1\right)\mathrm{i}ak_2\right] + B_{2m+1} \exp\left[-\left(2m+1\right)\mathrm{i}ak_2\right] = A_{2m} \exp\left(-k_1 a\right) + B_{2m} \tag{61a}$$

$$\left(k_2/\varepsilon_2\right)\left\{A_{2m+1} \exp\left[\left(2m+1\right)\mathrm{i}ak_2\right] - B_{2m+1} \exp\left[-\left(2m+1\right)\mathrm{i}ak_2\right]\right\} = \left(\mathrm{i}k_1/\varepsilon_1\right)\left[A_{2m} \exp\left(-k_1 a\right) - B_{2m}\right] \tag{61b}$$

from which we get $\left(A_{2m+1}, B_{2m+1}\right)$ in terms of $\left(A_{2m}, B_{2m}\right) m = 0,1\cdots M$.

$$\begin{aligned} 2A_{2m+1} \exp\left[\left(2m+1\right)\mathrm{i}ak_2\right] &= \left(1 + \mathrm{i}k_1\varepsilon_2\right) \exp\left(-k_1 a\right) A_{2m} + \left(1 - \mathrm{i}k_1\varepsilon_2/k_2\varepsilon_1\right) B_{2m} \\ 2B_{2m+1} \exp\left[-\left(2m+1\right)\mathrm{i}ak_2\right] &= \left(1 - \mathrm{i}k_1\varepsilon_2/k_2\varepsilon_1\right) \exp\left(-k_1 a\right) A_{2m} + \left(1 + \mathrm{i}k_1\varepsilon_2/k_2\varepsilon_1\right) B_{2m} \end{aligned} \tag{62}$$

These relations take the place of (C.6) while to get $\left(A_{2m}, B_{2m}\right)$ in terms of $\left(A_{2m-1}, B_{2m-1}\right)$, one has to change k_1 into $\mathrm{i}k_1$ in (C.3).

It is implicitly assumed that the first and last slabs of the stack have the permittivity ε_1. Then, the boundary conditions (45) at $z = 0$ become with evident notations

$$\begin{aligned} &A_r = A_0 + B_0 \exp\left(-k_1 a\right), \\ &\left(k_i/\varepsilon_0\right)\left(A_i - A_r\right) = -\left(\mathrm{i}k_1/\varepsilon_1\right)\left[A_0 - B_0 \exp\left(-k_1 a\right)\right] \end{aligned} \tag{63}$$

from which we get

$$\begin{aligned} 2A_0 &= \left(1 + \mathrm{i}k_i\varepsilon_1/k_1\varepsilon_0\right) A_i + \left(1 - \mathrm{i}k_i\varepsilon_1/k_1\varepsilon_0\right) A_r, \\ 2B_0 &= \left(1 - \mathrm{i}k_i\varepsilon_1/k_1\varepsilon_0\right) A_i + \left(1 + \mathrm{i}k_i\varepsilon_1/k_1\varepsilon_0\right) A_r \end{aligned} \tag{50a}$$

Finally, at $z = 2Ma$, output of the stack in a medium with permittivity ε_0 , we have similarly to (34)

$$\begin{aligned} &A_{2M} \exp\left(-k_1 a\right) + B_{2M} = A_t \exp\left(\mathrm{i}k_t a\right), \\ &\left(\mathrm{i}k_1/\varepsilon_1\left[A_{2M} \exp\left(-k_1 a\right) - B_{2M}\right] = \left(-k_t/\varepsilon_0\right) A_t \exp\left(\mathrm{i}k_t a\right)\right) \end{aligned} \tag{64}$$

Using (62), (C.3) and (63a), $\left(A_{2M}, B_{2M}\right)$ are obtained in terms of (A_i, A_r), so that the relations (51) supply A_r, A_t which achieves to determine the amplitudes (A_{2m-1}, B_{2m-1}), (A_{2m}, B_{2m}) and $(A_{2m+1}, B_{2m+1})\, m = 1,2,\cdots,M-1$ by running down the sequence of amplitudes.

There is also a frequency band gap if $\varepsilon_1 > \varepsilon_2$ for vertical propagation.

Remark 3: According to (36) the permittivity is periodic in the multilayer film $\varepsilon(z) = \varepsilon(z+2a)$ so that, since $\Omega^2(z) = \omega^2 c^{-2} \varepsilon(z) - k_{y^2} > 0$ the component $k_z = \Omega(z)$ of the wave vector is pe-riodic $k_z = k_{z+2a}$. So, if $\phi(z) = \exp(ik_z z)$ is a plane wave solution of the differential equation $\left[\mathrm{d}^2/\mathrm{d}z^2 + \Omega^2(z)\right]\phi(z) = 0$, then:

$$\phi(z+2a) = \exp\left[ik_{z+2a}(z+2a)\right] = \exp\left[ik_z(z+2a)\right] \quad (65)$$

and $\phi(z)$ is periodic if $2k_z a = 2n\pi, n = 0,1,2\cdots$

3.3. TE Plane Wave Propagation

For the TE plane wave (26), (29) impinging on a slab with the permittivity set (30), the wa-ve Equation (28) reduces to Equation (33) so that all the calculations of Secs, (2.1), (2.2) hold valid for the TE field. One has just to change H_x, E_y, E_z into E_x, H_y, H_z.

4. Discussion

As noticed in the introduction, the modern approach to harmonic plane wave propagation in multilayered films, made of a stack of slabs with different but constant permittivity, reposes on two principal techniques, both requiring important computational tools. The first one, mainly interested in frequency bands available for propagation, mixed solid state physics (Floquet-Bloch modes) and quantum mechanics (eigenstates of hermitian operators). The second method [5] working with the S-matrix propagation technique, an improved version of the T-matrix algorithm to analyse plane wave scattering from gratings, is mainly interested in the behaviour of high intensity lasers impinging on gratings made of 1D-photonic crystals. So, any comparison between the results supplied by the two techniques, both depending strongly on the performnces of their computational tools, is difficult.

The analysis performed in Sec.2 of TM and TE waves both polarized along ox, with fields of the type $\exp(ik_y y)\phi(z)$, suggests three comments. When these waves propagate in a slab with the permittivity set (30). First, in this situation, the Maxwell equations have the same solutions for TM and TE waves. Second, as discussed in Remark 3, the function $\phi(z)$ is not assumed periodic a-priori, leading to solutions of Maxwell's equations not taken into account in [1] so that, one may ask whether these extra-solutions play some role in propagation, specially for the frequency band gaps. What kind of incident fields is able to generate these aperiodic solutions? The third point concerns the existence of analytical expressions for the electromagnetic field amplitudes in each slab of the stack so that even if numerical calculations are needed to get them, they will not take the importance they have in [1] and [5].

It is easy to transpose the analysis of Sec.3 to harmonic plane waves $\exp(ik_y y + ik_z z)$ propagating in a photonic meta-film [9] made of alternate slabs and meta-slabs. It has been proved [10,11] that in a lossless meta-material with $\varepsilon < 0, \mu < 0$, the solutions of Maxwell's equa-tions have a classical behaviour provided that the refractive index n and the impedance Z are defined as $n = -(\varepsilon\mu)^{1/2}$ and $Z = (\mu/\varepsilon)^{1/2}$. As a consequence, when the TM plane wave (13), (16a) impinges on a meta-slab, one has just to change in (35), (35a) k and ε_1 into $-k$ and $-n_1^{1/2}$. Taking into account these conditions, the amplitudes of the electromagnetic field inside and outside the metaslab are still supplied in Appendix B suggesting that only minor differences exist for harmonic plane wave propagation in films and meta-films. Of course, the same properties hold valid for TE plane waves.

This result is confirmed in Appendix D where TM wave propagation in a two layered film with slabs made of dielectric or meta-dielectric is analyzed. The situation is particularly inte-resting when $\omega^2 n^2 c^{-2} - k_y^2 < 0$ so that kz is pure imaginary. Then, the two layered film behaves as a deforming mirror with a conjuguate complex distortion factor for slabs and meta-slabs. This analysis could be generalized to a stack of alternate slabs $\{2ma < z < (2m+1)a\}$ and meta-slabs $\{(2m+1)a < z < (2m+2)a\}$ when $\Omega_1^2 < 0$.

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Appendix A

Taking into account (10a), the expression $A = \mathrm{d} \wedge^{*} \varepsilon^{-1} \mathrm{d} \wedge \boldsymbol{H}$ becomes

$$A = \frac{1}{2} \mathrm{d}x_m \partial_m \wedge \varepsilon^{-1} \varepsilon_{jkl} \partial_1 H_k \mathrm{d}x_j \tag{A.1}$$

and, with A_2, A_3 deduced from A_1 by a circular permutation of x, y, z a simple calculation gives

$$A = A_1(x,y,z) + A_2(y,z,x) + A_3(z,x,y) \tag{A.2}$$

with

$$A_1(x,y,z) = \Psi_z(x,y,z)(\mathrm{d}x \wedge \mathrm{d}y) \tag{A.3}$$

in which

$$\begin{aligned}\Psi_z(x,y,z) &= \varepsilon^{-1}\left[\partial_z\left(\partial_x H_x + \partial_y H_y\right) - \partial_x^2 H_z - \partial_y^2 H_z\right] \\ &+ \varepsilon^{-2}\left(\partial_x \varepsilon \partial_x + \partial_y \varepsilon \partial_y\right) H_z - \varepsilon^{-2}\left(\partial_x \varepsilon \partial_z H_x + \partial_y \varepsilon \partial_z H_y\right)\end{aligned} \tag{A.4}$$

Using the divergence equation $\partial_j H_j = 0$ the first term of (A.4) becomes $-\partial_j \partial_j H_z$ while adding $\varepsilon^{-2} \partial_z \varepsilon \partial_z H_z$ to the last two terms, they become

$$\varepsilon^{-2} \partial_j \varepsilon \partial_j H_z - \varepsilon^{-2} \partial_j \varepsilon \partial_z H_j$$

Taking into account these results, $\Psi_z(x,y,z)$ has a simple expression in terms of the Laplacian and nabla operators Δ, ∇

$$\Psi_z(x,y,z) = -\varepsilon^{-1} \Delta H_z + \varepsilon^{-2} \nabla \varepsilon \nabla H_z - \varepsilon^{-2} \nabla \varepsilon \cdot \partial_z \mathbf{H} \tag{A.5}$$

so that

$$\begin{aligned}A_1(x,y,z) = \big[&-\varepsilon^{-1} \Delta H_z + \varepsilon^{-2} \nabla \varepsilon \nabla H_z \\ &-\varepsilon^{-2} \nabla \varepsilon \cdot \partial_z \mathbf{H}\big](dx \wedge dy)\end{aligned} \tag{A.6}$$

Substituting (A.6) into (A.2) gives finally with the ad hoc circular permutations

$$\begin{aligned}A &= \left[-\varepsilon^{-1} \Delta H_z + \varepsilon^{-2} \nabla \varepsilon \nabla H_z - \varepsilon^{-2} \nabla \varepsilon \cdot \partial_z \mathbf{H}\right](\mathrm{d}x \wedge \mathrm{d}y) \\ &+ \left[-\varepsilon^{-1} \Delta H_x + \varepsilon^{-2} \nabla \varepsilon \nabla H_x - \varepsilon^{-2} \nabla \varepsilon \cdot \partial_x \mathbf{H}\right](\mathrm{d}y \wedge \mathrm{d}z) \\ &+ \left[-\varepsilon^{-1} \Delta H_y + \varepsilon^{-2} \nabla \varepsilon \nabla H_y - \varepsilon^{-2} \nabla \varepsilon \cdot \partial_y \mathbf{H}\right](\mathrm{d}z \wedge \mathrm{d}x)\end{aligned} \tag{A.7}$$

Appendix B

We get from (37a) and (37b)

$$\begin{aligned}A &= \gamma_+ A_i + \gamma_- A_r, \\ B &= \gamma_- A_i + \gamma_+ A_r, \\ \gamma_\pm &= \frac{1}{2}\left(1 \pm \varepsilon_1 k_i / \varepsilon_0 k\right)\end{aligned} \tag{B.1}$$

$$\begin{aligned}&\alpha_- A + \alpha_+ B = 0, \\ &A_t = A \exp\left(\left[\mathrm{i}\left(k - k_t\right) a\right] + B \exp\left(\left[-\mathrm{i}\left(k + kt\right) a\right]\right)\right)\end{aligned} \tag{B.2}$$

in which

$$\alpha_\pm = \left(1 \pm \varepsilon_2 k / \varepsilon_1 k_t\right) \exp\left(-\pm \mathrm{i} k a\right) \tag{B.3}$$

substituting (B.1) into the first equation (B.2) gives A_r in terms of A_i and

$$\left(\alpha_- \gamma_+ + \alpha_+ \gamma_-\right) A_i + \left(\alpha_+ \gamma_+ + \alpha_- \gamma_-\right) A_r = 0 \tag{B.4}$$

Eliminating A_r between (B.1) and (B.4) gives A, B in terms of A_i and substituting this result into the second equation (B.2) supplies A_t.

Al these results are valid for $\mu = 1$, when $\mu \neq 1$, ε has to be changed into $n^2 = \varepsilon\mu$ so that according to (B.1) and (B.3)

$$\begin{aligned}\gamma_\pm &= \frac{1}{2}\left(1 + n_1^2 k_i / n_0^2 k\right), \\ \alpha_\pm &= \left(1 + n_2^2 k / n_1^2 k_t\right) \exp\left(-\pm \mathrm{i} k a\right)\end{aligned} \tag{B.5}$$

Finally, in a meta-slab k is transformed into $-k$ and these relations become

$$\begin{aligned}\gamma_\pm &= -\frac{1}{2}\left(-1 \pm n_1^2 k_i / n_0^2 k\right), \\ \alpha \pm &= -\left(-1 \pm n_2^2 k / n_1^2 k_t\right) \exp\left(\pm \mathrm{i} k a\right)\end{aligned} \tag{B.6}$$

while we get from the second relation (B.2)

$$A_t = A \exp\left(\left[-\mathrm{i}\left(k + k_t\right) a\right] + B \exp\left(\left[-\mathrm{i}\left(k - k_t\right) a\right]\right)\right) \tag{B.7}$$

Appendix C

Taking into account (53) and (53a) and introducing the functions

$$\beta_1^\pm = \exp\left(\pm 2 m i a k_1\right), \beta_2^\pm = \exp\left[\pm 2 m i a k_2\right] \tag{C.1}$$

the relations (54) become

$$\begin{aligned}&\beta_2^+ A_{2m-1} + \beta_2 - B_{2m-1} = \beta_1^+ A_{2m} + \beta_1^- B_{2m} \\ &\left(k_2/\varepsilon_2\right)\left[\beta_2^+ A_{2m-1} - \beta_2^- B_{2m-1}\right] = \left(k_1/\varepsilon_1\right)\left[\beta_1^+ A_{2m} - \beta_1^- B_{2m}\right]\end{aligned} \tag{C.2}$$

from which we get $\left(A_{2m}, B_{2m}\right)$ in terms of $\left(A_{2m-1}, B_{2m-1}\right)$

$$\begin{aligned}2\left(k_1/\varepsilon_1\right) \beta_1^- B_{2m} &= \beta_2^+\left(k_1/\varepsilon_1 - k_2/\varepsilon_2\right) A_{2m-1} + \beta_2^-\left(k_1/\varepsilon_1 + k_2/\varepsilon_2\right) B_{2m-1} \\ 2\left(k_1/\varepsilon_1\right) \beta_1^+ A_{2m} &= \beta_2^+\left(k_1/\varepsilon_1 + k_2/\varepsilon_2\right) A_{2m-1} + \beta_2^+\left(k_1/\varepsilon_1 - k_2/\varepsilon_2\right) B_{2m-1}\end{aligned} \tag{C.3}$$

Similarly, according to (52) and (53a), the boundary conditions at $z=(2m+1)a$ are

$$\delta_1^+ A_{2m} + \delta_1^- B_{2m} = \delta_2^+ A_{2m+1} + \delta_2^- B_{2m+1}$$
$$(k_1/\varepsilon_1)\left[\delta_1^+ A_{2m-1} - \delta_1^- B_{2m-1}\right] = (k_2/\varepsilon_2)\left[\delta_2^+ A_{2m} - \delta_2^- \mathbf{A}_{2m}\right] \quad \text{(C.4)}$$

in which

$$\delta_1^\pm = \exp\left[\pm(2m+1)\mathrm{i}ak_1\right], \quad \delta_2^\pm = \exp\left[\pm(2m+1)\mathrm{i}ak_2\right] \quad \text{(C.5)}$$

The relation (C.4) supplies (A_{2m+1}, B_{2m+1}) in terms of (A_{2m}, B_{2m})

$$2(k_2/\varepsilon_2)\delta_2^\pm A_{2m+1} = \delta_1^+ (k_1/\varepsilon_1 + k_2/\varepsilon_2) A_{2m} + \delta_1^- (k_1/\varepsilon_1 - k_2/\varepsilon_2) B_{2m}$$
$$2(k_2/\varepsilon_2)\delta_2^- B_{2m+1} = \delta_1^+ (k_1/\varepsilon_1 - k_2/\varepsilon_2) A_{2m} + \delta_1^- (k_1/\varepsilon_1 + k_2/\varepsilon_2) B_{2m} \quad \text{(C.6)}$$

Appendix D

TM wave propagation in a two layered film

We consider the propagation of a TM harmonic plane wave in a two layered film made of two slabs of thickness a with permittivity-permeability couples (ε_0, μ_0) outside the film, $(\varepsilon_1, \mu_1), (\varepsilon_2, \mu_2)$ in the first and second slab respectively. Then, the refractive index $n_j = \pm(\varepsilon_j \mu_j)^{1/2}$ $j = 0,1,2$, the plus sign for $\varepsilon_j > 0$, $\mu_j > 0$, the minus sign for $\varepsilon_j < 0, \mu_j < 0$.

For the TM wave (13), (16a), the components H_x, E_y of the electromagnetic field have the expressions (31)-(36) in which we use the following notations

$$k = \omega n\rho/c, \quad n = (\varepsilon\mu)^{1/2}, \quad \rho = \left[1 - k_y^2 c^2/\omega^2 n^2\right]^{1/2}, \quad v = n/\rho \quad \text{(D.1)}$$

ρ being either positive or pure imaginary. In addition, the coeffficient $\exp(\mathrm{i}k_y y)$ that appears in each component is deleted. Different situations exist according that ρ is real or not and n positive implying $k > 0$, $v > 0$ or negative with $k < 0, v < 0$.

Then, assuming $\rho_j > 0, n_j > 0$, H_x, E_y have the following expressions (*j* = 0, 1, 2)

$z < 0, k_i = \omega n_0 \rho_0/c$: incident and reflected fields

$$H_x^i = A\exp(\mathrm{i}k_i z),\ H_x^r = A_r \exp(-\mathrm{i}k_i z)$$
$$E_y^i = -1/v_0 A_i \exp(\mathrm{i}k_i z),\ E_y^r = -1/v_0 A_r \exp(-\mathrm{i}k_i z) \quad \text{(D.2)}$$

$0 < z < a, k_1 = \omega n_1 \rho_1/c$: incident and reflected fields

$$H_x^1 = A_1 \exp(\mathrm{i}k_1 z) + B_1 \exp(-\mathrm{i}k_1 z)$$
$$E_y^1 = -1/v_1\left[A_1 \exp(\mathrm{i}k_1 z) - B_1 \exp(-\mathrm{i}k_1 z)\right] \quad \text{(D.3)}$$

$a < z < 2a, k_2 = \omega n_2 \rho_2/c$: field in the second slab

$$H_x^2 = A_2 \exp(\mathrm{i}k_2 z) + B_2 \exp(-\mathrm{i}k_2 z)$$
$$E_y^2 = -1/v_2\left[A_2 \exp(\mathrm{i}k_2 z) - B_2 \exp(-\mathrm{i}k_2 z)\right] \quad \text{(D.4)}$$

$z > 2a, kt = \omega n_0 \rho_0/c$: field above the two layered film

$$H_x^t = A_t \exp(\mathrm{i}k_t z), \quad E_y^t = -1/v_0 A_t \exp(\mathrm{i}k_t z) \quad \text{(D.5)}$$

The boundary conditions at $z = 0, z = a, z = 2a$, give the following relations between the
the amplitudes A

$z = 0$:

$$A_i + A_r = A_1 + B_1, \quad 1/v_0\left[A_i - A_r\right] = 1/v_1\left[A_1 - B^1\right] \quad \text{(D.6)}$$

$z = a$:

$$A_1 \exp(\mathrm{i}k_1 a) + B_1 \exp(-\mathrm{i}k_1 a) = A_2 \exp(\mathrm{i}k_2 a) + B_2 \exp(-\mathrm{i}k_2 a)$$
$$1/v_1\left[A_1 \exp(\mathrm{i}k_1 a) - B_1 \exp(-\mathrm{i}k_1 a)\right] = 1/v_2\left[A_2 \exp(\mathrm{i}k_2 a) - B_2 \exp(-\mathrm{i}k_2 a)\right] \quad \text{(D.7)}$$

$z = 2a$:

$$A_2 \exp(2\mathrm{i}k_2 a) + B_2 \exp(-2\mathrm{i}k_2 a) = A_t \exp(2\mathrm{i}k_t a)$$
$$1/v_2\left[A_2 \exp(\mathrm{i}k_2 a) - B_2 \exp(-\mathrm{i}k_2 a)\right] = 1/v_0 A_t \exp(\mathrm{i}k_t a) \quad \text{(D.8)}$$

The fields (D.3), (D.4) are invariant under the inversions $(k, n, A, B) \Rightarrow (-k, -n, B, A)$ corres-ponding to the exchange slab $\Rightarrow$ meta-slab so that it does not matter whether the slabs are made of dielectric or meta-dielectric, the solutions of the equations (D.6)-(D.8) will be the same in any case.

We get at once from (D.8) in terms of an arbitrary amplitude X

$$A_2 = (v + v_2)\exp(-\mathrm{i}k_2 a) X, \quad B_2 = (v_0 - v_2)\exp(\mathrm{i}k_2 a) X \quad \text{(D.9)}$$

Substituting (D.9) into (D.7) gives

$$A_1\exp(\mathrm{i}k_1a)+B_1\exp(-\mathrm{i}k_1a)=\left[(v_0+v_2)\exp(-\mathrm{i}k_2a)+(v_0-v_2)\exp(\mathrm{i}k_2a)\right]X$$
$$1/v_1\left[A_1\exp(\mathrm{i}k_1a)-B_1\exp(-\mathrm{i}k_1a)\right]=1/v_2\left[(v_0+v_2)\exp(-\mathrm{i}k_2a)-(v_0-v_2)\exp(\mathrm{i}k_2a)\right]X \tag{D.10}$$

from which we get

$$2A_1=\gamma(k_1,v_1;k_2,v_2)X,\quad 2B_1=\gamma(-k_1,-v_1;k_2,v_2)X \tag{D.11}$$

$$\gamma(k_1,v_1;k_2,v_2)=\alpha(v_1,v_2)\exp\left[-\mathrm{i}(k_1+k_2)a\right]+\alpha(v_1,-v_2)\exp\left[-\mathrm{i}(k_1-k_2)a\right] \tag{D.11a}$$

$$\alpha(v_1,v_2)=(v_0+v_2)(v_2+v_1)v_2^{-1} \tag{D.11b}$$

But we get from (D.6)

$$2A_i=(1+v_0/v_1)A_1+(1-v_0/v_1)B_1\quad \text{a)}\quad 2A_r=(1-v_0/v_1)A_1+(1+v_0/v_1)B_1\quad \text{b)} \tag{D.12}$$

and, substituting (D.11) into (D.12a) gives

$$X=4\left[(1+v_0/v_1)\gamma(k_1,v_1;k_2,v_2)+(1-v_0/v_1)\gamma(-k_1,-v_1;k_2,v_2)\right]^{-1}A_i \tag{D.13}$$

Once X ob X tained from (D.13), the relations (D.11) and (D.12b) supply respectively (A_1,B_1) and A_r. Substituting (D.13) into (D.9) gives (A_2,B_2) and finally A_t is obtained from the first relation (D.8) which achieves to determine the amplitudes of the TM harmonic plane wave in the two layered film, a result that does not depend, as said earlier, whether one has to deal with slabs or meta-slabs.

We now assume that ρ in the first slab $0<z<a$, is pure imaginary giving birth to an evanes-cent wave so that since k_1, v_1 are changed into $\mathrm{i}k_1\cdot\mathrm{i}v_1$, the components H_x^1,E_y^1 in this slab become with k_1, v_1 >0

$$H_x^1=A_1\exp(-k_1z)+B_1\exp\left[-k_1(a-z)\right]$$
$$E_y^1=-\mathrm{i}/v_1\left\{A_1\exp(-k_1z)-B_1\exp\left[-k_1(a-z)\right]\right\}\ 0<z<a \tag{D.14}$$

{in a meta-slab where k_1, v_1 < 0, (D.14) is defined with $\exp(k_1z)$ and $\exp\left[k_1(a-z)\right]$}

The boundary conditions are at $z=0$

$$A_i+A_r=A_1+B_1\exp(-k_1a)X,$$
$$1/v_0\left[A_i-A_r\right]=\mathrm{i}/v_1\left[A_1-B_1\exp(-k_1a)\right]X \tag{D.15}$$

and at $z=a$

$$A_1\exp(-k_1a)+B_1=A_2\exp(\mathrm{i}k_2a)+B_2\exp(-\mathrm{i}k_2a)$$
$$\mathrm{i}/v_2 1\left[A_1\exp(-k_1a)+B_1\right]=1/v_2\left[A_2\exp(\mathrm{i}k_2a)-B_2\exp(-\mathrm{i}k_2a)\right] \tag{D.16}$$

Substituting (D.9) valid for a slab or a meta-slab, into (D.16) gives

$$2A_1=\exp(k_1a)\gamma^\dagger(v_1;k_2,v_2),$$
$$2B_1=\gamma^\dagger(-v_1;k_2,v_2) \tag{D.17}$$

$$\gamma^\dagger(v_1;k_2,v_2)=\alpha(v_1,v_2)\exp(-\mathrm{i}k_2a)$$
$$+\alpha(v_1,-v_2)\exp(\mathrm{i}k_2a) \tag{D.17a}$$

$$\alpha(v_1,v_2)=(v_0,v_2)(v_2,-\mathrm{i}v_1)v_2^{-1} \tag{D.17b}$$

But, we get from (D.15)

$$2A_i=(1+\mathrm{i}v_0/v_1)A_1+(1-\mathrm{i}v_0/v_1)B_1\exp(-k_1a) \tag{D.18a}$$

$$2A_r=(1-\mathrm{i}v_0/v_1)A_1+(1+\mathrm{i}v_0/v_1)B_1\exp(-k_1a) \tag{D.18b}$$

and, substituting (D.17) into (D.18a) gives

$$X=4\left[(1+\mathrm{i}v_0/v_1)\gamma^\dagger(v_1;k_2,v_2)\exp(k_1a)+(1-\mathrm{i}v_0/v_1)\gamma^\dagger(-v_2;k_2,v_2)\exp(-k_1a)\right]^{-1}A_i \tag{D.19}$$

Once X obtained from (D.19), the relations (D.17), (D.18b) supply respectively A_1,B_1 and A_r. Substituting (D.19) into (D.9) gives A_2,B_2 and finally A_t from the first relation (D.8).

Let us explicit the amplitudes A. Neglecting the second term that depends on exp(–k1a) in the coefficient of Ai, the expression (D.19) reduces to

$$X=4\left[(1+\mathrm{i}v_0/v_1)\gamma^\dagger(v_1;k_2,v_2)\right]^{-1}\exp(-k_1a) \tag{D.20}$$

so that according to (D.17) in which B_1 may be deleted for the same reason,

$$A_1=2(1+\mathrm{i}v_0/v_1)^{-1}A_i,B_1\cong 0 \tag{D.21}$$

and, substituting (D.21) into (D.18) gives

$$A_1 = \left(1 - \mathrm{i}v_0/v_1\right)\left(1 + \mathrm{i}v_0/v_1\right)^{-1} A_i \qquad \text{(D.22)}$$

while according to (D.9) and to the first relation (D.8), the amplitudes A_2, B_2, A_t, depend on $\exp\left(-k_1 a\right)$.

Then, we get from (D.22) $\left|A_r\right|^2 = A_i^2$ implying that the two layered film behaves as a mirror giving a distorted image because of the factor $\left(1 - \mathrm{i}v_0/v_1\right)\left(1 + \mathrm{i}v_0/v_1\right)^{-1}$ changed into its con-juguate complex for a meta-slab.