

Embedding Stochastic Correlation into the Pricing of FX Quanto Options under Stochastic Volatility Models*

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Abstract

The aim of the paper is to incorporate a stochastic correlation structure when pricing quanto options under the assumption that both the underlying asset and the foreign exchange (FX) rate follow a stochastic volatility model. This is reached not only assuming that the correlation between the underlying asset and its variance process is stochastic (and the same between the exchange rate and its variance process), but also assuming a stochastic correlation between the underlying asset and the exchange rate. Under different stochastic correlation processes specifications, by approximating non-affine terms, we derive a closed-form approximation for the characteristic function of the underlying asset. Numerical experiments and comparison with Monte Carlo simulations are discussed. The analytical tractability of the formulas allows for fast pricing and calibration purposes.

Keywords

Stochastic Correlation Processes, Stochastic Volatility Models, Characteristic Function, FX Derivatives Pricing, Quanto Options

1. Introduction

A quantity-adjusting option, also known as a *quanto* option, is a cash-settled cross currency derivative, where the underlying asset is denominated in a currency (the so-called *foreign currency*) which is different from the one in which the option is settled (the so-called *domestic currency*). Investors use quanto options when they believe a particular asset will perform well in a country but they fear that country's currency will not perform as well. Thus, they will buy an op-

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tion written on the foreign asset while keeping the payout in their domestic currency. This makes the option *free* from the foreign exchange rate fluctuation risk.

As pointed out in [1], the correlation between the underlying process and the exchange rate process plays a very important role in pricing. However, assuming a constant correlation between the underlying asset and the exchange rate is not realistic and might lead to correlation risk. In this paper, we choose to model correlation as a stochastic process.

In particular, our model specification attempts at unifying the setup of [2] and [1]. Indeed, in the first work, the authors extend the Heston model [3], incorporating a stochastic correlation between the underlying asset and its variance process, while in the second one, the same authors assume a stochastic correlation between the underlying asset and the exchange rate but within a constant volatility framework for both the underlying asset and the exchange rate.

In this paper, we relax the assumption of constant volatility for the underlying asset and the exchange rate, and we will assume that both of them follow a stochastic volatility model where, not only the correlation between the underlying asset and its variance is stochastic (and the same assumption holds for the exchange rate and its variance), but also the one between the underlying asset and the exchange rate is stochastic.

Adopting different stochastic correlation processes specifications, such as the Ornstein-Uhlenbeck and the bounded Jacobi processes, the characteristic function for the underlying asset can be derived in closed-form, by approximating all the non-affine terms in the model.

The remainder of the paper is organized as follows. In Section 2, we introduce the stochastic evolution for the underlying asset and for the exchange rate under the stochastic volatility model assumption with stochastic correlation structure. In Section 3 we show how to embed a stochastic correlation structure into the pricing of FX quanto options. In Section 4, we investigate the approximations of non-affine terms in the model, and by using different stochastic correlation processes specification, we derive the corresponding characteristic functions for the underlying asset in closed-form. In Section 5, we discuss how to discretize the model dynamics in order to run Monte Carlo simulations and in Section 6 we test the accuracy of the proposed approximation by comparing it with the Monte Carlo method. Conclusions are reported in Section 7.

2. A Joint Stochastic Volatility Model for the Underlying Price and the FX Rate with Stochastic Correlation Structure

Let $S(t)$ be the price at time t for a certain underlying asset denominated in a foreign currency and let $X(t)$ be the (spot) exchange rate at time t between the foreign and the domestic currencies. In what follows, we will assume that, under the historical measure \mathbb{P} , the stochastic dynamics for $S(t)$ and $X(t)$ are given by the following system of stochastic differential equations (SDEs):

$$\begin{cases} \frac{dS(t)}{S(t)} = \mu_s dt + \sqrt{V(t)} dW_s(t)^{\mathbb{P}}, S(0) > 0, \\ dV(t) = \kappa_v (\mu_v - V(t)) dt + \sigma_v \sqrt{V(t)} dW_v(t)^{\mathbb{P}}, V(0) > 0, \\ \frac{dX(t)}{X(t)} = \mu_x dt + \sqrt{U(t)} dW_x(t)^{\mathbb{P}}, X(0) > 0, \\ dU(t) = \kappa_u (\mu_u - U(t)) dt + \sigma_u \sqrt{U(t)} dW_u(t)^{\mathbb{P}}, U(0) > 0, \end{cases} \quad (1)$$

where $V(t)$ and $U(t)$ are the variance processes for $S(t)$ and $X(t)$ respectively, following the well-known CIR process, as introduced in [4]. In particular, the stochastic dynamics for the pairs $(S(t), V(t)), (X(t), U(t))$ is nothing less than the Heston dynamics, but instead of assuming a constant correlation coefficient as in [3], we impose a stochastic correlation structure, in the sense that

$$\text{Corr}(dW_s(t)^{\mathbb{P}}, dW_v(t)^{\mathbb{P}}) = \eta(t) dt \quad \text{and} \quad \text{Corr}(dW_x(t)^{\mathbb{P}}, dW_u(t)^{\mathbb{P}}) = \gamma(t) dt, \quad (2)$$

where $\eta(t)$ and $\gamma(t)$ are stochastic processes, whose dynamics is given by:

$$\begin{cases} d\eta(t) = a(t, \eta(t)) dt + b(t, \eta(t)) dW_{\eta}(t)^{\mathbb{P}}, \eta(0) \in [-1, 1], \\ d\gamma(t) = c(t, \gamma(t)) dt + d(t, \gamma(t)) dW_{\gamma}(t)^{\mathbb{P}}, \gamma(0) \in [-1, 1], \end{cases} \quad (3)$$

for opportune coefficients $a(t, \eta(t)), b(t, \eta(t)), c(t, \gamma(t)), d(t, \gamma(t))$, which will depend on the assumption for the stochastic correlation process and which will be defined in Section 4.

Therefore, as previously done in [2], we have assumed that the underlying asset $S(t)$ (respectively the exchange rate $X(t)$) is stochastically correlated with its own variance process $V(t)$ (resp. $U(t)$), with correlation process given by $\eta(t)$ (resp. $\gamma(t)$).

Besides that, we will assume that the underlying asset $S(t)$ and the exchange rate $X(t)$ are stochastically correlated, with correlation process $\beta(t)$, whose stochastic dynamics is given by:

$$d\beta(t) = f(t, \beta(t)) dt + g(t, \beta(t)) dW_{\beta}(t)^{\mathbb{P}}, \beta(0) \in [-1, 1], \quad (4)$$

once again for opportune coefficients $f(t, \beta(t)), g(t, \beta(t))$ to be specified.

We note that assuming a stochastic correlation between the underlying asset S and the exchange rate X has been proposed in [1], but in their work the authors assume a constant volatility structure for S and X . Therefore, our model specification attempts at unifying the setup of [2] and [1].

As far as the other correlations are concerned, we will assume that the only non-zero correlations are as follows:

$$\begin{aligned} \text{Corr}(W_s(t)^{\mathbb{P}}, W_{\beta}(t)^{\mathbb{P}}) &= \rho_{S\beta} dt, & \text{Corr}(W_s(t)^{\mathbb{P}}, W_{\eta}(t)^{\mathbb{P}}) &= \rho_{S\eta} dt, \\ \text{Corr}(W_x(t)^{\mathbb{P}}, W_{\beta}(t)^{\mathbb{P}}) &= \rho_{X\beta} dt, & \text{Corr}(W_x(t)^{\mathbb{P}}, W_{\gamma}(t)^{\mathbb{P}}) &= \rho_{X\gamma} dt, \end{aligned} \quad (5)$$

i.e. we assume that the three stochastic correlation processes $(\eta(t), \gamma(t), \beta(t))$ are not correlated among each other, and that they are not correlated with the

variance processes $(V(t), U(t))$ either. Finally, the variance processes themselves are not correlated between each other, and that $V(t)$ (resp. $U(t)$) is not correlated with $X(t)$ (resp. $S(t)$).

In summary, our model specification under the historical measure \mathbb{P} reads as:

$$\left\{ \begin{aligned} \frac{dS(t)}{S(t)} &= \mu_S dt + \sqrt{V(t)} dW_S(t)^\mathbb{P}, \\ dV(t) &= \kappa_V (\mu_V - V(t)) dt + \sigma_V \sqrt{V(t)} dW_V(t)^\mathbb{P}, \\ d\eta(t) &= a(t, \eta(t)) dt + b(t, \eta(t)) dW_\eta(t)^\mathbb{P}, \eta(0) \in [-1, 1], \\ \frac{dX(t)}{X(t)} &= \mu_X dt + \sqrt{U(t)} dW_X(t)^\mathbb{P}, \\ dU(t) &= \kappa_U (\mu_U - U(t)) dt + \sigma_U \sqrt{U(t)} dW_U(t)^\mathbb{P}, \\ d\gamma(t) &= c(t, \gamma(t)) dt + d(t, \gamma(t)) dW_\gamma(t)^\mathbb{P}, \gamma(0) \in [-1, 1], \\ d\beta(t) &= f(t, \beta(t)) dt + g(t, \beta(t)) dW_\beta(t)^\mathbb{P}, \beta(0) \in [-1, 1], \end{aligned} \right. \quad (6)$$

with the following correlation structure for the seven-dimensional Brownian motion

$$R = (\rho_{ij})_{ij} = \begin{pmatrix} W_S(t)^\mathbb{P}, W_V(t)^\mathbb{P}, W_\eta(t)^\mathbb{P}, W_X(t)^\mathbb{P}, W_U(t)^\mathbb{P}, W_\gamma(t)^\mathbb{P}, W_\beta(t)^\mathbb{P} \end{pmatrix}^\top$$

$$R = (\rho_{ij})_{ij} = \begin{bmatrix} 1 & \eta(t) & \rho_{S\eta} & \beta(t) & 0 & 0 & \rho_{S\beta} \\ \eta(t) & 1 & 0 & 0 & 0 & 0 & 0 \\ \rho_{S\eta} & 0 & 1 & 0 & 0 & 0 & 0 \\ \beta(t) & 0 & 0 & 1 & \gamma(t) & \rho_{X\gamma} & \rho_{\beta X} \\ 0 & 0 & 0 & \gamma(t) & 1 & 0 & 0 \\ 0 & 0 & 0 & \rho_{X\gamma} & 0 & 1 & 0 \\ \rho_{S\beta} & 0 & 0 & \rho_{\beta X} & 0 & 0 & 1 \end{bmatrix},$$

which satisfies the common requirements of being symmetric and positive semi-definite.

By the argument of change of measure and using the Girsanov theorem, the model in Equation (6) can be specified under the risk-neutral measure \mathbb{Q} as:

$$\left\{ \begin{aligned} \frac{dS(t)}{S(t)} &= (r_t - \beta(t) \sqrt{V(t)} \sqrt{U(t)}) dt + \sqrt{V(t)} dW_S(t)^\mathbb{Q}, \\ dV(t) &= [\kappa_V (\mu_V - V(t)) - \lambda_V(t)] dt + \sigma_V \sqrt{V(t)} dW_V(t)^\mathbb{Q}, \\ d\eta(t) &= (a(t, \eta(t)) - \lambda_\eta(t)) dt + b(t, \eta(t)) dW_\eta(t)^\mathbb{Q}, \eta(0) \in [-1, 1], \\ \frac{dX(t)}{X(t)} &= (r_d - r_t) dt + \sqrt{U(t)} dW_X(t)^\mathbb{Q}, \\ dU(t) &= [\kappa_U (\mu_U - U(t)) - \lambda_U(t)] dt + \sigma_U \sqrt{U(t)} dW_U(t)^\mathbb{Q}, \\ d\gamma(t) &= (c(t, \gamma(t)) - \lambda_\gamma(t)) dt + d(t, \gamma(t)) dW_\gamma(t)^\mathbb{Q}, \gamma(0) \in [-1, 1], \\ d\beta(t) &= (f(t, \beta(t)) - \lambda_\beta(t)) dt + g(t, \beta(t)) dW_\beta(t)^\mathbb{Q}, \beta(0) \in [-1, 1], \end{aligned} \right. \quad (7)$$

with the same correlation structure as the one above, and where $(\lambda_V(t), \lambda_U(t))$

and $(\lambda_\eta(t), \lambda_\gamma(t), \lambda_\beta(t))$ represent the price of volatility and correlation risk respectively.

In particular, in what follows, we will assume that the market price of volatility risk is linear in the variance process, namely $\lambda_v(t) = \lambda_v V(t)$ (resp. $\lambda_u(t) = \lambda_u U(t)$), and that the market price of correlation risk is constant, in the sense that $\lambda_\eta(t) = \lambda_\eta$ and the same for $\lambda_\gamma(t)$ and $\lambda_\beta(t)$ respectively.

The risk-neutral dynamics for the variance process V reads as¹

$$\begin{aligned} dV(t) &= (\kappa_v + \lambda_v) \left(\frac{\mu_v}{\kappa_v + \lambda_v} - V(t) \right) dt + \sigma_v \sqrt{V(t)} dW_v(t)^\mathbb{Q} \\ &= \tilde{\kappa}_v (\tilde{\mu}_v - V(t)) dt + \sigma_v \sqrt{V(t)} dW_v(t)^\mathbb{Q}, \end{aligned}$$

where we have defined $\tilde{\kappa}_v = \kappa_v + \lambda_v$, $\tilde{\mu}_v = \frac{\mu_v}{\kappa_v + \lambda_v}$ respectively.

Furthermore, under the log-transform for the underlying asset S and for the exchange rate X , *i.e.* $Y(t) = \ln(S(t))$, and $Z(t) = \ln(X(t))$, the risk-neutral dynamics for S and X in the model of Equation (7) can be re-written as:

$$\begin{cases} dY(t) = \left(r_f - \frac{1}{2}V(t) - \beta(t)\sqrt{V(t)}\sqrt{U(t)} \right) dt + \sqrt{V(t)}dW_s(t)^\mathbb{Q}, \\ dZ(t) = \left(r_d - r_f - \frac{1}{2}U(t) \right) dt + \sqrt{U(t)}dW_x(t)^\mathbb{Q}, \end{cases}$$

with the same dynamics for the variance and correlation processes as the one above.

3. FX Quanto Options Pricing under Stochastic Volatility and Correlation

Let $P(t) = P(t, Y(t), V(t), \eta(t), Z(t), U(t), \gamma(t), \beta(t))$ be the value at time t of any contract written on the underlying asset S , denominated in the foreign currency, but paid in the domestic one.

Applying the multi-dimensional Feynman-Kac formula gives the following partial differential equation (PDE) satisfied by P .

$$\begin{aligned} & \frac{\partial P}{\partial t} + \left(r_f - \frac{1}{2}V(t) - \beta(t)\sqrt{V(t)}\sqrt{U(t)} \right) \frac{\partial P}{\partial y} + \left(r_d - r_f - \frac{1}{2}U(t) \right) \frac{\partial P}{\partial z} \\ & + \tilde{a}(t, \eta(t)) \frac{\partial P}{\partial \eta} + \kappa_v (\mu_v - V(t)) \frac{\partial P}{\partial v} + \kappa_u (\mu_u - U(t)) \frac{\partial P}{\partial u} \\ & + \tilde{c}(t, \gamma(t)) \frac{\partial P}{\partial \gamma} + \tilde{f}(t, \beta(t)) \frac{\partial P}{\partial \beta} + \frac{1}{2}V(t) \frac{\partial^2 P}{\partial y^2} + \frac{1}{2}U(t) \frac{\partial^2 P}{\partial z^2} \\ & + \frac{1}{2}\sigma_v^2 V(t) \frac{\partial^2 P}{\partial v^2} + \frac{1}{2}\sigma_u^2 U(t) \frac{\partial^2 P}{\partial u^2} + \frac{1}{2}b^2(t, \eta(t)) \frac{\partial^2 P}{\partial \eta^2} \\ & + \frac{1}{2}d^2(t, \gamma(t)) \frac{\partial^2 P}{\partial \gamma^2} + \frac{1}{2}g^2(t, \beta(t)) \frac{\partial^2 P}{\partial \beta^2} + \beta(t)\sqrt{V(t)}\sqrt{U(t)} \frac{\partial^2 P}{\partial y \partial z} \end{aligned}$$

¹A similar discussion holds for the variance process U as well.

$$\begin{aligned}
 & + \sigma_v \eta(t) V(t) \frac{\partial^2 P}{\partial y \partial v} + \sigma_u \gamma(t) U(t) \frac{\partial^2 P}{\partial z \partial u} + \rho_{s\beta} \sqrt{V(t)} g(t, \beta(t)) \frac{\partial^2 P}{\partial y \partial \beta} \\
 & + \rho_{s\eta} \sqrt{V(t)} b(t, \eta(t)) \frac{\partial^2 P}{\partial y \partial \eta} + \rho_{x\beta} \sqrt{U(t)} g(t, \beta(t)) \frac{\partial^2 P}{\partial z \partial \beta} \\
 & + \rho_{x\gamma} \sqrt{U(t)} d(t, \gamma(t)) \frac{\partial^2 P}{\partial z \partial \gamma} - r_d P = 0.
 \end{aligned} \tag{8}$$

In case where P represents the price of a quanto (call) option, then its payoff (*i.e.* the terminal condition of the above PDE) at maturity time T , reads as:

$$P(T) = \max(S(T) - K, 0), \tag{9}$$

where K is the strike price of the option.

Risk-neutral arguments imply as well that

$$P(t) = P(t, S(t), X(t), T, K) = e^{-r_d(T-t)} X(t) \mathbb{E}^{\mathbb{Q}} \left[(S(T) - K)_+ \mid \mathcal{F}(t) \right], \tag{10}$$

where $\mathcal{F}(t)$ represents the flow of information available up to time t .

As done in [3], we assume that a solution of (10) has the following form:

$$P(t, S(t), X(t), T, K) = e^{-r_d(T-t)} \left(\mathbb{E}^{\mathbb{Q}} [S(T) \mid \mathcal{F}(t)] P_1 - K P_2 \right), \tag{11}$$

where the first term is the present value of the underlying asset upon optimal exercise, and the second term is the present value of the strike price payment (in the above equation we have also assumed that $X(t) \equiv 1$).

Being more explicit, the probabilities P_1, P_2 are defined as follows:

$$P_j = P_j(Y(T) \geq \ln K \mid \mathcal{F}(t)), \text{ for } j = 1, 2, \tag{12}$$

and they are nothing less than the conditional probabilities that the option expires in-the-money. It is well known that these probabilities are not immediately available in closed-form, however, we know that not only P_j , but also their corresponding characteristic functions $\phi_j = \mathbb{E} \left[e^{iuX(T)} \mid \mathcal{F}(t) \right]$ satisfy the same PDE (8), subject to the terminal condition

$$\phi_j(T) = e^{iuX(T)}, \text{ for } j = 1, 2. \tag{13}$$

From the knowledge of the characteristic functions ϕ_j , one can get the probabilities P_j by applying the following Fourier inversion formula

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left[\frac{e^{-i\omega \ln K} \phi_j}{i\omega} \right] d\omega. \tag{14}$$

If we look back at Equation (11), we can observe that the only remaining quantity to be discussed is $\mathbb{E}^{\mathbb{Q}} [S(T) \mid \mathcal{F}(t)]$, which takes the following form, under the model specification of Equation (7):

$$\mathbb{E}^{\mathbb{Q}} [S(T) \mid \mathcal{F}(t)] = e^{r(T-t) + \ln S(t)} \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T \beta(s) \sqrt{U(s)} \sqrt{V(s)} ds} \mid \mathcal{F}(t) \right]. \tag{15}$$

Since, in general, there is no closed-form solution for the above expectation, we propose to approximate the term $\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T \beta(s) \sqrt{U(s)} \sqrt{V(s)} ds} \mid \mathcal{F}(t) \right]$ as follows:

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T \beta(s) \sqrt{U(s)} \sqrt{V(s)} ds} \mid \mathcal{F}(t) \right] \\ & \approx \mathbb{E}^{\mathbb{Q}} \left[e^{-\sqrt{U(t)} \sqrt{V(t)} \int_t^T \beta(s) ds} \mid \mathcal{F}(t) \right] \approx \mathbb{E}^{\mathbb{Q}} \left[e^{-\sqrt{U(0)} \sqrt{V(0)} \int_t^T \beta(s) ds} \mid \mathcal{F}(t) \right], \end{aligned} \quad (16)$$

where the last term on the right hand side can be calculated by deriving the characteristic function ϕ_{ξ} of the integrated correlation process $\xi = \int_t^T \beta(s) ds$ under the risk-neutral measure and then set as argument $i\omega \sqrt{U(0)} \sqrt{V(0)}$.

Therefore, we have that

$$\mathbb{E}[S(T) \mid \mathcal{F}(t)] \approx S(t) e^{r(T-t)} \phi_{\xi} \left(i\omega \sqrt{U(0)} \sqrt{V(0)} \right), \quad (17)$$

from which it follows that our proposed solution would read as:

$$P(t) \approx e^{-r_d(T-t)} \left(S(t) e^{r(T-t)} \phi_{\xi} \left(i\omega \sqrt{U(0)} \sqrt{V(0)} \right) P_1 - KP_2 \right). \quad (18)$$

Obviously, the system of SDEs in Equation (7) is not in the affine form, as some of the coefficients in the corresponding PDE (8) are not linear in the state variables $(y, v, \eta, z, u, \gamma, \beta)$. Therefore, as done in [2] and in [1], we linearize the coefficients of the pricing PDE in order to generate a system of SDEs which is in affine form. As we leave the specification of the drift and volatility coefficients for the correlation processes in Section 4, we first consider the following terms:

$$\sigma_v \eta(t) V(t) \text{ and } \sigma_u \gamma(t) U(t), \quad (19)$$

which can be approximated (see for example [2]) as

$$\sigma_v \eta(t) V(t) \approx \sigma_v \eta(t) \mathbb{E}^{\mathbb{Q}}[V(t)], \sigma_u \gamma(t) U(t) \approx \sigma_u \gamma(t) \mathbb{E}^{\mathbb{Q}}[U(t)], \quad (20)$$

and this is justified from the fact that we have assumed zero correlation between the correlation process $\eta(t)$ (resp. $\gamma(t)$) and the variance process $V(t)$ (resp. $U(t)$).

Next, we propose to approximate the term $\beta(t) \sqrt{U(t)} \sqrt{V(t)}$ as follows:

$$\beta(t) \sqrt{U(t)} \sqrt{V(t)} \approx \beta(t) \mathbb{E}^{\mathbb{Q}}[U(t)] \mathbb{E}^{\mathbb{Q}}[V(t)], \quad (21)$$

where we use the fact that the correlation process $\beta(t)$ is not correlated with any of the variance processes V and U , and also that V and U are not correlated between each other.

As said, the affinity of the terms including the drift and volatility coefficients of the stochastic correlation processes $\eta(t), \gamma(t)$ and $\beta(t)$ is discussed in the next section, as it will depend on the chosen stochastic process for modelling the correlations.

4. Stochastic Correlation Processes Specification

In this section, we apply an Ornstein-Uhlenbeck (OU) process and a bounded Jacobi (BJ) process in order to model stochastic correlation. As done in [2], we will approximate the non-affine terms involving the drift and the volatility coefficients of the correlation processes in order to be able to derive a closed-form solution for the price of a quanto option under the model in Equation (7).

4.1. The Ornstein-Uhlenbeck Process

If we assume that the processes $\eta(t), \gamma(t)$ and $\beta(t)$ follow an OU process, then their drift and volatility coefficients are given by:

$$\begin{aligned} \tilde{a}(t, \eta(t)) &= \kappa_\eta (\mu_\eta - \eta(t)) - \lambda_\eta, \quad \tilde{c}(t, \gamma(t)) = \kappa_\gamma (\mu_\gamma - \gamma(t)) - \lambda_\gamma, \\ \tilde{f}(t, \beta(t)) &= \kappa_\beta (\mu_\beta - \beta(t)) - \lambda_\beta, \quad b(t, \eta(t)) = \sigma_\eta, \\ d(t, \gamma(t)) &= \sigma_\gamma, \quad g(t, \beta(t)) = \sigma_\beta. \end{aligned} \tag{22}$$

In particular, the drift coefficient for η can be re-written as follows:

$$\tilde{a}(t, \eta(t)) = \kappa_\eta (\tilde{\mu}_\eta - \eta(t)), \text{ with } \tilde{\mu}_\eta = \mu_\eta - \frac{\lambda_\eta}{\kappa_\eta}, \tag{23}$$

and the same holds for the drift coefficients for γ and β ².

Since the calibration of the model in Equation (7) is done directly under the \mathbb{Q} measure, the usage of $\tilde{\mu}$ instead of μ , is not relevant for the derivation of the results.

As pointed out in [2], one of the main drawbacks of using an OU process for stochastic correlation is the fact that the process is not bounded over the interval $[-1, 1]$, and this specially happens for a small value of the mean reversion rate and a large value of the volatility parameter. Nevertheless, because of its analytical tractability, we will derive the results also under the assumption that the stochastic correlations follow an OU process.

Under the OU assumption for the correlation processes, the remaining non-affine terms can be approximated as follows:

$$\begin{cases} \rho_{S\beta} \sigma_\beta \sqrt{V(t)} \approx \rho_{S\beta} \sigma_\beta \mathbb{E}^\mathbb{Q} [\sqrt{V(t)}], \quad \rho_{S\eta} \sigma_\eta \sqrt{V(t)} \approx \rho_{S\eta} \sigma_\eta \mathbb{E}^\mathbb{Q} [\sqrt{V(t)}], \\ \rho_{X\beta} \sigma_\beta \sqrt{U(t)} \approx \rho_{X\beta} \sigma_\beta \mathbb{E}^\mathbb{Q} [\sqrt{U(t)}], \quad \rho_{X\gamma} \sigma_\gamma \sqrt{U(t)} \approx \rho_{X\gamma} \sigma_\gamma \mathbb{E}^\mathbb{Q} [\sqrt{U(t)}], \end{cases} \tag{24}$$

where the above approximation has been proposed in [5].

In particular, the following expectations $\mathbb{E}^\mathbb{Q} [\sqrt{V(t)}], \mathbb{E}^\mathbb{Q} [\sqrt{U(t)}]$ involving the square root of the variance processes V and U can be approximated as stated in the next proposition.

Proposition 1 (3.1 in [2]) *Let $v(t)$ a stochastic process, who dynamics is given by:*

$$dv(t) = \kappa_v (\mu_v - v(t))dt + \sigma_v \sqrt{v(t)}dW_v(t)^\mathbb{Q}. \tag{25}$$

Then, $\mathbb{E}^\mathbb{Q} [\sqrt{v(t)}]$ can be approximated by:

$$\mathbb{E}^\mathbb{Q} [\sqrt{v(t)}] \approx m_v + n_v e^{-l_v t}, \tag{26}$$

where m_v, n_v and l_v are defined as:

²We, indeed, have that $\tilde{c}(t, \gamma(t)) = \kappa_\gamma (\tilde{\mu}_\gamma - \gamma(t))$, with $\tilde{\mu}_\gamma = \mu_\gamma - \frac{\lambda_\gamma}{\kappa_\gamma}$, and

$\tilde{f}(t, \beta(t)) = \kappa_\beta (\tilde{\mu}_\beta - \beta(t))$, with $\tilde{\mu}_\beta = \mu_\beta - \frac{\lambda_\beta}{\kappa_\beta}$.

$$m_v = \sqrt{\mu_v - \frac{\sigma_v^2}{8\kappa_v}}, n_v = \sqrt{v_0} - m_v, l_v = -\ln\left(\frac{\hat{d}_v - m_v}{n_v}\right), \tag{27}$$

with

$$\hat{d}_v = \sqrt{\left(v_0 e^{-\kappa_v} - \frac{\sigma_v^2(1 - e^{-\kappa_v})}{4\kappa_v}\right) + \mu_v(1 - e^{-\kappa_v}) + \frac{\sigma_v^2 \mu_v (1 - e^{-\kappa_v})^2}{8\kappa_v \mu_v + 8\kappa_v e^{-\kappa_v} (v_0 - \mu_v)}}. \tag{28}$$

Proof. See Appendix A.1.

We have therefore all the elements in order to find a solution for the pricing PDE (8), under the assumption that the correlation processes follow an OU process.

First, we observe that Equation (8) becomes

$$\begin{aligned} & \frac{\partial P}{\partial t} + \left(r_f - \frac{1}{2}V(t) - \beta(t)\mathbb{E}^{\mathbb{Q}}[\sqrt{V(t)}]\mathbb{E}^{\mathbb{Q}}[\sqrt{U(t)}]\right)\frac{\partial P}{\partial y} \\ & + \left(r_d - r_f - \frac{1}{2}U(t)\right)\frac{\partial P}{\partial z} + \kappa_\eta(\mu_\eta - \eta(t))\frac{\partial P}{\partial \eta} + \kappa_v(\mu_v - V(t))\frac{\partial P}{\partial v} \\ & + \kappa_u(\mu_u - U(t))\frac{\partial P}{\partial u} + \kappa_\gamma(\mu_\gamma - \gamma(t))\frac{\partial P}{\partial \gamma} + \kappa_\beta(\mu_\beta - \beta(t))\frac{\partial P}{\partial \beta} \\ & + \frac{1}{2}V(t)\frac{\partial^2 P}{\partial y^2} + \frac{1}{2}U(t)\frac{\partial^2 P}{\partial z^2} + \frac{1}{2}\sigma_v^2 V(t)\frac{\partial^2 P}{\partial v^2} + \frac{1}{2}\sigma_u^2 U(t)\frac{\partial^2 P}{\partial u^2} \\ & + \frac{1}{2}\sigma_\eta^2 \frac{\partial^2 P}{\partial \eta^2} + \frac{1}{2}\sigma_\gamma^2 \frac{\partial^2 P}{\partial \gamma^2} + \frac{1}{2}\sigma_\beta^2 \frac{\partial^2 P}{\partial \beta^2} + \beta(t)\mathbb{E}^{\mathbb{Q}}[\sqrt{V(t)}]\mathbb{E}^{\mathbb{Q}}[\sqrt{U(t)}]\frac{\partial^2 P}{\partial y \partial z} \\ & + \sigma_v \mathbb{E}^{\mathbb{Q}}[V(t)]\eta(t)\frac{\partial^2 P}{\partial y \partial v} + \sigma_u \mathbb{E}^{\mathbb{Q}}[U(t)]\gamma(t)\frac{\partial^2 P}{\partial z \partial u} \\ & + \rho_{s\beta}\sigma_\beta \mathbb{E}^{\mathbb{Q}}[\sqrt{V(t)}]\frac{\partial^2 P}{\partial y \partial \beta} + \rho_{s\eta}\sigma_\eta \mathbb{E}^{\mathbb{Q}}[\sqrt{V(t)}]\frac{\partial^2 P}{\partial y \partial \eta} \\ & + \rho_{x\beta}\sigma_\beta \mathbb{E}^{\mathbb{Q}}[\sqrt{U(t)}]\frac{\partial^2 P}{\partial z \partial \beta} + \rho_{x\gamma}\sigma_\gamma \mathbb{E}^{\mathbb{Q}}[\sqrt{U(t)}]\frac{\partial^2 P}{\partial z \partial \gamma} - r_d P = 0. \end{aligned} \tag{29}$$

By substituting (18) into the pricing PDE (29), we obtain the following PDEs in P_1 and P_2 respectively:

$$\begin{aligned} & \frac{\partial P_1}{\partial t} + \left(r_f + \frac{1}{2}V(t) - \beta(t)\mathbb{E}[\sqrt{V(t)}]\mathbb{E}[\sqrt{U(t)}]\right)\frac{\partial P_1}{\partial y} + \left(r_d - r_f - \frac{1}{2}U(t)\right. \\ & + \beta(t)\mathbb{E}[\sqrt{V(t)}]\mathbb{E}[\sqrt{U(t)}]\left.\right)\frac{\partial P_1}{\partial z} + \left(\kappa_\eta(\mu_\eta - \eta(t)) + \rho_{s\eta}\sigma_\eta \mathbb{E}[\sqrt{V(t)}]\right)\frac{\partial P_1}{\partial \eta} \\ & + \left(\kappa_v(\mu_v - V(t)) + \sigma_v \mathbb{E}[V(t)]\eta(t)\right)\frac{\partial P_1}{\partial v} + \kappa_u(\mu_u - U(t))\frac{\partial P_1}{\partial u} \\ & + \left(\kappa_\beta(\mu_\beta - \beta(t)) + \rho_{s\beta}\sigma_\beta \mathbb{E}[\sqrt{V(t)}]\right)\frac{\partial P_1}{\partial \beta} + \kappa_\gamma(\mu_\gamma - \gamma(t))\frac{\partial P_1}{\partial \gamma} \\ & + \frac{1}{2}V(t)\frac{\partial^2 P_1}{\partial y^2} + \frac{1}{2}U(t)\frac{\partial^2 P_1}{\partial z^2} + \frac{1}{2}\sigma_v^2 V(t)\frac{\partial^2 P_1}{\partial v^2} + \frac{1}{2}\sigma_u^2 U(t)\frac{\partial^2 P_1}{\partial u^2} \\ & + \frac{1}{2}\sigma_\beta^2 \frac{\partial^2 P_1}{\partial \beta^2} + \frac{1}{2}\sigma_\eta^2 \frac{\partial^2 P_1}{\partial \eta^2} + \frac{1}{2}\sigma_\gamma^2 \frac{\partial^2 P_1}{\partial \gamma^2} + \beta(t)\mathbb{E}[\sqrt{V(t)}]\mathbb{E}[\sqrt{U(t)}]\frac{\partial^2 P_1}{\partial y \partial z} \end{aligned}$$

$$\begin{aligned}
 & + \sigma_v \mathbb{E}[V(t)] \eta(t) \frac{\partial^2 P_1}{\partial y \partial v} + \sigma_u \mathbb{E}[U(t)] \gamma(t) \frac{\partial^2 P_1}{\partial z \partial u} \\
 & + \rho_{S\beta} \sigma_\beta \mathbb{E}[\sqrt{V(t)}] \frac{\partial^2 P_1}{\partial \beta \partial y} + \rho_{S\eta} \sigma_\eta \mathbb{E}[\sqrt{V(t)}] \frac{\partial^2 P_1}{\partial \eta \partial y} \\
 & + \rho_{X\beta} \sigma_\beta \mathbb{E}[\sqrt{U(t)}] \frac{\partial^2 P_1}{\partial z \partial \beta} + \rho_{X\gamma} \sigma_\gamma \mathbb{E}[\sqrt{U(t)}] \frac{\partial^2 P_1}{\partial z \partial \gamma} = 0,
 \end{aligned} \tag{30}$$

and

$$\begin{aligned}
 & \frac{\partial P_2}{\partial t} + \left(r_f - \frac{1}{2} V(t) - \beta(t) \mathbb{E}[\sqrt{V(t)}] \mathbb{E}[\sqrt{U(t)}] \right) \frac{\partial P_2}{\partial y} \\
 & + \left(r_d - r_f - \frac{1}{2} U(t) \right) \frac{\partial P_2}{\partial z} + \kappa_\eta (\mu_\eta - \eta(t)) \frac{\partial P_2}{\partial \eta} + \kappa_v (\mu_v - V(t)) \frac{\partial P_2}{\partial v} \\
 & + \kappa_u (\mu_u - U(t)) \frac{\partial P_2}{\partial u} + \kappa_\beta (\mu_\beta - \beta(t)) \frac{\partial P_2}{\partial \beta} + \kappa_\gamma (\mu_\gamma - \gamma(t)) \frac{\partial P_2}{\partial \gamma} \\
 & + \frac{1}{2} V(t) \frac{\partial^2 P_2}{\partial y^2} + \frac{1}{2} U(t) \frac{\partial^2 P_2}{\partial z^2} + \frac{1}{2} \sigma_v^2 V(t) \frac{\partial^2 P_2}{\partial v^2} + \frac{1}{2} \sigma_u^2 U(t) \frac{\partial^2 P_2}{\partial u^2} \\
 & + \frac{1}{2} \sigma_\beta^2 \frac{\partial^2 P_2}{\partial \beta^2} + \frac{1}{2} \sigma_\eta^2 \frac{\partial^2 P_2}{\partial \eta^2} + \frac{1}{2} \sigma_\gamma^2 \frac{\partial^2 P_2}{\partial \gamma^2} + \beta(t) \mathbb{E}[\sqrt{V(t)}] \mathbb{E}[\sqrt{U(t)}] \frac{\partial^2 P_2}{\partial y \partial z} \\
 & + \sigma_v \mathbb{E}[V(t)] \eta(t) \frac{\partial^2 P_2}{\partial y \partial v} + \sigma_u \mathbb{E}[U(t)] \gamma(t) \frac{\partial^2 P_2}{\partial z \partial u} \\
 & + \rho_{S\beta} \sigma_\beta \mathbb{E}[\sqrt{V(t)}] \frac{\partial^2 P_2}{\partial \beta \partial y} + \rho_{S\eta} \sigma_\eta \mathbb{E}[\sqrt{V(t)}] \frac{\partial^2 P_2}{\partial \eta \partial y} \\
 & + \rho_{X\beta} \sigma_\beta \mathbb{E}[\sqrt{U(t)}] \frac{\partial^2 P_2}{\partial z \partial \beta} + \rho_{X\gamma} \sigma_\gamma \mathbb{E}[\sqrt{U(t)}] \frac{\partial^2 P_2}{\partial z \partial \gamma} = 0.
 \end{aligned} \tag{31}$$

As pointed out in the previous section, we know that the corresponding characteristic functions $\phi = \phi_j(t, y, z, v, u, \beta, \eta, \gamma; \omega)$ of P_j , $j = 1, 2$, must also satisfy the PDEs (30) and (31), respectively. Their solutions can be found in a closed-form as stated in the next lemmas.

Lemma 1. *The characteristic function of P_1 in (18), under the assumption that the correlation processes $\beta(t), \eta(t)$ and $\gamma(t)$ follow an OU process, is given by:*

$$\begin{aligned}
 \phi_1^{OU} = \exp & \left(A_1(\tau, \omega) + B_1(\tau, \omega)v(t) + C_1(\tau, \omega)u(t) + D_1(\tau, \omega)z(t) \right. \\
 & \left. + E_1(\tau, \omega)\beta(t) + F_1(\tau, \omega)\eta(t) + G_1(\tau, \omega)\gamma(t) + i\omega y(t) \right),
 \end{aligned} \tag{32}$$

where $C_1(\tau, \omega) = D_1(\tau, \omega) = G_1(\tau, \omega) \equiv 0$ and where

$$B_1(\tau, \omega) = \frac{\kappa_v - \bar{B}_1}{\sigma_v^2} \frac{1 - e^{-\bar{B}_1 \tau}}{1 - \bar{B}_2 e^{-\bar{B}_1 \tau}}, \tag{33}$$

$$\begin{aligned}
 E_1(\tau, \omega) = -i\omega & \left[\frac{m_v m_u}{\kappa_\beta} \left(1 - e^{-\kappa_\beta \tau} \right) + \frac{m_v n_u}{\kappa_\beta + l_u} e^{-l_u(T-\tau)} \left(1 - e^{-(\kappa_\beta + l_u)\tau} \right) \right. \\
 & + \frac{m_u n_v}{\kappa_\beta + l_v} e^{-l_v(T-\tau)} \left(1 - e^{-(\kappa_\beta + l_v)\tau} \right) \\
 & \left. + \frac{n_u n_v}{\kappa_\beta + l_u + l_v} e^{-(l_u + l_v)(T-\tau)} \left(1 - e^{-(\kappa_\beta + l_u + l_v)\tau} \right) \right],
 \end{aligned} \tag{34}$$

$$F_1(\tau, \omega) = \frac{(\mu_V - v(0))\hat{B}_1}{\kappa_\eta + \kappa_V - \bar{B}_3} e^{(\kappa_V - \bar{B}_3)\tau - \kappa_V T} + \frac{(v(0) - \mu_V)\hat{B}_1}{\kappa_\eta + \kappa_V} e^{\kappa_V(\tau - T)} + \frac{\mu_V \hat{B}_1}{\kappa_\eta} - \frac{\mu_V \hat{B}_1}{\kappa_\eta - \bar{B}_3} e^{-\bar{B}_3 \tau} + \hat{B}_1 \hat{B}_2 e^{-\kappa_\eta \tau}, \tag{35}$$

$$A_1(\tau, \omega) = r_f i \omega \tau + \kappa_V \mu_V H_1(\tau, \omega) + (\kappa_\eta \mu_\eta + (i\omega + 1) \rho_{S\eta} \sigma_\eta m_V) H_2(\tau, \omega) + (i\omega + 1) \rho_{S\eta} \sigma_\eta n_V H_3(\tau, \omega) + (\kappa_\beta \mu_\beta + (i\omega + 1) \rho_{S\beta} \sigma_\beta m_V) H_4(\tau, \omega) + (i\omega + 1) \rho_{S\beta} \sigma_\beta n_V H_5(\tau, \omega) + \frac{1}{2} \sigma_\beta^2 H_6(\tau, \omega) + \frac{1}{2} \sigma_\eta^2 H_7(\tau, \omega). \tag{36}$$

The H functions in Equation (36) are defined as:

$$H_1(\tau, \omega) = \int_0^\tau B_1(s, \omega) ds, \quad H_2(\tau, \omega) = \int_0^\tau F_1(s, \omega) ds, \\ H_3(\tau, \omega) = \int_0^\tau e^{-l_V(T-s)} F_1(s, \omega) ds, \quad H_4(\tau, \omega) = \int_0^\tau E_1(s, \omega) ds, \\ H_5(\tau, \omega) = \int_0^\tau e^{-l_V(T-s)} E_1(s, \omega) ds, \quad H_6(\tau, \omega) = \int_0^\tau E_1(s, \omega)^2 ds, \\ H_7(\tau, \omega) = \int_0^\tau F_1(s, \omega)^2 ds, \tag{37}$$

and the coefficients $\bar{B}_1, \bar{B}_2, \bar{B}_3, \hat{B}_1, \hat{B}_2$ are respectively given by

$$\bar{B}_1 = \sqrt{\kappa_V^2 + \sigma_V^2 \omega(\omega - i)}, \tag{38}$$

$$\bar{B}_2 = \frac{-\kappa_V + \bar{B}_1}{\kappa_V + \bar{B}_1}, \tag{39}$$

$$\bar{B}_3 = -\ln\left(\frac{e^{-\bar{B}_1} - \bar{B}_2 e^{-\bar{B}_1}}{1 - \bar{B}_2 e^{-\bar{B}_1}}\right), \tag{40}$$

$$\hat{B}_1 = \frac{i\omega + 1}{\sigma_V} (\kappa_V - \bar{B}_1), \tag{41}$$

$$\hat{B}_2 = -\frac{\mu_V}{\kappa_V} - \frac{v(0) - \mu_V}{\kappa_\eta + \kappa_V} e^{-\kappa_V T} - \frac{\mu_V - v(0)}{\kappa_\eta + \kappa_V - \bar{B}_3} e^{-\kappa_V T} + \frac{\mu_V}{\kappa_\eta - \bar{B}_3}. \tag{42}$$

with $m_U, m_V, n_U, n_V, l_U, l_V$ as in Proposition 1.

Proof. See Appendix A.2.

Lemma 2. The characteristic function of P_2 in (18), under the assumption that the correlation processes $\beta(t), \eta(t)$ and $\gamma(t)$ follow an OU process, is given by:

$$\phi_2^{OU} = \exp(A_2(\tau, \omega) + B_2(\tau, \omega)v(t) + C_2(\tau, \omega)u(t) + D_2(\tau, \omega)z(t) + E_2(\tau, \omega)\beta(t) + F_2(\tau, \omega)\eta(t) + G_2(\tau, \omega)\gamma(t) + i\omega y(t)), \tag{43}$$

where $C_2(\tau, \omega) = D_2(\tau, \omega) = G_2(\tau, \omega) \equiv 0$ and where

$$B_2(\tau, \omega) = \frac{\kappa_V - \tilde{B}_1}{\sigma_V^2} \frac{1 - e^{-\tilde{B}_1 \tau}}{1 - \tilde{B}_2 e^{-\tilde{B}_1 \tau}}, \tag{44}$$

$$E_2(\tau, \omega) = i\omega \left[\frac{m_V m_U}{\kappa_\beta} \left(1 - e^{-\kappa_\beta \tau}\right) + \frac{m_V n_U}{\kappa_\beta + l_U} e^{-l_U(T-\tau)} \left(1 - e^{-(\kappa_\beta + l_U)\tau}\right) + \frac{m_U n_V}{\kappa_\beta + l_V} e^{-l_V(T-\tau)} \left(1 - e^{-(\kappa_\beta + l_V)\tau}\right) + \frac{n_U n_V}{\kappa_\beta + l_U + l_V} e^{-(l_U + l_V)(T-\tau)} \left(1 - e^{-(\kappa_\beta + l_U + l_V)\tau}\right) \right], \tag{45}$$

$$F_2(\tau, \omega) = \frac{(\mu_V - v(0))B_1^*}{\kappa_\eta + \kappa_V - \tilde{B}_3} e^{(\kappa_V - \tilde{B}_3)\tau - \kappa_V T} + \frac{(v(0) - \mu_V)B_1^*}{\kappa_\eta + \kappa_V} e^{\kappa_V(\tau - T)} + \frac{\mu_V B_1^*}{\kappa_\eta} - \frac{\mu_V B_1^*}{\kappa_\eta - \tilde{B}_3} e^{-\tilde{B}_3 \tau} + B_1^* B_2^* e^{-\kappa_\eta \tau}, \tag{46}$$

$$A_2(\tau, \omega) = r_f i \omega \tau + \kappa_V \mu_V \tilde{H}_1(\tau, \omega) + (\kappa_\eta \mu_\eta + i \omega \rho_{S_\eta} \sigma_\eta m_V) \tilde{H}_2(\tau, \omega) + i \omega \rho_{S_\eta} \sigma_\eta n_V \tilde{H}_3(\tau, \omega) + (\kappa_\beta \mu_\beta + i \omega \rho_{S_\beta} \sigma_\beta m_V) \tilde{H}_4(\tau, \omega) + i \omega \rho_{S_\beta} \sigma_\beta n_V \tilde{H}_5(\tau, \omega) + \frac{1}{2} \sigma_\beta^2 \tilde{H}_6(\tau, \omega) + \frac{1}{2} \sigma_\eta^2 \tilde{H}_7(\tau, \omega). \tag{47}$$

The \tilde{H} functions in Equation (47) are defined as:

$$\begin{aligned} \tilde{H}_1(\tau, \omega) &= \int_0^\tau B_2(s, \omega) ds, & \tilde{H}_2(\tau, \omega) &= \int_0^\tau F_2(s, \omega) ds, \\ \tilde{H}_3(\tau, \omega) &= \int_0^\tau e^{-l_V(T-s)} F_2(s, \omega) ds, & \tilde{H}_4(\tau, \omega) &= \int_0^\tau E_2(s, \omega) ds, \\ \tilde{H}_5(\tau, \omega) &= \int_0^\tau e^{-l_V(T-s)} E_2(s, \omega) ds, & \tilde{H}_6(\tau, \omega) &= \int_0^\tau E_2(s, \omega)^2 ds, \\ \tilde{H}_7(\tau, \omega) &= \int_0^\tau F_2(s, \omega)^2 ds, \end{aligned} \tag{48}$$

and the coefficients $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, B_1^*, B_2^*$ are respectively given by

$$\tilde{B}_1 = \sqrt{\kappa_V^2 + \sigma_V^2 \omega(\omega + i)}, \tag{49}$$

$$\tilde{B}_2 = \frac{-\kappa_V + \tilde{B}_1}{\kappa_V + \tilde{B}_1}, \tag{50}$$

$$\tilde{B}_3 = -\ln\left(\frac{e^{-\tilde{B}_1} - \tilde{B}_2 e^{-\tilde{B}_1}}{1 - \tilde{B}_2 e^{-\tilde{B}_1}}\right), \tag{51}$$

$$B_1^* = \frac{i\omega}{\sigma_V} (\kappa_V - \tilde{B}_1), \tag{52}$$

$$B_2^* = -\frac{\mu_V}{\kappa_V} - \frac{v(0) - \mu_V}{\kappa_\eta + \kappa_V} e^{-\kappa_V T} - \frac{\mu_V - v(0)}{\kappa_\eta + \kappa_V - \tilde{B}_3} e^{-\kappa_V T} + \frac{\mu_V}{\kappa_\eta - \tilde{B}_3}, \tag{53}$$

with $m_U, m_V, n_U, n_V, l_U, l_V$ as in Proposition 1.

The proof of Lemma 2 is similar to the one for Lemma 1 and is left to the reader.

Next, we turn to the calculation of the term

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T \sqrt{V(s)} \sqrt{U(s)} \int_t^s \beta(s) ds} \right],$$

in (18), where $\xi = \int_t^T \beta(s) ds$ is the integrated OU process. Since the OU

process is an affine process, we can calculate the above expectation in closed-form. This is presented in the next lemma.

Lemma 3. *Let*

$$\xi = \int_t^T \beta(s) ds$$

be an integrated correlation process, where $\beta(t)$ follows an OU process.

Then, we have that

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\sqrt{V(0)}\sqrt{U(0)} \int_0^t \beta(s) ds} \right] = e^{-\delta(t) - \beta(0)\chi(t)}, \quad (54)$$

with

$$\delta(t) = \frac{\sqrt{V(0)}\sqrt{U(0)}}{\kappa_\beta} (1 - e^{-\kappa_\beta t}), \quad (55)$$

$$\begin{aligned} \chi(t) = & \frac{V(0)U(0)\sigma_\beta^2}{4\kappa_\beta^3} (e^{-2\kappa_\beta t} - 4e^{-\kappa_\beta t} + 3) + \mu_\beta \frac{\sqrt{V(0)}\sqrt{U(0)}}{\kappa_\beta} (e^{-\kappa_\beta t} - 1) \\ & + \left(\mu_\beta \sqrt{V(0)}\sqrt{U(0)} - \frac{V(0)U(0)\sigma_\beta^2}{2\kappa_\beta^2} \right) t. \end{aligned} \quad (56)$$

Proof. The proof follows the same line as the corresponding result proven in [1], Lemma 3.3, with constant volatilities $\sqrt{V(0)}, \sqrt{U(0)}$ respectively.

4.2. The Bounded Jacobi Process

In this section, we assume that the processes $\eta(t), \gamma(t)$ and $\beta(t)$ follow a bounded Jacobi (BJ) process. In particular, the drift coefficient for a BJ process is exactly the same as the one for an OU process. What changes is the volatility coefficient, and in particular, we have that

$$\begin{aligned} b(t, \eta(t)) &= \sigma_\eta \sqrt{1 - \eta(t)^2}, \\ d(t, \gamma(t)) &= \sigma_\gamma \sqrt{1 - \gamma(t)^2}, \\ g(t, \beta(t)) &= \sigma_\beta \sqrt{1 - \beta(t)^2}. \end{aligned} \quad (57)$$

The BJ process is bounded to $(-1, 1)^3$ as long as the following restriction on the model parameters holds (for the detailed derivation, see [6]):

$$\kappa_\rho > \frac{\sigma_\rho^2}{1 \pm \mu_\rho}. \quad (58)$$

Similar to what we have done in Section 4.1, we observe that the non-affine terms under the BJ stochastic correlation assumption in the pricing PDE (8) can be approximated as follows:

$$\begin{aligned} \sigma_\eta^2 (1 - \eta(t)^2) &\approx \sigma_\eta^2 (1 - \mathbb{E}^{\mathbb{Q}} [\eta(t)^2]), \\ \sigma_\gamma^2 (1 - \gamma(t)^2) &\approx \sigma_\gamma^2 (1 - \mathbb{E}^{\mathbb{Q}} [\gamma(t)^2]), \\ \sigma_\beta^2 (1 - \beta(t)^2) &\approx \sigma_\beta^2 (1 - \mathbb{E}^{\mathbb{Q}} [\beta(t)^2]), \end{aligned} \quad (59)$$

³It means that the boundaries -1 and 1 are not attractive and unattainable.

and

$$\begin{cases} \rho_{S\beta}\sigma_\beta\sqrt{V(t)}\sqrt{1-\beta(t)^2} \approx \rho_{S\beta}\sigma_\beta\mathbb{E}^\mathbb{Q}\left[\sqrt{V(t)}\right]\mathbb{E}^\mathbb{Q}\left[\sqrt{1-\beta(t)^2}\right], \\ \rho_{S\eta}\sigma_\eta\sqrt{V(t)}\sqrt{1-\eta(t)^2} \approx \rho_{S\eta}\sigma_\eta\mathbb{E}^\mathbb{Q}\left[\sqrt{V(t)}\right]\mathbb{E}^\mathbb{Q}\left[\sqrt{1-\eta(t)^2}\right], \\ \rho_{X\beta}\sigma_\beta\sqrt{U(t)}\sqrt{1-\beta(t)^2} \approx \rho_{X\beta}\sigma_\beta\mathbb{E}^\mathbb{Q}\left[\sqrt{U(t)}\right]\mathbb{E}^\mathbb{Q}\left[\sqrt{1-\beta(t)^2}\right], \\ \rho_{X\gamma}\sigma_\gamma\sqrt{U(t)}\sqrt{1-\gamma(t)^2} \approx \rho_{X\gamma}\sigma_\gamma\mathbb{E}^\mathbb{Q}\left[\sqrt{U(t)}\right]\mathbb{E}^\mathbb{Q}\left[\sqrt{1-\gamma(t)^2}\right]. \end{cases} \quad (60)$$

Now, if we look at Equation (59), we can see that the proposed approximations involved the second moment of the BJ stochastic correlation process.

In particular, [7] has shown that, for a stochastic process $\rho(t)$ following the BJ process dynamics, the second moment is given by:

$$\begin{aligned} & \mathbb{E}^\mathbb{Q}\left[\rho(t)^2\right] \\ &= \frac{e^{-t(\sigma_\rho^2+2\kappa_\rho)}}{\sigma_\rho^4+3\kappa_\rho\sigma_\rho^2+2\kappa_\rho^2}\left(\rho(0)^2\left(\sigma_\rho^4+3\kappa_\rho\sigma_\rho^2+2\kappa_\rho^2\right)\right. \\ & \quad \left.+2\mu_\rho\kappa_\rho\rho(0)\left(\sigma_\rho^2+2\kappa_\rho\right)\left(e^{t(\sigma_\rho^2+\kappa_\rho)}-1\right)+\sigma_\rho^2\left(\sigma_\rho^2+\kappa_\rho\right)\left(e^{t(\sigma_\rho^2+2\kappa_\rho)}-1\right)\right. \\ & \quad \left.-2\mu_\rho^2\kappa_\rho\left(\kappa_\rho\left(2e^{t(\sigma_\rho^2+\kappa_\rho)}-e^{t(\sigma_\rho^2+2\kappa_\rho)}-1\right)-\sigma_\rho^2e^{t(\sigma_\rho^2+\kappa_\rho)}\left(e^{t\kappa_\rho}-1\right)\right)\right). \end{aligned} \quad (61)$$

As Equation (61) is rather complicated and not convenient for further calculations, we rely instead on an approximation, which has been proposed in [2], and used as well in [1]. The result is reported in the following proposition.

Proposition 2 (3.2 in [2] or alternatively 3.1 in [1]) *Let $\rho(t)$ a stochastic process, whose dynamics is given by:*

$$d\rho(t) = \kappa_\rho(\mu_\rho - \rho(t))dt + \sigma_\rho\sqrt{1-\rho(t)^2}dW_\rho(t)^\mathbb{Q}. \quad (62)$$

Then, the function $h_\rho(t) \equiv \mathbb{E}^\mathbb{Q}\left[\rho(t)^2\right]$ can be approximated by:

$$\mathbb{E}^\mathbb{Q}\left[\rho(t)^2\right] \approx e^{-s_\rho t} + r_\rho e^{-p_\rho t} + q_\rho, \quad (63)$$

where

$$q_\rho = \frac{(\sigma_\rho^2 + \kappa_\rho)(\sigma_\rho^2 + 2\kappa_\rho\mu_\rho^2)}{\sigma_\rho^4 + 3\kappa_\rho\sigma_\rho^2 + 2\kappa_\rho^2}, \quad r_\rho = \rho(0)^2 - q_\rho - 1, \quad (64)$$

$$s_\rho = -2\ln\left(\alpha_\rho - r_\rho e^{\frac{p_\rho}{2}}\right), \quad p_\rho = -2\ln\left(\frac{r_\rho\alpha_\rho - \sqrt{r_\rho^2\alpha_\rho^2 - \theta_\rho\psi_\rho}}{\theta_\rho}\right), \quad (65)$$

with

$$\alpha_\rho = h_\rho(0.5) - q_\rho, \quad \theta_\rho = r_\rho(1 + r_\rho), \quad \psi_\rho = \alpha_\rho^2 + q_\rho - h_\rho(1). \quad (66)$$

Proof. See Appendix B.2 in [2].

Finally, if we look at Equation (60), the only remaining quantity we need to calculate is $\mathbb{E}^{\mathbb{Q}}\left[\sqrt{1-\rho(t)^2}\right]$, where $\rho \in \{\eta, \gamma, \beta\}$. In order to calculate this expectation, we state the result in [2], where it is shown that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\sqrt{1-\rho(t)^2}\right] &= \sqrt{\mathbb{E}^{\mathbb{Q}}\left[1-\rho(t)^2\right] - \frac{\mathbb{E}^{\mathbb{Q}}\left[\rho(t)\right]^2}{1-\mathbb{E}^{\mathbb{Q}}\left[\rho(t)\right]^2} \text{Var}^{\mathbb{Q}}\left[\rho(t)\right]} \\ &= \sqrt{1 - \frac{\mathbb{E}^{\mathbb{Q}}\left[\rho(t)^2\right] - \mathbb{E}^{\mathbb{Q}}\left[\rho(t)\right]^4}{1-\mathbb{E}^{\mathbb{Q}}\left[\rho(t)\right]^2}}.\end{aligned}\quad (67)$$

The next proposition contains the relevant approximation.

Proposition 3 (3.3 in [2] or alternatively 3.2 in [1]) *Let $\rho(t)$ a stochastic process, whose dynamics is given by:*

$$d\rho(t) = \kappa_{\rho}(\mu_{\rho} - \rho(t))dt + \sigma_{\rho}\sqrt{1-\rho(t)^2}dW_{\rho}(t)^{\mathbb{Q}}. \quad (68)$$

Then, the function $\hat{h}_{\rho}(t) \equiv \mathbb{E}^{\mathbb{Q}}\left[\sqrt{1-\rho(t)^2}\right]$ can be approximated by:

$$\mathbb{E}^{\mathbb{Q}}\left[\sqrt{1-\rho(t)^2}\right] \approx e^{-\hat{s}_{\rho}t} + \hat{r}_{\rho}e^{-\hat{p}_{\rho}t} + \hat{q}_{\rho}, \quad (69)$$

where

$$\hat{q}_{\rho} = \sqrt{1 - \frac{(\sigma_{\rho}^2 + \kappa_{\rho})(\sigma_{\rho}^2 + 2\kappa_{\rho}\mu_{\rho}^2) - \mu_{\rho}^4(\sigma_{\rho}^4 + 3\kappa_{\rho}\sigma_{\rho}^2 + 2\kappa_{\rho}^2)}{(1-\mu_{\rho}^2)(\sigma_{\rho}^4 + 3\kappa_{\rho}\sigma_{\rho}^2 + 2\kappa_{\rho}^2)}}, \quad (70)$$

$$\hat{r}_{\rho} = \sqrt{1-\rho(0)^2} - \hat{q}_{\rho} - 1, \quad \hat{s}_{\rho} = -2\ln\left(\hat{\alpha}_{\rho} - \hat{r}_{\rho}e^{-\frac{\hat{p}_{\rho}}{2}}\right), \quad (71)$$

$$\hat{p}_{\rho} = -2\ln\left(\frac{\hat{r}_{\rho}\hat{\alpha}_{\rho} - \sqrt{\hat{r}_{\rho}^2\hat{\alpha}_{\rho}^2 - \hat{\theta}_{\rho}\hat{\psi}_{\rho}}}{\hat{\theta}_{\rho}}\right), \quad (72)$$

with

$$\hat{\alpha}_{\rho} = \hat{h}_{\rho}(0.5) - \hat{q}_{\rho}, \quad \hat{\theta}_{\rho} = \hat{r}_{\rho}(1 + \hat{r}_{\rho}), \quad \hat{\psi}_{\rho} = \hat{\alpha}_{\rho}^2 + \hat{q}_{\rho} - \hat{h}_{\rho}(1). \quad (73)$$

Proof. See Appendix B.3 in [2].

We have therefore all the elements in order to find a solution for the pricing PDE (8), under the assumption that the correlation processes follow a bounded Jacobi process.

First, we observe that Equation (8) becomes

$$\begin{aligned}\frac{\partial P}{\partial t} &+ \left(r_f - \frac{1}{2}V(t) - \beta(t)\mathbb{E}^{\mathbb{Q}}\left[\sqrt{V(t)}\right]\mathbb{E}^{\mathbb{Q}}\left[\sqrt{U(t)}\right]\right)\frac{\partial P}{\partial y} \\ &+ \left(r_d - r_f - \frac{1}{2}U(t)\right)\frac{\partial P}{\partial z} + \kappa_{\eta}(\mu_{\eta} - \eta(t))\frac{\partial P}{\partial \eta} + \kappa_{\nu}(\mu_{\nu} - \nu(t))\frac{\partial P}{\partial \nu} \\ &+ \kappa_U(\mu_U - U(t))\frac{\partial P}{\partial u} + \kappa_{\gamma}(\mu_{\gamma} - \gamma(t))\frac{\partial P}{\partial \gamma} + \kappa_{\beta}(\mu_{\beta} - \beta(t))\frac{\partial P}{\partial \beta}\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2}V(t)\frac{\partial^2 P}{\partial y^2} + \frac{1}{2}U(t)\frac{\partial^2 P}{\partial z^2} + \frac{1}{2}\sigma_v^2 V(t)\frac{\partial^2 P}{\partial v^2} + \frac{1}{2}\sigma_u^2 U(t)\frac{\partial^2 P}{\partial u^2} \\
 & + \frac{1}{2}\sigma_\eta^2 \left(1 - \mathbb{E}^\mathbb{Q}[\eta(t)^2]\right)\frac{\partial^2 P}{\partial \eta^2} + \frac{1}{2}\sigma_\gamma^2 \left(1 - \mathbb{E}^\mathbb{Q}[\gamma(t)^2]\right)\frac{\partial^2 P}{\partial \gamma^2} \\
 & + \frac{1}{2}\sigma_\beta^2 \left(1 - \mathbb{E}^\mathbb{Q}[\beta(t)^2]\right)\frac{\partial^2 P}{\partial \beta^2} + \beta(t)\mathbb{E}^\mathbb{Q}[\sqrt{V(t)}]\mathbb{E}^\mathbb{Q}[\sqrt{U(t)}]\frac{\partial^2 P}{\partial y \partial z} \\
 & + \sigma_v \mathbb{E}^\mathbb{Q}[V(t)]\eta(t)\frac{\partial^2 P}{\partial y \partial v} + \sigma_u \mathbb{E}^\mathbb{Q}[U(t)]\gamma(t)\frac{\partial^2 P}{\partial z \partial u} \\
 & + \rho_{S\beta}\sigma_\beta \mathbb{E}^\mathbb{Q}[\sqrt{V(t)}]\mathbb{E}^\mathbb{Q}[\sqrt{1-\beta(t)^2}]\frac{\partial^2 P}{\partial y \partial \beta} \\
 & + \rho_{S\eta}\sigma_\eta \mathbb{E}^\mathbb{Q}[\sqrt{V(t)}]\mathbb{E}^\mathbb{Q}[\sqrt{1-\eta(t)^2}]\frac{\partial^2 P}{\partial y \partial \eta} \\
 & + \rho_{X\beta}\sigma_\beta \mathbb{E}^\mathbb{Q}[\sqrt{U(t)}]\mathbb{E}^\mathbb{Q}[\sqrt{1-\beta(t)^2}]\frac{\partial^2 P}{\partial z \partial \beta} \\
 & + \rho_{X\gamma}\sigma_\gamma \mathbb{E}^\mathbb{Q}[\sqrt{U(t)}]\mathbb{E}^\mathbb{Q}[\sqrt{1-\gamma(t)^2}]\frac{\partial^2 P}{\partial z \partial \gamma} - r_d P = 0.
 \end{aligned} \tag{74}$$

By substituting (18) into the pricing PDE (74), we obtain the following PDEs in P_1 and P_2 respectively:

$$\begin{aligned}
 & \frac{\partial P_1}{\partial t} + \left(r_f + \frac{1}{2}V(t) - \beta(t)\mathbb{E}[\sqrt{V(t)}]\mathbb{E}[\sqrt{U(t)}]\right)\frac{\partial P_1}{\partial y} \\
 & + \left(r_d - r_f - \frac{1}{2}U(t) + \beta(t)\mathbb{E}[\sqrt{V(t)}]\mathbb{E}[\sqrt{U(t)}]\right)\frac{\partial P_1}{\partial z} \\
 & + \left(\kappa_\eta(\mu_\eta - \eta(t)) + \rho_{S\eta}\sigma_\eta \mathbb{E}[\sqrt{V(t)}]\mathbb{E}[\sqrt{1-\eta(t)^2}]\right)\frac{\partial P_1}{\partial \eta} \\
 & + \left(\kappa_v(\mu_v - V(t)) + \sigma_v \mathbb{E}[V(t)]\eta(t)\right)\frac{\partial P_1}{\partial v} + \kappa_u(\mu_u - U(t))\frac{\partial P_1}{\partial u} \\
 & + \left(\kappa_\beta(\mu_\beta - \beta(t)) + \rho_{S\beta}\sigma_\beta \mathbb{E}[\sqrt{V(t)}]\mathbb{E}[\sqrt{1-\beta(t)^2}]\right)\frac{\partial P_1}{\partial \beta} \\
 & + \kappa_\gamma(\mu_\gamma - \gamma(t))\frac{\partial P_1}{\partial \gamma} + \frac{1}{2}V(t)\frac{\partial^2 P_1}{\partial y^2} + \frac{1}{2}U(t)\frac{\partial^2 P_1}{\partial z^2} + \frac{1}{2}\sigma_v^2 V(t)\frac{\partial^2 P_1}{\partial v^2} \\
 & + \frac{1}{2}\sigma_u^2 U(t)\frac{\partial^2 P_1}{\partial u^2} + \frac{1}{2}\sigma_\beta^2 \left(1 - \mathbb{E}[\beta(t)^2]\right)\frac{\partial^2 P_1}{\partial \beta^2} + \frac{1}{2}\sigma_\eta^2 \left(1 - \mathbb{E}[\eta(t)^2]\right)\frac{\partial^2 P_1}{\partial \eta^2} \\
 & + \frac{1}{2}\sigma_\gamma^2 \left(1 - \mathbb{E}[\gamma(t)^2]\right)\frac{\partial^2 P_1}{\partial \gamma^2} + \beta(t)\mathbb{E}[\sqrt{V(t)}]\mathbb{E}[\sqrt{U(t)}]\frac{\partial^2 P_1}{\partial y \partial z} \\
 & + \sigma_v \mathbb{E}[V(t)]\eta(t)\frac{\partial^2 P_1}{\partial y \partial v} + \sigma_u \mathbb{E}[U(t)]\gamma(t)\frac{\partial^2 P_1}{\partial z \partial u} \\
 & + \rho_{S\beta}\sigma_\beta \mathbb{E}[\sqrt{V(t)}]\mathbb{E}[\sqrt{1-\beta(t)^2}]\frac{\partial^2 P_1}{\partial \beta \partial y} \\
 & + \rho_{S\eta}\sigma_\eta \mathbb{E}[\sqrt{V(t)}]\mathbb{E}[\sqrt{1-\eta(t)^2}]\frac{\partial^2 P_1}{\partial \eta \partial y} \\
 & + \rho_{X\beta}\sigma_\beta \mathbb{E}[\sqrt{U(t)}]\mathbb{E}[\sqrt{1-\beta(t)^2}]\frac{\partial^2 P_1}{\partial z \partial \beta} \\
 & + \rho_{X\gamma}\sigma_\gamma \mathbb{E}[\sqrt{U(t)}]\mathbb{E}[\sqrt{1-\gamma(t)^2}]\frac{\partial^2 P_1}{\partial z \partial \gamma} = 0,
 \end{aligned} \tag{75}$$

and

$$\begin{aligned}
 & \frac{\partial P_2}{\partial t} + \left(r_f - \frac{1}{2}V(t) - \beta(t) \mathbb{E}[\sqrt{V(t)}] \mathbb{E}[\sqrt{U(t)}] \right) \frac{\partial P_2}{\partial y} \\
 & + \left(r_d - r_f - \frac{1}{2}U(t) \right) \frac{\partial P_2}{\partial z} + \kappa_\eta (\mu_\eta - \eta(t)) \frac{\partial P_2}{\partial \eta} + \kappa_v (\mu_v - V(t)) \frac{\partial P_2}{\partial v} \\
 & + \kappa_u (\mu_u - U(t)) \frac{\partial P_2}{\partial u} + \kappa_\beta (\mu_\beta - \beta(t)) \frac{\partial P_2}{\partial \beta} + \kappa_\gamma (\mu_\gamma - \gamma(t)) \frac{\partial P_2}{\partial \gamma} \\
 & + \frac{1}{2}V(t) \frac{\partial^2 P_2}{\partial y^2} + \frac{1}{2}U(t) \frac{\partial^2 P_2}{\partial z^2} + \frac{1}{2}\sigma_v^2 V(t) \frac{\partial^2 P_2}{\partial v^2} + \frac{1}{2}\sigma_u^2 U(t) \frac{\partial^2 P_2}{\partial u^2} \\
 & + \frac{1}{2}\sigma_\beta^2 (1 - \mathbb{E}[\beta(t)^2]) \frac{\partial^2 P_2}{\partial \beta^2} + \frac{1}{2}\sigma_\eta^2 (1 - \mathbb{E}[\eta(t)^2]) \frac{\partial^2 P_2}{\partial \eta^2} \\
 & + \frac{1}{2}\sigma_\gamma^2 (1 - \mathbb{E}[\gamma(t)^2]) \frac{\partial^2 P_2}{\partial \gamma^2} + \beta(t) \mathbb{E}[\sqrt{V(t)}] \mathbb{E}[\sqrt{U(t)}] \frac{\partial^2 P_2}{\partial y \partial z} \\
 & + \sigma_v \mathbb{E}[V(t)] \eta(t) \frac{\partial^2 P_2}{\partial y \partial v} + \sigma_u \mathbb{E}[U(t)] \gamma(t) \frac{\partial^2 P_2}{\partial z \partial u} \\
 & + \rho_{S\beta} \sigma_\beta \mathbb{E}[\sqrt{V(t)}] \mathbb{E}[\sqrt{1 - \beta(t)^2}] \frac{\partial^2 P_2}{\partial \beta \partial y} \\
 & + \rho_{S\eta} \sigma_\eta \mathbb{E}[\sqrt{V(t)}] \mathbb{E}[\sqrt{1 - \eta(t)^2}] \frac{\partial^2 P_2}{\partial \eta \partial y} \\
 & + \rho_{X\beta} \sigma_\beta \mathbb{E}[\sqrt{U(t)}] \mathbb{E}[\sqrt{1 - \beta(t)^2}] \frac{\partial^2 P_2}{\partial z \partial \beta} \\
 & + \rho_{X\gamma} \sigma_\gamma \mathbb{E}[\sqrt{U(t)}] \mathbb{E}[\sqrt{1 - \gamma(t)^2}] \frac{\partial^2 P_2}{\partial z \partial \gamma} = 0.
 \end{aligned} \tag{76}$$

As pointed out in the previous section, we know that the corresponding characteristic functions $\phi = \phi_j(t, y, z, v, u, \beta, \eta, \gamma; \omega)$ of P_j , $j = 1, 2$, must also satisfy the PDEs (75) and (76), respectively. Their solutions can be found in a closed-form as stated in the next lemmas.

Lemma 4. *The characteristic function of P_1 in (18), under the assumption that the correlation processes $\beta(t), \eta(t)$ and $\gamma(t)$ follow a BJ process, is given by:*

$$\begin{aligned}
 \phi_1^{\text{BJ}} = & \exp(\tilde{A}_1(\tau, \omega) + B_1(\tau, \omega)v(t) + C_1(\tau, \omega)u(t) + D_1(\tau, \omega)z(t) \\
 & + E_1(\tau, \omega)\beta(t) + F_1(\tau, \omega)\eta(t) + G_1(\tau, \omega)\gamma(t) + i\omega y(t)),
 \end{aligned} \tag{77}$$

where B_1, C_1, D_1, E_1, F_1 and G_1 are as in Lemma 1, and where \tilde{A}_1 is given by:

$$\begin{aligned}
 \tilde{A}_1(\tau, \omega) = & r_f i \omega \tau + \kappa_v \mu_v H_1(\tau, \omega) + (\kappa_\eta \mu_\eta + (i\omega + 1) \rho_{S\eta} \sigma_\eta m_v \hat{q}_\eta) H_2(\tau, \omega) \\
 & + (i\omega + 1) \rho_{S\eta} \sigma_\eta (m_v H_3(\tau, \omega) + m_v \hat{r}_\eta H_4(\tau, \omega) \\
 & + n_v H_5(\tau, \omega) + n_v \hat{r}_\eta H_6(\tau, \omega) + n_v \hat{q}_\eta H_7(\tau, \omega)) \\
 & + (\kappa_\beta \mu_\beta + (i\omega + 1) \rho_{S\beta} \sigma_\beta m_v \hat{q}_\beta) H_8(\tau, \omega) \\
 & + (i\omega + 1) \rho_{S\beta} \sigma_\beta (m_v H_9(\tau, \omega) + m_v \hat{r}_\beta H_{10}(\tau, \omega) + n_v H_{11}(\tau, \omega)
 \end{aligned}$$

$$\begin{aligned}
 &+ n_v \hat{r}_\beta H_{12}(\tau, \omega) + n_v \hat{q}_\beta H_{13}(\tau, \omega) + \frac{1}{2} (\sigma_\beta^2 (1 - q_\beta) H_{14}(\tau, \omega) \\
 &+ \sigma_\eta^2 (1 - q_\eta) H_{15}(\tau, \omega)) - \frac{1}{2} (\sigma_\beta^2 (H_{16}(\tau, \omega) + r_\beta H_{17}(\tau, \omega)) \\
 &+ \sigma_\eta^2 (H_{18}(\tau, \omega) + r_\eta H_{19}(\tau, \omega))).
 \end{aligned} \tag{78}$$

The H functions in Equation (78) are defined as:

$$\begin{aligned}
 H_1(\tau, \omega) &= \int_0^\tau B_1(s, \omega) ds, \quad H_2(\tau, \omega) = \int_0^\tau F_1(s, \omega) ds, \\
 H_3(\tau, \omega) &= \hat{H}(\tau, \omega, \hat{s}_\eta), \quad H_4(\tau, \omega) = \hat{H}(\tau, \omega, \hat{p}_\eta), \\
 H_5(\tau, \omega) &= \hat{H}(\tau, \omega, l_v + \hat{s}_\eta), \quad H_6(\tau, \omega) = \hat{H}(\tau, \omega, l_v + \hat{p}_\eta), \\
 H_7(\tau, \omega) &= \hat{H}(\tau, \omega, l_v), \quad H_8(\tau, \omega) = \int_0^\tau E_1(s, \omega) ds, \\
 H_9(\tau, \omega) &= \bar{H}(\tau, \omega, \hat{s}_\beta), \quad H_{10}(\tau, \omega) = \bar{H}(\tau, \omega, \hat{p}_\beta), \\
 H_{11}(\tau, \omega) &= \bar{H}(\tau, \omega, l_v + \hat{s}_\beta), \quad H_{12}(\tau, \omega) = \bar{H}(\tau, \omega, l_v + \hat{p}_\beta), \\
 H_{13}(\tau, \omega) &= \bar{H}(\tau, \omega, l_v), \quad H_{14}(\tau, \omega) = \int_0^\tau E_1(s, \omega)^2 ds, \\
 H_{15}(\tau, \omega) &= \int_0^\tau F_1(s, \omega)^2 ds, \quad H_{16}(\tau, \omega) = H^*(\tau, \omega, s_\beta), \\
 H_{17}(\tau, \omega) &= H^*(\tau, \omega, p_\beta), \quad H_{18}(\tau, \omega) = H^\circ(\tau, \omega, s_\eta), \\
 H_{19}(\tau, \omega) &= H^\circ(\tau, \omega, p_\eta),
 \end{aligned} \tag{79}$$

where the functions \hat{H}, \bar{H}, H^* and H° are given by:

$$\begin{aligned}
 \hat{H}(\tau, \omega, a) &= \int_0^\tau e^{-a(T-s)} F_1(s, \omega) ds, \quad \bar{H}(\tau, \omega, a) = \int_0^\tau e^{-a(T-s)} E_1(s, \omega) ds, \\
 H^*(\tau, \omega, a) &= \int_0^\tau e^{-a(T-s)} E_1^2(s, \omega) ds, \quad H^\circ(\tau, \omega, a) = \int_0^\tau e^{-a(T-s)} F_1^2(s, \omega) ds,
 \end{aligned} \tag{80}$$

and all the other parameters have been already defined in Lemma 1.

Proof. See Appendix A.3.

Lemma 5. *The characteristic function of P_2 in (18), under the assumption that the correlation processes $\beta(t), \eta(t)$ and $\gamma(t)$ follow an BJ process, is given by:*

$$\begin{aligned}
 \phi_2^{BJ} &= \exp(\tilde{A}_2(\tau, \omega) + B_2(\tau, \omega)v(t) + C_2(\tau, \omega)u(t) + D_2(\tau, \omega)z(t) \\
 &+ E_2(\tau, \omega)\beta(t) + F_2(\tau, \omega)\eta(t) + G_2(\tau, \omega)\gamma(t) + i\omega y(t)),
 \end{aligned} \tag{81}$$

where B_2, C_2, D_2, E_2, F_2 and G_2 are as in Lemma 2, and where \tilde{A}_2 is given by:

$$\begin{aligned}
 \tilde{A}_2(\tau, \omega) &= r_i i \omega \tau + \kappa_v \mu_v \tilde{H}_1(\tau, \omega) + (\kappa_\eta \mu_\eta + i \omega \rho_{S\eta} \sigma_\eta m_v \hat{q}_\eta) \tilde{H}_2(\tau, \omega) \\
 &+ i \omega \rho_{S\eta} \sigma_\eta (m_v \tilde{H}_3(\tau, \omega) + m_v \hat{r}_\eta \tilde{H}_4(\tau, \omega)) \\
 &+ n_v \tilde{H}_5(\tau, \omega) + n_v \hat{r}_\eta \tilde{H}_6(\tau, \omega) + n_v \hat{q}_\eta \tilde{H}_7(\tau, \omega) \\
 &+ (\kappa_\beta \mu_\beta + i \omega \rho_{S\beta} \sigma_\beta m_v \hat{q}_\beta) \tilde{H}_8(\tau, \omega) \\
 &+ i \omega \rho_{S\beta} \sigma_\beta (m_v \tilde{H}_9(\tau, \omega) + m_v \hat{r}_\beta \tilde{H}_{10}(\tau, \omega) + n_v \tilde{H}_{11}(\tau, \omega))
 \end{aligned}$$

$$\begin{aligned}
 &+ n_v \hat{r}_\beta \tilde{H}_{12}(\tau, \omega) + n_v \hat{q}_\beta \tilde{H}_{13}(\tau, \omega) + \frac{1}{2} (\sigma_\beta^2 (1 - q_\beta) \tilde{H}_{14}(\tau, \omega) \\
 &+ \sigma_\eta^2 (1 - q_\eta) \tilde{H}_{15}(\tau, \omega)) - \frac{1}{2} (\sigma_\beta^2 (\tilde{H}_{16}(\tau, \omega) + r_\beta \tilde{H}_{17}(\tau, \omega)) \\
 &+ \sigma_\eta^2 (\tilde{H}_{18}(\tau, \omega) + r_\eta \tilde{H}_{19}(\tau, \omega))). \tag{82}
 \end{aligned}$$

The \tilde{H} functions in Equation (82) are defined as:

$$\begin{aligned}
 \tilde{H}_1(\tau, \omega) &= \int_0^\tau B_2(s, \omega) ds, \quad \tilde{H}_2(\tau, \omega) = \int_0^\tau F_2(s, \omega) ds, \\
 \tilde{H}_3(\tau, \omega) &= \hat{K}(\tau, \omega, \hat{s}_\eta), \quad \tilde{H}_4(\tau, \omega) = \hat{K}(\tau, \omega, \hat{p}_\eta), \\
 \tilde{H}_5(\tau, \omega) &= \hat{K}(\tau, \omega, l_v + \hat{s}_\eta), \quad \tilde{H}_6(\tau, \omega) = \hat{K}(\tau, \omega, l_v + \hat{p}_\eta), \\
 \tilde{H}_7(\tau, \omega) &= \hat{K}(\tau, \omega, l_v), \quad \tilde{H}_8(\tau, \omega) = \int_0^\tau E_2(s, \omega) ds, \\
 \tilde{H}_9(\tau, \omega) &= \bar{K}(\tau, \omega, \hat{s}_\beta), \quad \tilde{H}_{10}(\tau, \omega) = \bar{K}(\tau, \omega, \hat{p}_\beta), \\
 \tilde{H}_{11}(\tau, \omega) &= \bar{K}(\tau, \omega, l_v + \hat{s}_\beta), \quad \tilde{H}_{12}(\tau, \omega) = \bar{K}(\tau, \omega, l_v + \hat{p}_\beta), \\
 \tilde{H}_{13}(\tau, \omega) &= \bar{K}(\tau, \omega, l_v), \quad \tilde{H}_{14}(\tau, \omega) = \int_0^\tau E_2(s, \omega)^2 ds, \\
 \tilde{H}_{15}(\tau, \omega) &= \int_0^\tau F_2(s, \omega)^2 ds, \quad \tilde{H}_{16}(\tau, \omega) = K^*(\tau, \omega, s_\beta), \\
 \tilde{H}_{17}(\tau, \omega) &= K^*(\tau, \omega, p_\beta), \quad \tilde{H}_{18}(\tau, \omega) = K^\circ(\tau, \omega, s_\eta), \\
 \tilde{H}_{19}(\tau, \omega) &= K^\circ(\tau, \omega, p_\eta), \tag{83}
 \end{aligned}$$

where the functions \hat{K}, \bar{K}, K^* and K° are given by:

$$\begin{aligned}
 \hat{K}(\tau, \omega, a) &= \int_0^\tau e^{-a(T-s)} F_2(s, \omega) ds, \quad \bar{K}(\tau, \omega, a) = \int_0^\tau e^{-a(T-s)} E_2(s, \omega) ds, \\
 K^*(\tau, \omega, a) &= \int_0^\tau e^{-a(T-s)} E_2^2(s, \omega) ds, \quad K^\circ(\tau, \omega, a) = \int_0^\tau e^{-a(T-s)} F_2^2(s, \omega) ds, \tag{84}
 \end{aligned}$$

and where all the other parameters have been already defined in Lemma 2.

The proof of Lemma 5 is similar to the one for Lemma 4 and is left to the reader.

Lemma 6. Let

$$\xi = \int_t^T \beta(s) ds$$

be an integrated correlation process, where $\beta(t)$ follows a BJ process.

Then, we have that

$$\mathbb{E}^{\mathbb{Q}} \left[e^{-\sqrt{V(0)}\sqrt{U(0)} \int_0^t \beta(s) ds} \right] = e^{-\tilde{\delta}(t) - \beta(0)\tilde{\chi}(t)}, \tag{85}$$

with

$$\begin{aligned}
 \tilde{\delta}(t) &= \frac{\sqrt{V(0)}\sqrt{U(0)}}{\kappa_\beta} (1 - e^{-\kappa_\beta t}), \tag{86} \\
 \tilde{\chi}(t) &= \frac{V(0)U(0)\sigma_\beta^2}{2\kappa_\beta^2} \left(\frac{2(e^{-(\kappa_\beta + s_\beta)t} - 1)}{\kappa_\beta + s_\beta} + \frac{2r_\beta(e^{-(\kappa_\beta + p_\beta)t} - 1)}{\kappa_\beta + p_\beta} \right) \\
 &\quad - \frac{e^{(2\kappa_\beta + s_\beta)t} - 1}{2\kappa_\beta + s_\beta} - \frac{r_\beta(e^{-(2\kappa_\beta + p_\beta)t} - 1)}{2\kappa_\beta + p_\beta}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{2(q_\beta - 1)(e^{-\kappa_\beta t} - 1)}{\kappa_\beta} - \frac{(q_\beta - 1)(e^{-2\kappa_\beta t} - 1)}{2\kappa_\beta} \\
 & - \frac{e^{-s_\beta t} - 1}{s_\beta} - \frac{r_\beta(e^{-p_\beta t} - 1)}{p_\beta} + (q_\beta - 1)t \Bigg) \\
 & + \frac{\sqrt{V(0)}\sqrt{U(0)}t}{\kappa_\beta} + \frac{\sqrt{U(0)}\sqrt{V(0)}(e^{-\kappa_\beta t} - 1)}{\kappa_\beta^2},
 \end{aligned} \tag{87}$$

where q, r, s, p have been defined in Proposition 2.

5. Monte Carlo Model Simulation: A Numerical Scheme

In this section, we discuss how to simulate the paths for the model in Equation (7) in order to compute the price of FX quanto options applying Monte Carlo simulation. In particular, we need to generate random paths of

$$(U(t), V(t), \eta(t), \gamma(t), \beta(t), Z(t), Y(t)) \text{ for all } t \in \{t_i\}_{i=1}^N \equiv \mathcal{T}^4.$$

To be more precise, for an arbitrary time increment Δ , we need to generate a random sample of

$$(U(t + \Delta), V(t + \Delta), \eta(t + \Delta), \gamma(t + \Delta), \beta(t + \Delta), Z(t + \Delta), Y(t + \Delta)) \text{ for given } (U(t), V(t), \eta(t), \gamma(t), \beta(t), Y(t)).$$

Repeated application of the resulting one period scheme will generate a full path:

$$(U(t), V(t), \eta(t), \gamma(t), \beta(t), Z(t), Y(t)), t \in \mathcal{T},$$

for the model

$$\begin{cases}
 dU(t) = [\kappa_U(\mu_U - U(t)) - \lambda_U(t)]dt + \sigma_U\sqrt{U(t)}dW_U(t)^\mathbb{Q}, \\
 dV(t) = [\kappa_V(\mu_V - V(t)) - \lambda_V(t)]dt + \sigma_V\sqrt{V(t)}dW_V(t)^\mathbb{Q}, \\
 d\gamma(t) = (c(t, \gamma(t)) - \lambda_\gamma(t))dt + d(t, \gamma(t))dW_\gamma(t)^\mathbb{Q}, \gamma(0) \in [-1, 1], \\
 d\eta(t) = (a(t, \eta(t)) - \lambda_\eta(t))dt + b(t, \eta(t))dW_\eta(t)^\mathbb{Q}, \eta(0) \in [-1, 1], \\
 d\beta(t) = (f(t, \beta(t)) - \lambda_\beta(t))dt + g(t, \beta(t))dW_\beta(t)^\mathbb{Q}, \beta(0) \in [-1, 1], \\
 dZ(t) = \left(r_d - r_f - \frac{1}{2}U(t)\right)dt + \sqrt{U(t)}dW_X(t)^\mathbb{Q}, \\
 dY(t) = \left(r_f - \frac{1}{2}V(t) - \beta(t)\sqrt{V(t)}\sqrt{U(t)}\right)dt + \sqrt{V(t)}dW_S(t)^\mathbb{Q},
 \end{cases} \tag{88}$$

5.1. Cholesky Decomposition

As pointed out in [8], a straight discretization of the model in Equation (88) may lead to the problem of *leaking correlation*. To tackle this problem, we reformulated the system of SDEs (88) with respect to independent Brownian motions as defined below:

⁴As mentioned in Section 2, we apply the log-transform to the underlying asset S , $Y(t) = \ln S(t)$, and to the exchange rate X , $Z(t) = \ln X(t)$.

$$\begin{cases} dW_U(t) = d\hat{W}_U(t), dW_V(t) = d\hat{W}_V(t), dW_\gamma(t) = d\hat{W}_\gamma(t), dW_\eta(t) = d\hat{W}_\eta(t), dW_\beta(t) = d\hat{W}_\beta(t), \\ dW_X(t) = \gamma(t)d\hat{W}_U(t) + \rho_{X\gamma}d\hat{W}_\gamma(t) + \rho_{X\beta}d\hat{W}_\beta(t) + \sqrt{1 - \gamma(t)^2 - \rho_{X\gamma}^2 - \rho_{X\beta}^2}d\hat{W}_X(t), \\ dW_S(t) = \eta(t)d\hat{W}_V(t) + \rho_{S\beta}d\hat{W}_\beta(t) + \tilde{\rho}_{SX}d\hat{W}_X(t) + \sqrt{1 - \eta(t)^2 - \rho_{S\beta}^2 - \tilde{\rho}_{SX}^2}d\hat{W}_S(t), \end{cases} \quad (89)$$

where

$$\tilde{\rho}_{SX} \equiv \frac{\beta(t) - \rho_{X\beta}\rho_{S\beta}}{\sqrt{1 - \gamma(t)^2 - \rho_{X\gamma}^2 - \rho_{X\beta}^2}} \in [-1, 1], \quad (90)$$

under the conditions that

$$\gamma(t)^2 + \rho_{X\gamma}^2 + \rho_{X\beta}^2 < 1, \quad \eta(t)^2 + \rho_{S\beta}^2 + \tilde{\rho}_{SX}^2 < 1. \quad (91)$$

Therefore, the Cholesky decomposition of the correlation matrix R defined in Section 2 can be easily calculated. In the following sections we will discuss the discretization schemes for each of the processes in Equation (88).

5.2. Discretization Scheme for $U(t)$ and $V(t)$

To discretize the variance processes $U(t)$ and $V(t)$, we employ the full truncation scheme as in [9], which reads as:

$$\begin{cases} \hat{v}(t + \Delta) = v(t) + \mu_v \kappa_v \Delta - \kappa_v \Delta \max(v(t), 0) + \sigma_v \sqrt{\max(v(t), 0)} \sqrt{\Delta} Z_v, \\ v(t + \Delta) = \max(\hat{v}(t + \Delta), 0), \end{cases} \quad (92)$$

which is a valid alternative to the Quadratic-Exponential (QE) scheme proposed in [8].

5.3. Discretization Scheme for $\eta(t)$, $\gamma(t)$ and $\beta(t)$

Let us assume first that the correlation processes $\eta(t), \gamma(t)$ and $\beta(t)$ follow an OU process. It is well known that if the process $\rho(t)$ follows an OU dynamics

$$d\rho(t) = \kappa_\rho (\mu_\rho - \rho(t))dt + \sigma_\rho dW_\rho(t),$$

then its exact solution at time $t + \Delta$, given the information available at time t is given by:

$$\rho(t + \Delta) = \rho(t)e^{-\kappa_\rho \Delta} + \mu_\rho (1 - e^{-\kappa_\rho \Delta}) + \sigma_\rho \sqrt{\frac{1 - e^{-2\kappa_\rho \Delta}}{2\kappa_\rho}} Z_\rho, \quad (93)$$

where Z_ρ is a standard Gaussian random variable.

If we instead assume that the correlation processes $\gamma(t), \eta(t)$ and $\beta(t)$, follow a BJ dynamics, then its SDE is given by:

$$d\rho(t) = \kappa_\rho (\mu_\rho - \rho(t))dt + \sigma_\rho \sqrt{1 - \rho(t)^2} dW_\rho(t), \text{ for } \rho \in \{\gamma, \eta, \beta\}.$$

Given the lack of an explicit solution for the above SDE, we propose to discretize the dynamics of $\rho(t)$ via either the Euler scheme

$$\rho(t+\Delta) \approx \hat{\rho}(t+\Delta) = \kappa_\rho \mu_\rho \Delta + (1 - \kappa_\rho \Delta) \hat{\rho}(t) + \sigma_\rho \sqrt{1 - \hat{\rho}(t)^2} \sqrt{\Delta} Z_\rho, \quad (94)$$

or via the Milstein scheme

$$\begin{aligned} \rho(t+\Delta) &\approx \hat{\rho}(t+\Delta) \\ &= \kappa_\rho \mu_\rho \Delta + (1 - \kappa_\rho \Delta) \hat{\rho}(t) + \sigma_\rho \sqrt{1 - \hat{\rho}(t)^2} \sqrt{\Delta} Z_\rho \\ &\quad - \frac{1}{2} \sigma_\rho^2 \hat{\rho}(t) \Delta (Z_\rho^2 - 1). \end{aligned} \quad (95)$$

5.4. Discretization Scheme for $Y(t)$ and $Z(t)$

To discretize the dynamics for the log-underlying price $Y(t) = \ln S(t)$ and for the log-FX rate $Z(t) = \ln X(t)$, we employ the Euler scheme based on the Cholesky decomposition presented in Section 5.1. In particular, we have that

$$\begin{aligned} \hat{Z}(t+\Delta) &= \hat{Z}(t) + \left(r_d - r_f - \frac{1}{2} \hat{U}(t) \right) \Delta \\ &\quad + \sqrt{\hat{U}(t)} \sqrt{\Delta} \left(\hat{\gamma}(t) \hat{Z}_U + \rho_{X_Y} \hat{Z}_Y + \rho_{X_\beta} \hat{Z}_\beta + \sqrt{1 - \hat{\gamma}(t)^2 - \rho_{X_Y}^2 - \rho_{X_\beta}^2} \hat{Z}_X \right), \end{aligned} \quad (96)$$

and

$$\begin{aligned} \hat{Y}(t+\Delta) &= \hat{Y}(t) + \left(r_f - \frac{1}{2} \hat{V}(t) - \hat{\beta}(t) \sqrt{\hat{U}(t)} \sqrt{\hat{V}(t)} \right) \Delta \\ &\quad + \sqrt{\hat{V}(t)} \sqrt{\Delta} \left(\hat{\eta}(t) \hat{Z}_V + \rho_{S_\beta} \hat{Z}_\beta + \tilde{\rho}_{S_X} \hat{Z}_X + \sqrt{1 - \hat{\eta}(t)^2 - \rho_{S_\beta}^2 - \tilde{\rho}_{S_X}^2} \hat{Z}_S \right). \end{aligned} \quad (97)$$

We conclude this section by remarking that, although the FX rate does not explicitly enter in the payoff of a quanto option, we do still need to simulate it, as the standard Gaussian random variable \hat{Z}_X enters explicitly in the dynamics for the log-underlying price Y , as shown in Equation (97).

6. Numerical Experiments

In this section we compare the option prices using the closed-form approximation Formula (18) to the prices computed by performing a Monte Carlo simulation according to the numerical scheme presented in Section 5. In particular, we have considered quanto call options on a foreign stock priced in the domestic currency, where the majority of the model parameters are defined at the bottom of **Table 1**.

In **Table 1** we have reported the results using different stochastic processes specification (namely the Ornstein-Uhlenbeck and the bounded Jacobi processes) and for different strikes. Six different scenarios have been considered under the following assumptions (as said all the other model parameters are at the bottom of the table and they have been kept fixed along the scenarios):

- 1) $\beta(0) = \mu_\beta = 0$ and $\rho_{S_\beta} = \rho_{X_\beta} = \rho_{S_\eta} = \rho_{X_Y} = 0$,
- 2) $\beta(0) = 0, \mu_\beta = 0.5$ and $\rho_{S_\beta} = \rho_{X_\beta} = \rho_{S_\eta} = \rho_{X_Y} = 0$,
- 3) $\beta(0) = 0, \mu_\beta = -0.5$ and $\rho_{S_\beta} = \rho_{X_\beta} = \rho_{S_\eta} = \rho_{X_Y} = 0$,
- 4) $\beta(0) = \mu_\beta = 0$, $\rho_{S_\beta} = \rho_{X_\beta} = 0.5$ and $\rho_{S_\eta} = \rho_{X_Y} = 0$,

Table 1. The other parameters are assumed as: $S(0)=100$, $X(0)=1$, $r_d=0.03$, $r_f=0.05$, $T=1$, $U(0)=V(0)=0.02$, $\mu_U=\mu_V=0.03$, $\kappa_U=\kappa_V=2.1$, $\sigma_U=\sigma_V=0.1$, and $\eta(0)=\gamma(0)=-0.2$, $\mu_\eta=\mu_\gamma=-0.3$, $\kappa_\eta=\kappa_\gamma=3.4$, $\sigma_\eta=\sigma_\gamma=0.1$. The numbers in round brackets represent the standard deviations. The number of Monte Carlo simulations is equal to 100,000.

Scenario	Model Parameters			OU Stochastic Correlation			BJ Stochastic Correlation					
	$\beta(0)$	μ_β	Strike	MC Price	Approx	Rel Error	MC Price	Approx	Rel Error			
1	0	0.0	80	24.6479 (0.0506)	24.6472	-0.00%	24.6483 (0.0506)	24.6476	-0.00%			
			85	20.1757 (0.0489)	20.1426	-0.16%	20.1761 (0.0489)	20.1430	-0.16%			
			90	16.0172 (0.0463)	15.8726	-0.91%	16.0177 (0.0463)	15.8731	-0.91%			
			95	12.2973 (0.0423)	12.0183	-2.32%	12.2976 (0.0427)	12.0186	-2.32%			
			100	9.1217 (0.0384)	8.7531	-4.21%	9.1221 (0.0384)	8.7535	-4.21%			
			105	6.5438 (0.0335)	6.1505	-6.40%	6.5440 (0.0335)	6.1507	-6.39%			
			110	4.5422 (0.0285)	4.1780	-8.72%	4.5425 (0.0285)	4.1784	-8.71%			
			115	3.0571 (0.0236)	2.7525	-11.07%	3.0573 (0.0236)	2.7527	-11.07%			
2	0	0.5	80	23.8150 (0.0498)	23.8102	-0.02%	23.8138 (0.0498)	23.8090	-0.02%			
			85	19.3741 (0.0480)	19.3008	-0.38%	19.3729 (0.0480)	19.2995	-0.38%			
			90	15.2703 (0.0453)	15.0387	-1.54%	15.2693 (0.0453)	15.0376	-1.54%			
			95	11.6337 (0.0416)	11.2522	-3.39%	11.6329 (0.0416)	11.2513	-3.39%			
			100	8.5556 (0.0371)	8.0958	-5.68%	8.5549 (0.0371)	8.0949	-5.68%			
			105	6.0753 (0.0321)	5.6123	-8.25%	6.0748 (0.0321)	5.6117	-8.25%			
			110	4.1717 (0.0271)	3.7607	-10.93%	4.1713 (0.0271)	3.7602	-10.93%			
			115	2.7783 (0.0223)	2.4455	-13.61%	2.7781 (0.0223)	2.4452	-13.61%			
3	0	-0.5	80	25.4883 (0.0514)	25.4847	-0.01%	25.4900 (0.0514)	25.4864	-0.01%			
			85	20.9794 (0.0498)	20.9712	-0.04%	20.9811 (0.0498)	20.9728	-0.04%			
			90	16.773 (0.0473)	16.7199	-0.32%	16.7746 (0.0473)	16.7217	-0.32%			
			95	12.9872 (0.0439)	12.8213	-1.29%	12.9885 (0.0439)	12.8226	-1.29%			
			100	9.7213 (0.0396)	9.4572	-2.79%	9.7224 (0.0396)	9.4585	-2.79%			
			105	7.0384 (0.0348)	6.7269	-4.63%	7.0394 (0.0348)	6.7281	-4.63%			
			110	4.9338 (0.0298)	4.6267	-6.64%	4.9345 (0.0298)	4.6275	-6.63%			
			115	3.3552 (0.0249)	3.0870	-8.69%	3.3557 (0.0249)	3.0876	-8.68%			
4	0.5	0.5	80	24.6494 (0.0506)	24.6487	-0.00%	24.6492 (0.0506)	24.6485	-0.00%			
			85	20.1723 (0.0489)	20.1386	-0.02%	20.1718 (0.0489)	20.1380	-0.18%			
			90	16.0182 (0.0463)	15.8726	-0.92%	16.0172 (0.0463)	15.8712	-0.92%			
			95	12.3098 (0.0427)	12.0310	-2.32%	12.3086 (0.0427)	12.0292	-2.32%			
			100	9.1403 (0.0383)	8.7727	-4.19%	9.1389 (0.0383)	8.7705	-4.20%			
				$\rho_{S\beta}$	$\rho_{x\beta}$	Strike	MC Price	Approx	Rel Error	MC Price	Approx	Rel Error

Continued

			105	6.5570 (0.0334)	6.1636	-6.38%	6.5555 (0.0334)	6.1614	-6.40%
			110	4.5517 (0.0284)	4.1869	-8.71%	4.5502 (0.0284)	4.1847	-8.73%
			115	3.0590 (0.0235)	2.7530	-11.12%	3.0576 (0.0235)	2.7510	-11.14%
			120	1.9949 (0.0191)	1.7552	-13.65%	1.9935 (0.0191)	1.7534	-13.69%
5	-0.5	-0.5	80	24.6577 (0.0508)	24.6571	-0.00%	24.6594 (0.0508)	24.6588	-0.00%
			85	20.1758 (0.0492)	20.1431	-0.16%	20.1776 (0.0492)	20.1451	-0.16%
			90	16.0215 (0.0465)	15.8785	-0.90%	16.0236 (0.0466)	15.8811	-0.90%
			95	12.3172 (0.0430)	12.0432	-2.27%	12.3196 (0.0430)	12.0464	-2.27%
			100	9.1541 (0.0386)	8.7932	-4.10%	9.1568 (0.0386)	8.7969	-4.09%
			105	6.5792 (0.0337)	6.1939	-6.22%	6.5818 (0.0337)	6.1974	-6.20%
			110	4.5789 (0.0286)	4.2220	-8.45%	4.5811 (0.0286)	4.2249	-8.43%
			115	3.0899 (0.0238)	2.7908	-10.72%	3.0919 (0.0238)	2.7934	-10.69%
			120	2.0307 (0.0194)	1.7970	-13.01%	2.0324 (0.0194)	1.7991	-12.96%
	$\rho_{S\eta}$	$\rho_{X\gamma}$	Strike	MC Price	Approx	Rel Error	MC Price	Approx	Rel Error
6	-0.5	-0.5	80	24.6515 (0.0506)	24.6508	-0.00%	24.6515 (0.0506)	24.6508	-0.00%
			85	20.1751 (0.0490)	20.1416	-0.16%	20.1748 (0.0490)	20.1412	-0.18%
			90	16.0203 (0.0463)	15.8749	-0.92%	16.0195 (0.0463)	15.8738	-0.93%
			95	12.3113 (0.0427)	12.0326	-2.32%	12.3101 (0.0427)	12.0308	-2.32%
			100	9.1420 (0.0383)	8.7744	-4.19%	9.1406 (0.0383)	8.7723	-4.20%
			105	6.5574 (0.0335)	6.1638	-6.39%	6.5562 (0.0334)	6.1619	-6.40%
			110	4.5516 (0.0284)	4.1866	-8.72%	4.5562 (0.0284)	4.1900	-8.74%
			115	3.0605 (0.0236)	2.7545	-11.11%	3.0592 (0.0236)	2.7527	-11.14%
			120	1.9973 (0.0192)	1.7578	-13.62%	1.9959 (0.0192)	1.7560	-13.66%

5) $\beta(0) = \mu_\beta = 0$, $\rho_{S\beta} = \rho_{X\beta} = -0.5$ and $\rho_{S\eta} = \rho_{X\gamma} = 0$,

6) $\beta(0) = \mu_\beta = 0$, $\rho_{S\beta} = \rho_{X\beta} = 0.5$ and $\rho_{S\eta} = \rho_{X\gamma} = -0.5$.

As pointed out in [2] and in [1], one should choose a large value for the mean reversion rate of the correlation processes and a small value for the volatility coefficient in order to ensure that the generated correlations by the OU process lie in the interval $(-1, 1)$, while for the BJ process the condition in Equation (58) has to be fulfilled. Besides that, care has to be taken also in the choice of the other model parameters for the stochastic processes (namely the initial value and the long term mean level) because of the conditions implied by the Cholesky decomposition, see Equation (91). We remark that in all the scenarios considered in this Section all the model constraints have been taken into account.

If we look at the relative errors reported in Table 1, we can see that the ap-

proximations in both models give accurate results, but the accuracy of the approximation decreases when passing from in-the-money to out-of-the-money options. If we consider Scenarios 1 - 3 first, we can see the effect of changing the long-term mean level for the correlation process $\beta(t)$ on the option value. Indeed, the fact that the price in Scenario 2 (resp. 3) is lower (higher) than the one in Scenario 1 can be explained from the fact that the drift of the underlying process decreases (increases), under the same volatility assumption. If we compare Scenarios 4 and 5 with Scenario 1, we can see that increasing (decreasing) the correlations between the stochastic drivers for the underlying price (resp. FX rate) and the correlation process β does not have a huge impact on the option price. This has been already noted in [2] and it is due to the fact that we are pricing the option under a low volatility regime for the correlation process β . Also, this explains as well why the option price given by the two model specifications for the correlation process are very close. The same conclusion holds for Scenario 6 where we have assumed a non-zero correlation between the stochastic drivers for the underlying price (resp. FX rate) and the correlation process η (resp. γ).

7. Conclusion and Future Work

In this paper we have incorporated a stochastic correlation structure into the pricing of FX quanto options where both the dynamics for the underlying asset and for the exchange rate are given by a stochastic volatility model. This has been done not only assuming a stochastic correlation between the underlying asset and its variance process (and the same between the exchange rate and its variance process), but also assuming a stochastic correlation between the underlying asset and the exchange rate. In particular, under the assumption that the set of stochastic correlation processes follow an Ornstein-Uhlenbeck and a bounded Jacobi process respectively, we have derived a closed-form approximation for the characteristic function of the underlying asset, by approximating non-affine terms in the model dynamics. The comparison of both approximations with the Monte Carlo method has also been discussed. Given the fact that the analytical tractability of the formulas allows for fast pricing and calibration purposes, improvements of the present work could be to focus on the calibration of the model parameters by looking at real market quotes from the FX market and also to run more scenario analysis under stressed market conditions. We leave them as future work.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

A.1. The Proof of Proposition 1

In order to find a first order approximation for $\mathbb{E}[\sqrt{\nu(t)}]$ we can apply the so-called *delta method* which states that a function $\phi(X)$ can be approximated by a first order Taylor expansion around $\mathbb{E}[X]$, for a given random variable X , with expectation $\mathbb{E}[X]$ and variance $\text{Var}[X]$, under the assumption that the first two moments of X exist and that the first derivative of $\phi(X)$ with respect to X exists and is sufficiently smooth.

Therefore, we obtain by first order Taylor expansion:

$$\phi(X) \approx \phi(\mathbb{E}[X]) + (X - \mathbb{E}[X]) \frac{\partial}{\partial X} \phi(\mathbb{E}[X]). \quad (98)$$

Now, the variance of $\phi(X)$ can be approximated by taking the variance of the right-hand side of Equation (98), which reads as follows:

$$\begin{aligned} \text{Var}[\phi(X)] &\approx \text{Var}\left[\phi(\mathbb{E}[X]) + (X - \mathbb{E}[X]) \frac{\partial}{\partial X} \phi(\mathbb{E}[X])\right] \\ &= \left(\frac{\partial}{\partial X} \phi(\mathbb{E}[X])\right)^2 \text{Var}[X]. \end{aligned} \quad (99)$$

Now, if we apply this result to the function $\phi(\nu(t)) = \sqrt{\nu(t)}$, we have that

$$\text{Var}[\sqrt{\nu(t)}] \approx \left(\frac{1}{2\sqrt{\mathbb{E}[\nu(t)]}}\right)^2 \text{Var}[\nu(t)] = \frac{1}{4} \frac{\text{Var}[\nu(t)]}{\mathbb{E}[\nu(t)]}. \quad (100)$$

Besides it, we know from the definition of variance that

$$\text{Var}[\sqrt{\nu(t)}] = \mathbb{E}[\nu(t)] - (\mathbb{E}[\nu(t)])^2, \quad (101)$$

and combining Equations (100) and (101), we obtain the following approximation for $\mathbb{E}[\sqrt{\nu(t)}]$:

$$\mathbb{E}[\sqrt{\nu(t)}] \approx \sqrt{\mathbb{E}[\nu(t)] - \frac{1}{4} \frac{\text{Var}[\nu(t)]}{\mathbb{E}[\nu(t)]}}. \quad (102)$$

Since $\nu(t)$ is a square root process, then its explicit solution at time t is given by:

$$\nu(t) = \nu(0)e^{-\kappa_v t} + \mu_v (1 - e^{-\kappa_v t}) + \sigma_v \int_0^t e^{\kappa_v(s-t)} \sqrt{\nu(s)} dW_v(s). \quad (103)$$

Further calculations show that the expectation and the variance of $\nu(t)$ can be expressed as follows:

$$\mathbb{E}[\nu(t)] = \tilde{a}_v(t) (\tilde{b}_v + \tilde{c}_v(t)), \quad \text{Var}[\nu(t)] = \tilde{a}_v(t)^2 (2\tilde{b}_v + 4\tilde{c}_v(t)), \quad (104)$$

where the coefficients $\tilde{a}_v(t), \tilde{b}_v, \tilde{c}_v(t)$ are given by:

$$\tilde{a}_v(t) = \frac{\sigma_v^2}{4\kappa_v} (1 - e^{-\kappa_v t}), \quad \tilde{b}_v = \frac{4\kappa_v \mu_v}{\sigma_v^2}, \quad \tilde{c}_v(t) = \frac{4\kappa_v \nu(0) e^{-\kappa_v t}}{\sigma_v^2 (1 - e^{-\kappa_v t})}. \quad (105)$$

Now, substituting these expressions in Equation (102) gives

$$\begin{aligned} \mathbb{E}[\sqrt{v(t)}] &\approx \sqrt{\tilde{a}_v(t)(\tilde{b}_v + \tilde{c}_v(t)) - \frac{1}{4} \frac{\tilde{a}_v(t)^2 (2\tilde{b}_v + 4\tilde{c}_v(t))}{\tilde{a}_v(t)(\tilde{b}_v + \tilde{c}_v(t))}} \\ &= \sqrt{\tilde{a}_v(t)(\tilde{c}_v(t) - 1) + \tilde{a}_v(t)\tilde{b}_v + \frac{\tilde{a}_v(t)\tilde{b}_v}{2(\tilde{b}_v + \tilde{c}_v(t))}}. \end{aligned} \tag{106}$$

Finally, if we denote by $\Lambda(t)$ the above expectation, then $\Lambda(t)$ can be further approximated by a function of the following form:

$$\Lambda(t) \approx \tilde{\Lambda}(t) = m_v + n_v e^{-l_v t}, \tag{107}$$

where the coefficients m_v, n_v, l_v are found by matching the functions $\Lambda(t)$ and $\tilde{\Lambda}(t)$ for $t \rightarrow +\infty$, $t \rightarrow 0$ and for $t \rightarrow 1$.

In particular, it can be proved that (see Result 3.3 in [5])

$$\begin{cases} \lim_{t \rightarrow +\infty} \Lambda(t) = \sqrt{\mu_v - \frac{\sigma_v^2}{8\kappa_v}} = m_v = \lim_{t \rightarrow +\infty} \tilde{\Lambda}(t), \\ \lim_{t \rightarrow 0} \Lambda(t) = \sqrt{v(0)} = m_v + n_v = \lim_{t \rightarrow 0} \tilde{\Lambda}(t), \\ \lim_{t \rightarrow 1} \Lambda(t) = \Lambda(1) = m_v + n_v e^{-l_v} = \lim_{t \rightarrow 1} \tilde{\Lambda}(t), \end{cases} \tag{108}$$

from which it follows that

$$m_v = \sqrt{\mu_v - \frac{\sigma_v^2}{8\kappa_v}}, \quad n_v = \sqrt{v_0} - m_v, \quad l_v = -\ln\left(\frac{\hat{d}_v - m_v}{n_v}\right), \tag{109}$$

where we have defined

$$\hat{d}_v = \sqrt{\left(v_0 e^{-\kappa_v} - \frac{\sigma_v^2 (1 - e^{-\kappa_v})}{4\kappa_v}\right) + \mu_v (1 - e^{-\kappa_v}) + \frac{\sigma_v^2 \mu_v (1 - e^{-\kappa_v})^2}{8\kappa_v \mu_v + 8\kappa_v e^{-\kappa_v} (v_0 - \mu_v)}}. \tag{110}$$

This completes the proof.

A.2. The Proof of Lemma 1

Since the system generating the pricing PDE for P_1 (30) is of affine form, we look for a solution $\phi_1 = \phi_1(\tau, \omega)$ of the following form:

$$\begin{aligned} \phi_1^{\text{OU}} &= \exp(A_1(\tau, \omega) + B_1(\tau, \omega)v(t) + C_1(\tau, \omega)u(t) + D_1(\tau, \omega)z(t) \\ &\quad + E_1(\tau, \omega)\beta(t) + F_1(\tau, \omega)\eta(t) + G_1(\tau, \omega)\gamma(t) + i\omega y(t)), \end{aligned} \tag{111}$$

with

$$A_1(0, \omega) = B_1(0, \omega) = C_1(0, \omega) = D_1(0, \omega) = E_1(0, \omega) = F_1(0, \omega) = G_1(0, \omega) = 0.$$

Substituting all the partial derivatives of ϕ_1 into (30) gives:

$$\begin{aligned} &-\left(\frac{\partial A_1}{\partial \tau} + v \frac{\partial B_1}{\partial \tau} + u \frac{\partial C_1}{\partial \tau} + z \frac{\partial D_1}{\partial \tau} + \beta \frac{\partial E_1}{\partial \tau} + \eta \frac{\partial F_1}{\partial \tau} + \gamma \frac{\partial G_1}{\partial \tau}\right) \\ &+ \left(r_f + \frac{1}{2}v - \beta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{u}]\right) i\omega + \left(r_d - r_f - \frac{1}{2}u + \beta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{u}]\right) D_1 \\ &+ \left(\kappa_\eta (\mu_\eta - \eta) + \rho_{S\eta} \sigma_\eta \mathbb{E}[\sqrt{v}]\right) F_1 + \left(\kappa_v (\mu_v - v) + \sigma_v \mathbb{E}[v] \eta\right) B_1 \\ &+ \kappa_u (\mu_u - u) C_1 + \left(\kappa_\beta (\mu_\beta - \beta) + \rho_{S\beta} \sigma_\beta \mathbb{E}[\sqrt{v}]\right) E_1 + \kappa_\gamma (\mu_\gamma - \gamma) G_1 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\omega^2v + \frac{1}{2}uD_1^2 + \frac{1}{2}\sigma_v^2vB_1^2 + \frac{1}{2}\sigma_u^2uC_1^2 + \frac{1}{2}\sigma_\beta^2E_1^2 + \frac{1}{2}\sigma_\eta^2F_1^2 + \frac{1}{2}\sigma_\gamma^2G_1^2 \\
& + \beta\mathbb{E}[\sqrt{v}][\sqrt{u}]i\omega D_1 + \sigma_v\mathbb{E}[v]\eta i\omega B_1 + \sigma_u\mathbb{E}[u]\gamma D_1 C_1 \\
& + \rho_{S\beta}\sigma_\beta\mathbb{E}[\sqrt{v}]i\omega E_1 + \rho_{S\eta}\sigma_\eta\mathbb{E}[\sqrt{v}]i\omega F_1 + \rho_{X\beta}\sigma_\beta\mathbb{E}[\sqrt{u}]D_1 E_1 \\
& + \rho_{X\gamma}\sigma_\gamma\mathbb{E}[\sqrt{u}]D_1 G_1 = 0.
\end{aligned} \tag{112}$$

Now, collecting together homogenous terms in $y, z, v, u, \beta, \eta, \gamma$ gives the following system of equations:

$$\frac{\partial B_1}{\partial \tau} + \kappa_v B_1 - \frac{1}{2}\sigma_v^2 B_1^2 + \frac{1}{2}\omega(\omega - i) = 0, \tag{113}$$

$$\frac{\partial C_1}{\partial \tau} + \kappa_u C_1 - \frac{1}{2}\sigma_u^2 C_1^2 - \frac{1}{2}D_1(D_1 - 1) = 0, \tag{114}$$

$$\frac{\partial D_1}{\partial \tau} = 0, \tag{115}$$

$$\frac{\partial E_1}{\partial \tau} + \kappa_\beta E_1 - \mathbb{E}[\sqrt{u}]\mathbb{E}[\sqrt{v}](i\omega(D_1 - 1) + D_1) = 0, \tag{116}$$

$$\frac{\partial F_1}{\partial \tau} + \kappa_\eta F_1 - \sigma_v\mathbb{E}[v]B_1(i\omega + 1) = 0, \tag{117}$$

$$\frac{\partial G_1}{\partial \tau} + \kappa_\gamma G_1 - \sigma_u\mathbb{E}[u]D_1 C_1 = 0, \tag{118}$$

and

$$\begin{aligned}
\frac{\partial A_1}{\partial \tau} = & r_f i\omega + \left(\kappa_\eta \mu_\eta + \rho_{S\eta} \sigma_\eta \mathbb{E}[\sqrt{v}]\right)(i\omega + 1)F_1 + \kappa_v \mu_v B_1 + \kappa_u \mu_u C_1 \\
& + \left(\kappa_\beta \mu_\beta + \rho_{S\beta} \sigma_\beta \mathbb{E}[\sqrt{v}]\right)(i\omega + 1)E_1 + \kappa_\gamma \mu_\gamma G_1 + \frac{1}{2}\sigma_\beta^2 E_1^2 + \frac{1}{2}\sigma_\eta^2 F_1^2 \\
& + \frac{1}{2}\sigma_\gamma^2 G_1^2 + \rho_{X\beta} \sigma_\beta \mathbb{E}[\sqrt{u}]D_1 E_1 + \rho_{X\gamma} \sigma_\gamma \mathbb{E}[\sqrt{u}]D_1 G_1 + (r_d - r_f)D_1.
\end{aligned} \tag{119}$$

From Equation (115), due to the initial condition $D_1(0, \omega) = 0$, we obtain $D_1(\tau, \omega) \equiv 0$, which is consistent with the fact that the option payoff does not depend explicitly on the exchange rate $Z(t) = \ln X(t)$.

The fact that $D_1(\tau, \omega) = 0$ implies that also $G_1(\tau, \omega) = 0$, because of the initial condition $G_1(0, \omega) = 0$.

Calculation of $B_1(\tau, \omega)$

Equation (113) is a Riccati equation, whose corresponding second order ordinary differential equation (ODE) reads as follows:

$$\theta'' + \kappa_v \theta' - \frac{1}{4}\sigma_v^2 \omega(\omega - i) = 0. \tag{120}$$

The solution of the above equation is:

$$\theta_{1,2} = \frac{-\kappa_v \pm \sqrt{\kappa_v^2 + \sigma_v^2 \omega(\omega - i)}}{2},$$

and therefore, the solution of the Riccati Equation (113) is given by:

$$B_1(\tau, \omega) = -\frac{2}{\sigma_V^2} \left(\frac{\theta_3 \theta_1 e^{\theta_1 \tau} + \theta_2 e^{\theta_2 \tau}}{\theta_3 e^{\theta_1 \tau} + e^{\theta_2 \tau}} \right).$$

Using the initial condition $B(0, \omega) = 0$, we find that the constant is equal to $\theta_3 = -\frac{\theta_2}{\theta_1}$.

Further calculations show that $B_1(\tau, \omega)$ can be expressed as follows:

$$B_1(\tau, \omega) = -\frac{2\theta_2}{\sigma_V^2} \frac{1 - e^{(\theta_1 - \theta_2)\tau}}{1 - \frac{\theta_2}{\theta_1} e^{(\theta_1 - \theta_2)\tau}}. \tag{121}$$

Now, it can be shown that

$$\frac{\theta_2}{\theta_1} = \bar{B}_2 = \frac{-\kappa_V + \sqrt{\kappa_V^2 + \sigma_V^2 \omega(\omega - i)}}{\kappa_V + \sqrt{\kappa_V^2 + \sigma_V^2 \omega(\omega - i)}} = \frac{-\kappa_V + \bar{B}_1}{\kappa_V + \bar{B}_1}, \tag{122}$$

$$\theta_1 - \theta_2 = -\bar{B}_1, \quad -\frac{2\theta_2}{\sigma_V^2} = \frac{\kappa_V - \bar{B}_1}{\sigma_V^2}, \quad \text{with } \bar{B}_1 = \sqrt{\kappa_V^2 + \sigma_V^2 \omega(\omega - i)}, \tag{123}$$

and hence $B_1(\tau, \omega)$ reads as:

$$B_1(\tau, \omega) = \frac{\kappa_V - \bar{B}_1}{\sigma_V^2} \frac{1 - e^{-\bar{B}_1 \tau}}{1 - \bar{B}_2 e^{-\bar{B}_1 \tau}}. \tag{124}$$

Calculation of $C_1(\tau, \omega)$

If we focus now on Equation (114), this is again a Riccati-type equation in C_1 and its corresponding second order ODE reads as:

$$\theta'' + \kappa_U \theta' = 0, \tag{125}$$

and $\theta_1 = -\kappa_U, \theta_2 = 0$.

A general solution for C_1 can therefore be written as follows:

$$C_1(\tau, \omega) = -\frac{2}{\sigma_U^2} \frac{\theta_3 \theta_1 e^{\theta_1 \tau} + \theta_2 e^{\theta_2 \tau}}{\theta_1 e^{\theta_1 \tau} + e^{\theta_2 \tau}}, \tag{126}$$

but due to the initial condition $C_1(0, \omega) = 0$, we have $\theta_3 = 0$, and hence $C_1(\tau, \omega) = 0$.

Calculation of $E_1(\tau, \omega)$

Equation (116) can be re-written as follows:

$$\frac{\partial E_1}{\partial \tau} + \kappa_\beta E_1 + \mathbb{E}[\sqrt{u}] \mathbb{E}[\sqrt{v}] i \omega = 0,$$

where we know that (see Proposition 1)

$$\mathbb{E}[\sqrt{v}(t)] \approx m_V + n_V e^{-l_V(T-t)} \quad \text{and} \quad \mathbb{E}[\sqrt{u}(t)] \approx m_U + n_U e^{-l_U(T-t)}.$$

Since

$$\begin{aligned} &\mathbb{E}[\sqrt{v}(t)] \mathbb{E}[\sqrt{u}(t)] \\ &= m_V m_U + m_V n_U e^{-l_U(T-t)} + m_U n_V e^{-l_V(T-t)} + n_U n_V e^{-(l_U + l_V)(T-t)}, \end{aligned} \tag{127}$$

the solution for the first-order ODE in E_1 reads as:

$$E_1(\tau, \omega) = (-i\omega) e^{-\kappa_\beta \tau} \int_0^\tau e^{\kappa_\beta s} \left(m_V m_U + m_V n_U e^{-l_U(T-s)} + m_U n_V e^{-l_V(T-s)} + n_U n_V e^{-(l_U+l_V)(T-s)} \right) ds. \tag{128}$$

Further calculations show that

$$E_1(\tau, \omega) = -i\omega \left[\frac{m_V m_U}{\kappa_\beta} \left(1 - e^{-\kappa_\beta \tau} \right) + \frac{m_V n_U}{\kappa_\beta + l_U} e^{-l_U(T-\tau)} \left(1 - e^{-(\kappa_\beta + l_U)\tau} \right) + \frac{m_U n_V}{\kappa_\beta + l_V} e^{-l_V(T-\tau)} \left(1 - e^{-(\kappa_\beta + l_V)\tau} \right) + \frac{n_U n_V}{\kappa_\beta + l_U + l_V} e^{-(l_U+l_V)(T-\tau)} \left(1 - e^{-(\kappa_\beta + l_U + l_V)\tau} \right) \right]. \tag{129}$$

Calculation of $F_1(\tau, \omega)$

We look then at Equation (117):

$$\frac{\partial F_1}{\partial \tau} + \kappa_\eta F_1 = \sigma_V \mathbb{E}[v] B_1(i\omega + 1),$$

where we know that

$$\mathbb{E}[v(t)] = (v(0) - \mu_V) e^{-\kappa_V(T-t)} + \mu_V \text{ and } B_1(\tau, \omega) = \frac{\kappa_V - \bar{B}_1}{\sigma_V^2} \frac{1 - e^{-\bar{B}_1 \tau}}{1 - \bar{B}_2 e^{-\bar{B}_1 \tau}}.$$

As done in Teng *et al.* (2016) (Appendix B.1) we approximate the term

$$\frac{1 - e^{-\bar{B}_1 \tau}}{1 - \bar{B}_2 e^{-\bar{B}_1 \tau}} \approx 1 - e^{-\bar{B}_3 \tau}, \text{ with } \bar{B}_3 = -\ln \left(\frac{e^{-\bar{B}_1} - \bar{B}_2 e^{-\bar{B}_1}}{1 - \bar{B}_2 e^{-\bar{B}_1}} \right),$$

from which it follows that

$$\frac{\partial F_1}{\partial \tau} + \kappa_\eta F_1 = \frac{\kappa_V - \bar{B}_1}{\sigma_V} (i\omega + 1) (1 - e^{-\bar{B}_3 \tau}) (\mu_V + (v(0) - \mu_V) e^{-\kappa_V(T-\tau)}), \tag{130}$$

whose solution reads as (since we have $F_1(0, \omega) = 0$):

$$F_1(\tau, \omega) = \frac{(\mu_V - v(0)) \hat{B}_1}{\kappa_\eta + \kappa_V - \bar{B}_3} e^{(\kappa_V - \bar{B}_3)\tau - \kappa_V T} + \frac{(v(0) - \mu_V) \hat{B}_1}{\kappa_\eta + \kappa_V} e^{\kappa_V(\tau - T)} + \frac{\mu_V \hat{B}_1}{\kappa_\eta} - \frac{\mu_V \hat{B}_1}{\kappa_\eta - \bar{B}_3} e^{-\bar{B}_3 \tau} + \hat{B}_1 \hat{B}_2 e^{-\kappa_\eta \tau}, \tag{131}$$

with

$$\hat{B}_1 = \frac{i\omega + 1}{\sigma_V} (\kappa_V - \bar{B}_1), \tag{132}$$

and

$$\hat{B}_2 = -\frac{\mu_V}{\kappa_V} - \frac{v(0) - \mu_V}{\kappa_\eta + \kappa_V} e^{-\kappa_V T} - \frac{\mu_V - v(0)}{\kappa_\eta + \kappa_V - \bar{B}_3} e^{-\kappa_V T} + \frac{\mu_V}{\kappa_\eta - \bar{B}_3}. \tag{133}$$

Calculation of $A_1(\tau, \omega)$

The last term to be computed is A_1 :

$$\begin{aligned}
 A_1(\tau, \omega) = & r_i i \omega \tau + \int_0^\tau \left(\kappa_\eta \mu_\eta + (i\omega + 1) \rho_{S_\eta} \sigma_\eta \mathbb{E} \left[\sqrt{v(T-s)} \right] \right) F_1(s, \omega) ds \\
 & + \kappa_V \mu_V \int_0^\tau B_1(s, \omega) ds + \int_0^\tau \left(\kappa_\beta \mu_\beta + (i\omega + 1) \rho_{S_\beta} \sigma_\beta \mathbb{E} \left[\sqrt{v(T-s)} \right] \right) \\
 & \times E_1(s, \omega) ds + \frac{1}{2} \left[\sigma_\beta^2 \int_0^\tau E_1^2(s, \omega) ds + \sigma_\eta^2 \int_0^\tau F_1^2(s, \omega) ds \right].
 \end{aligned} \tag{134}$$

In particular, it can be proved (after tedious calculations) that

$$\kappa_V \mu_V \int_0^\tau B_1(s, \omega) ds = \frac{\kappa_V \mu_V}{\sigma_V^2} \left((\kappa_V - \bar{B}_1) \tau - 2 \ln \left(\frac{1 - \bar{B}_2 e^{-\bar{B}_1 \tau}}{1 - \bar{B}_2} \right) \right), \tag{135}$$

$$\begin{aligned}
 \kappa_\eta \mu_\eta \int_0^\tau F_1(s, \omega) ds = & \kappa_\eta \mu_\eta \left[\frac{(v(0) - \mu_V) \hat{B}_1 e^{-\kappa_V T}}{\kappa_\eta + \kappa_V - \bar{B}_3} \frac{1 - e^{(\kappa_V - \bar{B}_3) \tau}}{\kappa_V - \bar{B}_3} \right. \\
 & + \frac{(\mu_V - v(0)) \hat{B}_1 e^{-\kappa_V T}}{\kappa_\eta + \kappa_V} (1 - e^{-\kappa_V \tau}) + \frac{\mu_V \hat{B}_1}{\kappa_\eta} \tau \\
 & \left. - \frac{\mu_V \hat{B}_1}{\bar{B}_3 (\kappa_\eta - \bar{B}_3)} (1 - e^{-\bar{B}_3 \tau}) + \frac{\hat{B}_1 \hat{B}_2}{\kappa_\eta} (1 - e^{-\kappa_\eta \tau}) \right].
 \end{aligned} \tag{136}$$

Besides that, we notice that

$$\int_0^\tau \mathbb{E} \left[\sqrt{v(T-s)} \right] F_1(s, \omega) ds \approx m_V \int_0^\tau F_1(s, \omega) ds + n_V \int_0^\tau e^{-l_V(T-s)} F_1(s, \omega) ds, \tag{137}$$

where

$$\begin{aligned}
 & \int_0^\tau e^{-l_V(T-s)} F_1(s, \omega) ds \\
 = & \frac{(\mu_V - v(0)) e^{-(\kappa_V + l_V) T}}{\kappa_\eta + \kappa_V - \bar{B}_3} \frac{e^{(l_V + \kappa_V - \bar{B}_3) \tau} - 1}{l_V + \kappa_V - \bar{B}_3} \\
 & + \frac{(v(0) - \mu_V) e^{-(l_V + \kappa_V) T}}{\kappa_\eta + \kappa_V} \frac{e^{(\kappa_V + l_V) \tau} - 1}{l_V + \kappa_V} \\
 & + \frac{\mu_V \hat{B}_1 e^{-l_V T}}{\kappa_\eta l_V} (e^{l_V \tau} - 1) - \frac{\mu_V \hat{B}_1 e^{-l_V T}}{\kappa_\eta - \bar{B}_3} \frac{e^{(l_V - \bar{B}_3) \tau} - 1}{l_V - \bar{B}_3} \\
 & + \frac{\hat{B}_1 \hat{B}_2 e^{-l_V T}}{l_V - \kappa_\eta} (e^{(l_V - \kappa_\eta) \tau} - 1).
 \end{aligned} \tag{138}$$

Now, further calculations show that

$$\begin{aligned}
 & \kappa_\beta \mu_\beta \int_0^\tau E_1(s, \omega) ds \\
 = & -i \omega \kappa_\beta \mu_\beta \left[\frac{m_U m_V}{\kappa_\beta} \left(\tau + \frac{e^{-\kappa_\beta \tau} - 1}{\kappa_\beta} \right) + \frac{m_U m_V}{\kappa_\beta + l_U} e^{-l_U T} \left(\frac{e^{l_U \tau} - 1}{l_U} \right. \right. \\
 & \left. \left. + \frac{e^{-\kappa_\beta \tau} - 1}{\kappa_\beta} \right) + \frac{m_U n_V}{\kappa_\beta + l_V} e^{-l_V T} \left(\frac{e^{l_V \tau} - 1}{l_V} + \frac{e^{-\kappa_\beta \tau} - 1}{\kappa_\beta} \right) \right. \\
 & \left. + \frac{n_U n_V}{\kappa_\beta + l_U + l_V} e^{-(l_U + l_V) T} \left(\frac{e^{(l_U + l_V) \tau} - 1}{l_U + l_V} + \frac{e^{-\kappa_\beta \tau} - 1}{\kappa_\beta} \right) \right],
 \end{aligned} \tag{139}$$

and

$$\int_0^\tau \mathbb{E} \left[\sqrt{v(T-s)} \right] E_1(s, \omega) ds \approx m_v \int_0^\tau E_1(s, \omega) ds + n_v \int_0^\tau e^{-l_v(T-s)} E_1(s, \omega) ds, \quad (140)$$

where

$$\begin{aligned} & \int_0^\tau e^{-l_v(T-s)} E_1(s, \omega) ds \\ &= -(i\omega) \left[\frac{m_v m_U}{\kappa_\beta} e^{-l_v T} \left(\frac{e^{l_v \tau} - 1}{l_v} - \frac{e^{(l_v - \kappa_\beta) \tau} - 1}{l_v - \kappa_\beta} \right) \right. \\ & \quad + \frac{m_v n_U}{\kappa_\beta + l_U} e^{-(l_U + l_v) T} \left(\frac{e^{(l_U + l_v) \tau} - 1}{l_U + l_v} - \frac{e^{(l_v - \kappa_\beta) \tau} - 1}{l_v - \kappa_\beta} \right) \\ & \quad + \frac{m_U n_v}{\kappa_\beta + l_v} e^{-2l_v T} \left(\frac{e^{2l_v \tau} - 1}{2l_v} - \frac{e^{(l_v - \kappa_\beta) \tau} - 1}{l_v - \kappa_\beta} \right) \\ & \quad \left. + \frac{n_U n_v}{\kappa_\beta + l_U + l_v} e^{-(l_U + 2l_v) T} \left(\frac{e^{(l_U + 2l_v) \tau} - 1}{l_U + 2l_v} - \frac{e^{(l_v - \kappa_\beta) \tau} - 1}{l_v - \kappa_\beta} \right) \right]. \end{aligned} \quad (141)$$

Finally, it can be proved that

$$\begin{aligned} & \int_0^\tau E_1(s, \omega)^2 ds \\ &= -\omega^2 \left[\frac{m_v^2 m_U^2}{\kappa_\beta^2} \left(\tau + \frac{1 - e^{-2\kappa_\beta \tau}}{2\kappa_\beta} - \frac{2}{\kappa_\beta} (1 - e^{-\kappa_\beta \tau}) \right) + \frac{m_v^2 n_U^2}{(\kappa_\beta + \kappa_U)^2} e^{-2l_U T} \right. \\ & \quad \times \left(\frac{1}{2l_U} (e^{2l_U \tau} - 1) - \frac{1}{2\kappa_\beta} (e^{-2\kappa_\beta \tau} - 1) - \frac{2}{\kappa_\beta} (e^{\kappa_\beta \tau} - 1) \right) \\ & \quad + \frac{m_U^2 n_v^2}{(\kappa_\beta + l_v)^2} e^{-2l_v T} \left(\frac{e^{2l_v \tau} - 1}{2l_v} - \frac{1}{2\kappa_\beta} (e^{-2\kappa_\beta \tau} - 1) \right. \\ & \quad \left. - \frac{2}{l_v - \kappa_\beta} (e^{(l_v - \kappa_\beta) \tau} - 1) \right) + \frac{n_U^2 n_v^2}{(\kappa_\beta + l_U + l_v)^2} e^{-2(l_U + l_v) T} \left(\frac{e^{2(l_U + l_v) \tau} - 1}{2(l_U + l_v)} \right. \\ & \quad \left. - \frac{e^{-2\kappa_\beta \tau} - 1}{2\kappa_\beta} - \frac{2}{l_v + l_U - \kappa_\beta} (e^{(l_v + l_U - \kappa_\beta) \tau} - 1) \right) \\ & \quad + 2 \frac{m_v^2 m_U n_U}{\kappa_\beta (\kappa_\beta + l_U)} e^{-l_U T} \left(\frac{e^{l_U \tau} - 1}{l_U} + \frac{e^{-\kappa_\beta} - 1}{\kappa_\beta} - \frac{e^{(l_U - \kappa_\beta) \tau} - 1}{l_U - \kappa_\beta} - \frac{e^{-2\kappa_\beta \tau} - 1}{2\kappa_\beta} \right) \\ & \quad + 2 \frac{m_v m_U^2 n_v}{\kappa_\beta (\kappa_\beta + l_v)} e^{-l_v T} \left(\frac{e^{l_v \tau} - 1}{l_v} + \frac{e^{-\kappa_\beta \tau} - 1}{\kappa_\beta} - \frac{e^{(l_v - \kappa_\beta) \tau} - 1}{l_v - \kappa_\beta} - \frac{e^{-2\kappa_\beta \tau} - 1}{2\kappa_\beta} \right) \\ & \quad + 2 \frac{m_v m_U n_v n_U}{\kappa_\beta (\kappa_\beta + l_U + l_v)} e^{-(l_U + l_v) T} \left(\frac{e^{(l_U + l_v) \tau} - 1}{l_U + l_v} - \frac{e^{(l_U + l_v - \kappa_\beta) \tau} - 1}{l_U + l_v - \kappa_\beta} + \frac{e^{-\kappa_\beta \tau} - 1}{\kappa_\beta} \right. \\ & \quad \left. - \frac{e^{-2\kappa_\beta \tau} - 1}{2\kappa_\beta} \right) + 2 \frac{m_v n_U m_U n_v}{(\kappa_\beta + l_U) (\kappa_\beta + l_v)} e^{-(l_U + l_v) T} \left(\frac{e^{(l_U + l_v) \tau} - 1}{l_U + l_v} - \frac{e^{(l_v - \kappa_\beta) \tau} - 1}{l_v - \kappa_\beta} \right. \end{aligned}$$

$$\begin{aligned}
 & \left. - \frac{e^{(l_U - \kappa_\beta)\tau} - 1}{l_U - \kappa_\beta} - \frac{e^{-2\kappa_\beta\tau} - 1}{2\kappa_\beta} \right) + 2 \frac{m_V n_U^2 n_V}{(\kappa_\beta + l_U)(\kappa_\beta + l_U + l_V)} e^{-(2l_U + l_V)\tau} \\
 & \times \left(\frac{e^{(2l_U + l_V)\tau} - 1}{2l_U + l_V} - \frac{e^{(l_U - \kappa_\beta)\tau} - 1}{l_U - \kappa_\beta} - \frac{e^{(l_U + l_V - \kappa_\beta)\tau} - 1}{l_U + l_V - \kappa_\beta} - \frac{e^{-2\kappa_\beta\tau} - 1}{2\kappa_\beta} \right) \\
 & + 2 \frac{m_U n_U n_V^2}{(\kappa_\beta + l_V)(\kappa_\beta + l_U + l_V)} e^{-(l_U + 2l_V)\tau} \left[\frac{e^{(l_U + 2l_V)\tau} - 1}{l_U + 2l_V} - \frac{e^{(l_U + l_V - \kappa_\beta)\tau} - 1}{l_U + l_V - \kappa_\beta} \right. \\
 & \left. - \frac{e^{(l_V - \kappa_\beta)\tau} - 1}{l_V - \kappa_\beta} - \frac{e^{-2\kappa_\beta\tau} - 1}{2\kappa_\beta} \right], \tag{142}
 \end{aligned}$$

and that

$$\begin{aligned}
 & \int_0^\tau F_1(s, \omega)^2 ds \\
 & = \frac{(\mu_V - v(0))^2}{(\kappa_\eta + \kappa_V - \bar{B}_3)^2} \hat{B}_1^2 e^{-2\kappa_V T} \frac{e^{2(\kappa_V - \bar{B}_3)\tau} - 1}{2(\kappa_V - \bar{B}_3)} + \frac{(v(0) - \mu_V)^2}{(\kappa_\eta + \kappa_V)^2} \hat{B}_1^2 e^{-2\kappa_V T} \frac{e^{2\kappa_V\tau} - 1}{2\kappa_V} \\
 & + \frac{\mu_V^2 \hat{B}_1^2}{\kappa_\eta^2} \tau - \frac{\mu_V^2 \hat{B}_1^2}{(\kappa_\eta - \bar{B}_3)^2} \frac{e^{-2\bar{B}_3\tau} - 1}{2\bar{B}_3} - \frac{\hat{B}_1^2 \hat{B}_2^2}{2\kappa_\eta} (e^{-2\kappa_\eta\tau} - 1) \\
 & + 2 \frac{(\mu_V - v(0))(v(0) - \mu_V) \hat{B}_1^2}{(\kappa_\eta + \kappa_V - \bar{B}_3)(\kappa_V + \kappa_\eta)} \frac{e^{2(\kappa_V - \bar{B}_3)\tau} - 1}{2(\kappa_V - \bar{B}_3)} + 2 \frac{\mu_V (\mu_V - v(0))}{\kappa_\eta (\kappa_\eta + \kappa_V - \bar{B}_3)} \hat{B}_1^2 e^{-\kappa_V T} \\
 & \times \frac{e^{(\kappa_V - \bar{B}_3)\tau} - 1}{\kappa_V - \bar{B}_3} - 2 \frac{\mu_V (\mu_V - v(0)) \hat{B}_1 e^{-\kappa_V T}}{(\kappa_\eta - \bar{B}_3)(\kappa_\eta + \kappa_V - \bar{B}_3)} \frac{e^{(\kappa_V - 2\bar{B}_3)\tau} - 1}{\kappa_V - 2\bar{B}_3} \\
 & + 2 \frac{(\mu_V - v(0)) \hat{B}_1^2 \hat{B}_2 e^{-\kappa_V T}}{\kappa_\eta + \kappa_V - \bar{B}_3} \frac{e^{(\kappa_V - \bar{B}_3 - \kappa_\eta)\tau} - 1}{\kappa_V - \bar{B}_3 - \kappa_\eta} + 2 \frac{\mu_V (v(0) - \mu_V) \hat{B}_1^2 e^{-\kappa_V T}}{\kappa_\eta (\kappa_\eta + \kappa_V)} \frac{e^{\kappa_V\tau} - 1}{\kappa_V} \\
 & - 2 \frac{\mu_V (v(0) - \mu_V) \hat{B}_1^2 e^{-\kappa_V T}}{(\kappa_\eta + \kappa_V)(\kappa_\eta - \bar{B}_3)} \frac{e^{(\kappa_V - \bar{B}_3)\tau} - 1}{\kappa_V - \bar{B}_3} + 2 \frac{(v(0) - \mu_V) \hat{B}_1^2 \hat{B}_2 e^{-\kappa_V T}}{\kappa_V + \kappa_\eta} \frac{e^{(\kappa_V - \kappa_\eta)\tau} - 1}{\kappa_V - \kappa_\eta} \\
 & - 2 \frac{\mu_V^2 \hat{B}_1^2}{\kappa_\eta (\kappa_\eta - \bar{B}_3)} \frac{e^{-\bar{B}_3\tau} - 1}{\bar{B}_3} - 2 \frac{\mu_V \hat{B}_1^2 \hat{B}_2}{\kappa_\eta} \frac{e^{-\kappa_\eta\tau} - 1}{\kappa_\eta} + 2 \frac{\mu_V \hat{B}_1^2 \hat{B}_2}{\kappa_\eta - \bar{B}_3} \frac{e^{-\bar{B}_3\tau} - 1}{\bar{B}_3}. \tag{143}
 \end{aligned}$$

Collecting all the terms together completes the proof.

A.3. The Proof of Lemma 4

Since the system generating the pricing PDE for P_1 (75) is of affine form, we look for a solution $\phi_1 = \phi_1(\tau, \omega)$ of the following form:

$$\begin{aligned}
 \phi_1^{\text{BJ}} = & \exp(T_1(\tau, \omega) + L_1(\tau, \omega)v(t) + M_1(\tau, \omega)u(t) + N_1(\tau, \omega)z(t) \\
 & + Q_1(\tau, \omega)\beta(t) + R_1(\tau, \omega)\eta(t) + S_1(\tau, \omega)\gamma(t) + i\omega y(t)), \tag{144}
 \end{aligned}$$

with

$$T_1(0, \omega) = L_1(0, \omega) = M_1(0, \omega) = N_1(0, \omega) = Q_1(0, \omega) = R_1(0, \omega) = S_1(0, \omega) = 0$$

Substituting all the partial derivatives of ϕ_1 into (75) gives:

$$\begin{aligned}
 & -\left(\frac{\partial T_1}{\partial \tau} + v \frac{\partial L_1}{\partial \tau} + u \frac{\partial M_1}{\partial \tau} + z \frac{\partial N_1}{\partial \tau} + \beta \frac{\partial Q_1}{\partial \tau} + \eta \frac{\partial R_1}{\partial \tau} + \gamma \frac{\partial S_1}{\partial \tau}\right) \\
 & + \left(r_f + \frac{1}{2}v - \beta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{u}]\right) i\omega + \left(r_d - r_f - \frac{1}{2}u + \beta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{u}]\right) N_1 \\
 & + \left(\kappa_\eta (\mu_\eta - \eta) + \rho_{s\eta} \sigma_\eta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{1-\eta^2}]\right) R_1 + \left(\kappa_v (\mu_v - v) + \sigma_v \mathbb{E}[v] \eta\right) L_1 \\
 & + \kappa_u (\mu_u - u) M_1 + \left(\kappa_\beta (\mu_\beta - \beta) + \rho_{s\beta} \sigma_\beta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{1-\beta^2}]\right) Q_1 \\
 & + \kappa_\gamma (\mu_\gamma - \gamma) S_1 - \frac{1}{2} \omega^2 v + \frac{1}{2} u N_1^2 + \frac{1}{2} \sigma_v^2 v L_1^2 + \frac{1}{2} \sigma_u^2 u M_1^2 + \frac{1}{2} \sigma_\beta^2 (1 - \mathbb{E}[\beta^2]) Q_1^2 \\
 & + \frac{1}{2} \sigma_\eta^2 (1 - \mathbb{E}[\eta^2]) R_1^2 + \frac{1}{2} \sigma_\gamma^2 (1 - \mathbb{E}[\gamma^2]) S_1^2 + \beta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{u}] i\omega N_1 \\
 & + \sigma_v \mathbb{E}[v] \eta i\omega L_1 + \sigma_u \mathbb{E}[u] \gamma N_1 M_1 + \rho_{s\beta} \sigma_\beta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{1-\beta^2}] i\omega Q_1 \\
 & + \rho_{s\eta} \sigma_\eta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{1-\eta^2}] i\omega R_1 + \rho_{x\beta} \sigma_\beta \mathbb{E}[\sqrt{u}] \mathbb{E}[\sqrt{1-\beta^2}] N_1 Q_1 \\
 & + \rho_{x\gamma} \sigma_\gamma \mathbb{E}[\sqrt{u}] \mathbb{E}[\sqrt{1-\gamma^2}] N_1 S_1 = 0.
 \end{aligned} \tag{145}$$

Now, collecting together homogenous terms in $y, z, v, u, \beta, \eta, \gamma$ gives the following system of equations:

$$\frac{\partial L_1}{\partial \tau} + \kappa_v L_1 - \frac{1}{2} \sigma_v^2 L_1^2 + \frac{1}{2} \omega(\omega - i) = 0, \tag{146}$$

$$\frac{\partial M_1}{\partial \tau} + \kappa_u M_1 - \frac{1}{2} \sigma_u^2 M_1^2 - \frac{1}{2} N_1 (N_1 - 1) = 0, \tag{147}$$

$$\frac{\partial N_1}{\partial \tau} = 0, \tag{148}$$

$$\frac{\partial Q_1}{\partial \tau} + \kappa_\beta Q_1 - \mathbb{E}[\sqrt{u}] \mathbb{E}[\sqrt{v}] (i\omega (N_1 - 1) + N_1) = 0, \tag{149}$$

$$\frac{\partial R_1}{\partial \tau} + \kappa_\eta R_1 - \sigma_v \mathbb{E}[v] L_1 (i\omega + 1) = 0, \tag{150}$$

$$\frac{\partial S_1}{\partial \tau} + \kappa_\gamma S_1 - \sigma_u \mathbb{E}[u] N_1 M_1 = 0, \tag{151}$$

and

$$\begin{aligned}
 \frac{\partial T_1}{\partial \tau} &= r_f i\omega + \left(\kappa_\eta \mu_\eta + \rho_{s\eta} \sigma_\eta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{1-\eta^2}]\right) (i\omega + 1) R_1 + \kappa_v \mu_v L_1 \\
 &+ \kappa_u \mu_u M_1 + \left(\kappa_\beta \mu_\beta + \rho_{s\beta} \sigma_\beta \mathbb{E}[\sqrt{v}] \mathbb{E}[\sqrt{1-\beta^2}]\right) (i\omega + 1) Q_1 \\
 &+ \kappa_\gamma \mu_\gamma S_1 + \frac{1}{2} \sigma_\beta^2 (1 - \mathbb{E}[\beta^2]) Q_1^2 + \frac{1}{2} \sigma_\eta^2 (1 - \mathbb{E}[\eta^2]) R_1^2 \\
 &+ \frac{1}{2} \sigma_\gamma^2 (1 - \mathbb{E}[\gamma^2]) S_1^2 + \rho_{x\beta} \sigma_\beta \mathbb{E}[\sqrt{u}] \mathbb{E}[\sqrt{1-\beta^2}] N_1 Q_1 \\
 &+ \rho_{x\gamma} \sigma_\gamma \mathbb{E}[\sqrt{u}] \mathbb{E}[\sqrt{1-\gamma^2}] N_1 S_1 + (r_d - r_f) N_1.
 \end{aligned} \tag{152}$$

If we compare Equations (146)-(151) with Equations (113)-(118), we can conclude that

$$\begin{aligned}
 L_1(\tau, \omega) &\equiv B_1(\tau, \omega), M_1(\tau, \omega) \equiv C_1(\tau, \omega), N_1(\tau, \omega) \equiv D_1(\tau, \omega), \\
 Q_1(\tau, \omega) &\equiv E_1(\tau, \omega), R(\tau, \omega) \equiv F_1(\tau, \omega) \text{ and } G_1(\tau, \omega) \equiv S_1(\tau, \omega),
 \end{aligned}
 \tag{153}$$

and therefore the only term to be calculated is $T_1(\tau, \omega)$, which can be re-written as follows:

$$\begin{aligned}
 T_1(\tau, \omega) &= r_i i \omega \tau + \int_0^\tau \left(\kappa_\eta \mu_\eta + (i\omega + 1) \rho_{S\eta} \sigma_\eta \mathbb{E} \left[\sqrt{v(T-s)} \right] \mathbb{E} \left[\sqrt{1-\eta(T-s)^2} \right] \right) \\
 &\quad \times F_1(s, \omega) ds + \kappa_v \mu_v \int_0^\tau B_1(s, \omega) ds \\
 &\quad + \int_0^\tau \left(\kappa_\beta \mu_\beta + (i\omega + 1) \rho_{S\beta} \sigma_\beta \mathbb{E} \left[\sqrt{v(T-s)} \right] \mathbb{E} \left[\sqrt{1-\beta(T-s)^2} \right] \right) \\
 &\quad \times E_1(s, \omega) ds + \frac{1}{2} \left[\sigma_\beta^2 \int_0^\tau \left(1 - \mathbb{E} \left[\beta(T-s)^2 \right] \right) E_1^2(s, \omega) ds \right. \\
 &\quad \left. + \sigma_\eta^2 \int_0^\tau \left(1 - \mathbb{E} \left[\eta(T-s)^2 \right] \right) F_1^2(s, \omega) ds \right].
 \end{aligned}
 \tag{154}$$

Now, from Propositions 1, 2, and 3 respectively, we know that $\mathbb{E} \left[\sqrt{v} \right]$, $\mathbb{E} \left[\rho^2 \right]$, and $\mathbb{E} \left[\sqrt{1-\rho^2} \right]$, with $\rho \in \{ \beta, \eta \}$, can be approximated as follows:

$$\mathbb{E} \left[\sqrt{v(T-s)} \right] \approx m_v + n_v e^{-l_v(T-s)}, \tag{155}$$

$$\mathbb{E} \left[\sqrt{1-\beta(T-s)^2} \right] \approx e^{-\hat{s}_\beta(T-s)} + \hat{r}_\beta e^{-\hat{p}_\beta(T-s)} + \hat{q}_\beta, \tag{156}$$

$$\mathbb{E} \left[\sqrt{1-\eta(T-s)^2} \right] \approx e^{-\hat{s}_\eta(T-s)} + \hat{r}_\eta e^{-\hat{p}_\eta(T-s)} + \hat{q}_\eta, \tag{157}$$

$$\mathbb{E} \left[\beta(T-s)^2 \right] \approx e^{-s_\beta(T-s)} + r_\beta e^{-p_\beta(T-s)} + q_\beta, \tag{158}$$

$$\mathbb{E} \left[\eta(T-s)^2 \right] \approx e^{-s_\eta(T-s)} + r_\eta e^{-p_\eta(T-s)} + q_\eta, \tag{159}$$

where the coefficients $m_v, n_v, l_v, s_\rho, r_\rho, p_\rho, q_\rho, \hat{s}_\rho, \hat{r}_\rho, \hat{p}_\rho, \hat{q}_\rho$, for $\rho \in \{ \beta, \eta \}$, are defined in the relevant propositions.

Therefore, further calculations show that

$$\begin{aligned}
 &\int_0^\tau \mathbb{E} \left[\sqrt{v(T-s)} \right] \mathbb{E} \left[\sqrt{1-\eta^2(T-s)} \right] F_1(s, \omega) ds \\
 &= m_v \int_0^\tau e^{-\hat{s}_\eta(T-s)} F_1(s, \omega) ds + m_v \hat{r}_\eta \int_0^\tau e^{-\hat{p}_\eta(T-s)} F_1(s, \omega) ds \\
 &\quad + m_v \hat{q}_\eta \int_0^\tau F_1(s, \omega) ds + n_v \int_0^\tau e^{-(l_v + \hat{s}_\eta)(T-s)} F_1(s, \omega) ds \\
 &\quad + n_v \hat{r}_\eta \int_0^\tau e^{-(l_v + \hat{p}_\eta)(T-s)} F_1(s, \omega) ds + n_v \hat{q}_\eta \int_0^\tau e^{-l_v(T-s)} F_1(s, \omega) ds.
 \end{aligned}
 \tag{160}$$

Repeating the same calculation for the same integral with β and E_1 instead of η and F_1 gives

$$\begin{aligned}
 &\int_0^\tau \mathbb{E} \left[\sqrt{v(T-s)} \right] \mathbb{E} \left[\sqrt{1-\beta^2(T-s)} \right] E_1(s, \omega) ds \\
 &= m_v \int_0^\tau e^{-\hat{s}_\beta(T-s)} E_1(s, \omega) ds + m_v \hat{r}_\beta \int_0^\tau e^{-\hat{p}_\beta(T-s)} E_1(s, \omega) ds \\
 &\quad + m_v \hat{q}_\beta \int_0^\tau E_1(s, \omega) ds + n_v \int_0^\tau e^{-(l_v + \hat{s}_\beta)(T-s)} E_1(s, \omega) ds \\
 &\quad + n_v \hat{r}_\beta \int_0^\tau e^{-(l_v + \hat{p}_\beta)(T-s)} E_1(s, \omega) ds + n_v \hat{q}_\beta \int_0^\tau e^{-l_v(T-s)} E_1(s, \omega) ds.
 \end{aligned}
 \tag{161}$$

Besides it, it can be shown as well that

$$\begin{aligned} & \int_0^\tau \mathbb{E} \left[\beta(T-s)^2 \right] E_1^2(s, \omega) ds \\ &= \int_0^\tau e^{-s\beta(T-s)} E_1^2(s, \omega) ds + r_\beta \int_0^\tau e^{-p_\beta(T-s)} E_1^2(s, \omega) ds + q_b \int_0^\tau E_1^2(s, \omega) ds, \end{aligned} \tag{162}$$

and

$$\begin{aligned} & \int_0^\tau \mathbb{E} \left[\eta(T-s)^2 \right] F_1^2(s, \omega) ds \\ &= \int_0^\tau e^{-s\eta(T-s)} F_1^2(s, \omega) ds + r_\eta \int_0^\tau e^{-p_\eta(T-s)} F_1^2(s, \omega) ds + q_\eta \int_0^\tau F_1^2(s, \omega) ds. \end{aligned} \tag{163}$$

Hence, the only integrals to be computed are of the following form:

$$\int_0^\tau e^{-a(T-s)} E_1^2(s, \omega) ds \text{ and } \int_0^\tau e^{-a(T-s)} F_1^2(s, \omega) ds \tag{164}$$

since all the other integrals

$$\begin{aligned} & \int_0^\tau E_1(s, \omega) ds, \int_0^\tau F_1(s, \omega) ds, \int_0^\tau e^{-a(T-s)} E_1(s, \omega) ds, \\ & \int_0^\tau e^{-a(T-s)} F_1(s, \omega) ds, \int_0^\tau E_1^2(s, \omega) ds, \int_0^\tau F_1^2(s, \omega) ds \end{aligned} \tag{165}$$

have been already computed in Appendix A.2.

In particular, it can be proved that

$$\begin{aligned} & \int_0^\tau e^{-a(T-s)} F_1^2(s, \omega) ds \\ &= \frac{(\mu_V - v(0))^2}{(\kappa_\eta + \kappa_V - \bar{B}_3)^2} \hat{B}_1^2 e^{-(a+2\kappa_V)T} \frac{e^{(a+2(\kappa_V - \bar{B}_3))\tau} - 1}{a + 2(\kappa_V - \bar{B}_3)} \\ &+ \frac{(v(0) - \mu_V)^2}{(\kappa_\eta + \kappa_V)^2} \hat{B}_1^2 e^{-(a+2\kappa_V)T} \frac{e^{(a+2\kappa_V)\tau} - 1}{a + 2\kappa_V} + \frac{\mu_V^2}{\kappa_\eta^2} \hat{B}_1^2 e^{-aT} \frac{e^{a\tau} - 1}{a} \\ &+ \frac{\mu_V^2}{(\kappa_\eta - \bar{B}_3)^2} \hat{B}_1^2 e^{-aT} \frac{e^{(a-2\bar{B}_3)\tau} - 1}{a - 2\bar{B}_3} + \frac{\hat{B}_1^2 \hat{B}_2^2 e^{-aT} (e^{(a-\kappa_\eta)\tau} - 1)}{a - \kappa_\eta} \\ &+ 2 \frac{(\mu_V - v(0))(v(0) - \mu_V)}{(\kappa_\eta + \kappa_V)(\kappa_\eta + \kappa_V - \bar{B}_3)} \hat{B}_1^2 e^{-(a+2\kappa_V)T} \frac{e^{(a+2\kappa_V - \bar{B}_3)\tau} - 1}{a + 2\kappa_V - \bar{B}_3} \\ &+ 2 \frac{\mu_V - v(0)\mu_V}{\kappa_\eta(\kappa_\eta + \kappa_V - \bar{B}_3)} \hat{B}_1^2 e^{-(a+\kappa_V)T} \frac{e^{(a+\kappa_V - \bar{B}_3)\tau} - 1}{a + \kappa_V - \bar{B}_3} \\ &- 2 \frac{\mu_V(\mu_V - v(0))}{(\kappa_\eta - \bar{B}_3)(\kappa_\eta + \kappa_V - \bar{B}_3)} \hat{B}_1^2 e^{-(\kappa_V+a)T} \frac{e^{(a+\kappa_V - 2\bar{B}_3)\tau} - 1}{a + \kappa_V - 2\bar{B}_3} \\ &+ 2 \frac{\mu_V - v(0)}{\kappa_\eta + \kappa_V - \bar{B}_3} \hat{B}_1^2 \hat{B}_2^2 e^{-(a+\kappa_V)T} \frac{e^{(a+\kappa_V - \bar{B}_3 - \kappa_\eta)\tau} - 1}{a + \kappa_V - \bar{B}_3 - \kappa_\eta} \\ &+ 2 \frac{(v(0) - \mu_V)\mu_V}{\kappa_\eta(\kappa_V + \kappa_\eta)} \hat{B}_1^2 e^{-(a+\kappa_V)T} \frac{e^{a\tau} - 1}{a} - 2 \frac{\mu_V(v(0) - \mu_V)}{(\kappa_\eta + \kappa_V)(\kappa_\eta - \bar{B}_3)} \hat{B}_1^2 \\ &\times e^{-(a+\kappa_V)T} \frac{e^{(a+\kappa_V - \kappa_\eta)\tau} - 1}{a + \kappa_V - \kappa_\eta} + 2 \frac{v(0) - \mu_V}{\kappa_\eta + \kappa_V} \hat{B}_1^2 \hat{B}_2^2 e^{-(a+\kappa_V)T} \end{aligned}$$

$$\begin{aligned} & \times \frac{e^{(a+\kappa_V-\kappa_\eta)\tau} - 1}{a + \kappa_V - \kappa_\eta} - 2 \frac{\mu_V^2}{\kappa_\eta (\kappa_\eta - \bar{B}_3)} \hat{B}_1^2 e^{-aT} \frac{e^{(a-\bar{B}_3)\tau} - 1}{a - \bar{B}_3} \\ & + 2 \frac{\mu_V}{\kappa_\eta} \hat{B}_1^2 \hat{B}_2 e^{-aT} \frac{e^{(a-\kappa_\eta)\tau} - 1}{a - \kappa_\eta} - 2 \frac{\mu_V}{\kappa_\eta - \bar{B}_3} \hat{B}_1^2 \hat{B}_2 e^{-aT} \frac{e^{(a-\bar{B}_3-\kappa_\eta)\tau} - 1}{a - \bar{B}_3 - \kappa_\eta}. \end{aligned} \tag{166}$$

and that

$$\begin{aligned} & \int_0^\tau e^{-a(T-s)} E_1^2(s, \omega) ds \\ & = -\omega^2 \left[\frac{m_V^2 m_U^2}{\kappa_\beta^2} e^{-aT} \left(\frac{e^{a\tau} - 1}{a} + \frac{e^{(a-2\kappa_\beta)\tau} - 1}{a - 2\kappa_\beta} - 2 \frac{e^{(a-\kappa_\beta)\tau} - 1}{a - \kappa_\beta} \right) \right. \\ & \quad + \frac{m_V^2 n_U^2}{(\kappa_\beta + l_U)^2} e^{-(a+2l_U)T} \left(\frac{e^{(a+2l_U)\tau} - 1}{a + 2l_U} + \frac{e^{(a-2\kappa_\beta)\tau} - 1}{a - 2\kappa_\beta} - 2 \frac{e^{(a+l_U-\kappa_\beta)\tau} - 1}{a + l_U - \kappa_\beta} \right) \\ & \quad + \frac{m_U^2 n_V^2}{(\kappa_\beta + l_V)^2} e^{-(a+2l_V)T} \left(\frac{e^{(a+2l_V)\tau} - 1}{a + 2l_V} + \frac{e^{(a-2\kappa_\beta)\tau} - 1}{a - 2\kappa_\beta} - 2 \frac{e^{(a+l_V-\kappa_\beta)\tau} - 1}{a + l_V - \kappa_\beta} \right) \\ & \quad + \frac{n_V^2 n_U^2}{(\kappa_\beta + l_U + l_V)^2} e^{-(a+2(l_U+l_V))T} \left(\frac{e^{(a+2(l_U+l_V))\tau} - 1}{a + 2(l_U + l_V)} + \frac{e^{(a-2\kappa_\beta)\tau} - 1}{a - 2\kappa_\beta} \right. \\ & \quad \left. - 2 \frac{e^{(a+l_U+l_V-\kappa_\beta)\tau} - 1}{a + l_U + l_V - \kappa_\beta} \right) + 2 \frac{m_V^2 m_U n_U}{\kappa_\beta (\kappa_\beta + l_U)} e^{-(a+l_U)T} \left(\frac{e^{(a+l_U)\tau} - 1}{a + l_U} \right. \\ & \quad \left. - \frac{e^{(a-\kappa_\beta)\tau} - 1}{a - \kappa_\beta} - \frac{e^{(a+l_U-\kappa_\beta)\tau} - 1}{a + l_U - \kappa_\beta} + \frac{e^{(a-2\kappa_\beta)\tau} - 1}{a - 2\kappa_\beta} \right) + 2 \frac{m_V m_U^2 n_V}{\kappa_\beta (\kappa_\beta + l_V)} \\ & \quad \times e^{-(a+l_V)T} \left(\frac{e^{(a+l_V)\tau} - 1}{a + l_V} - \frac{e^{(a-\kappa_\beta)\tau} - 1}{a - \kappa_\beta} - \frac{e^{(a+l_V-\kappa_\beta)\tau} - 1}{a + l_V - \kappa_\beta} \right. \\ & \quad \left. + \frac{e^{(a-2\kappa_\beta)\tau} - 1}{a - 2\kappa_\beta} \right) + 2 \frac{m_U m_V n_V n_U}{\kappa_\beta (\kappa_\beta + l_U + l_V)} e^{-(a+l_U+l_V)T} \left(\frac{e^{(a+l_U+l_V)\tau} - 1}{a + l_U + l_V} \right. \\ & \quad \left. - \frac{e^{(a-\kappa_\beta)\tau} - 1}{a - \kappa_\beta} - \frac{e^{(a+l_U+l_V-\kappa_\beta)\tau} - 1}{a + l_U + l_V - \kappa_\beta} + \frac{e^{(a-2\kappa_\beta)\tau} - 1}{a - 2\kappa_\beta} \right) \\ & \quad + 2 \frac{m_V m_U n_V n_U}{(\kappa_\beta + l_U)(\kappa_\beta + l_V)} e^{-(a+l_U+l_V)T} \left(\frac{e^{(a+l_U+l_V)\tau} - 1}{a + l_U + l_V} \right. \\ & \quad \left. - \frac{e^{(a+l_U-\kappa_\beta)\tau} - 1}{a + l_U - \kappa_\beta} - \frac{e^{(a+l_V-\kappa_\beta)\tau} - 1}{a + l_V - \kappa_\beta} + \frac{e^{(a-2\kappa_\beta)\tau} - 1}{a - 2\kappa_\beta} \right) \\ & \quad + 2 \frac{m_V n_U^2 n_V}{(\kappa_\beta + l_U)(\kappa_\beta + l_U + l_V)} e^{-(a+2l_U+l_V)T} \left(\frac{e^{(a+2l_U+l_V)\tau} - 1}{a + 2l_U + l_V} \right. \\ & \quad \left. - \frac{e^{(a+l_U-\kappa_\beta)\tau} - 1}{a + l_U - \kappa_\beta} - \frac{e^{(a+l_U+l_V-\kappa_\beta)\tau} - 1}{a + l_U + l_V - \kappa_\beta} + \frac{e^{(a-2\kappa_\beta)\tau} - 1}{a - 2\kappa_\beta} \right) \end{aligned}$$

$$\begin{aligned}
& + 2 \frac{m_U n_V^2 n_U}{(\kappa_\beta + l_V)(\kappa_\beta + l_U + l_V)} e^{-(a+2l_V+l_U)T} \left(\frac{e^{(a+2l_V+l_U)\tau} - 1}{a + 2l_V + l_U} \right. \\
& \left. - \frac{e^{(a+l_V-\kappa_\beta)\tau} - 1}{a + l_V - \kappa_\beta} - \frac{e^{(a+l_U+l_V-\kappa_\beta)\tau} - 1}{a + l_U + l_V - \kappa_\beta} + \frac{e^{(a-2\kappa_\beta)\tau} - 1}{a - 2\kappa_\beta} \right) \Bigg], \tag{167}
\end{aligned}$$

Collecting all the terms together completes the proof.