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Optimal Reciprocal Reinsurance under GlueVaR Distortion Risk Measures

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Abstract

This article investigates the optimal reciprocal reinsurance strategies when the risk is measured by a general risk measure, namely the GlueVaR distortion risk measures, which can be expressed as a linear combination of two tail value at risk (TVaR) and one value at risk (VaR) risk measures. When we consider the reciprocal reinsurance, the linear combination of three risk measures can be difficult to deal with. In order to overcome difficulties, we give a new form of the GlueVaR distortion risk measures. This paper not only derives the necessary and sufficient condition that guarantees the optimality of marginal indemnification functions (MIF), but also obtains explicit solutions of the optimal reinsurance design. This method is easy to understand and can be simplified calculation. To further illustrate the applicability of our results, we give a numerical example.

Keywords

Distortion Risk Measure, VaR, TVaR, GlueVaR, Marginal Indemnification Function (MIF), Optimal Reciprocal Reinsurance

1. Introduction

Reinsurance is an effective risk management tool for the insurer to transfer part of its risk to the reinsurer. Let X be the original loss, if the insurer cedes a part of loss $f(X)$ (f is called the ceded loss function, or indemnification function) to the reinsurer and pays reinsurance premium $\delta_f(X)$, then the insurer's total liability $T_{I_f}(X)$ contains two parts: one is the retained loss risk $I_f(X)=X-f(X)$ and the other is the reinsurance premium $\delta_f(X)$, that is

$$T_{I_f}(X)=X-f(X)+\delta_f(X). \tag{1.1}$$

The reinsurer's total liability $T_{R_f}(X)$ also contains two parts: one is the ceded loss risk $R_f(X) = f(X)$ and the other is the received reinsurance premium $\delta_f(X)$, that is

$$T_{R_f}(X) = f(X) - \delta_f(X). \tag{1.2}$$

For any $\lambda \in [0,1]$, we define total risks $T_f(X)$ in the presence of an insurer and a reinsurer as

$$T_f(X) = \lambda T_{I_f}(X) + (1-\lambda) T_{R_f}(X). \tag{1.3}$$

Due to the development and application of risk measures in finance and insurance, many workers formulate the optimal reinsurance problem with Value at Risk (VaR) and Tail Value at Risk (TVaR). [1] proposed two optimization criterion that minimize total loss of the insurer by the Value at Risk (VaR) and the Conditional Tail Expectation (CTE). [2] showed that quota-share and stop-loss reinsurance are optimal when they studied a class of increasing convex ceded loss functions by VaR and CTE under the expected value principle. Many works extended the fundamental results, for example, [3]-[15]. [16] extended the conclusion of [15] to the general convex risk measure that satisfied regular invariance. Recently, there has a surge of interest in more generally distortion risk measures. [17] discussed the general model of the distortion risk measure and assumed that the distortion function is piecewise convex or concave. [18] studied the general model with distortion risk measures under general reinsurance premium principles. [19] expended the model of [18] under the cost-benefit framework. [20] studied the optimal reinsurance model of [18] without the premium constraint by a marginal indemnification function (MIF) formula. [21] studied the optimal reinsurance with premium constraint by combining the MIF formula and the Lagrangian dual method. [22] and [23] studied the optimal reinsurance with constraints under the distortion risk measure.

VaR has been adopted as the standard tool for assessing the risks and calculating the capital requirements in finance and insurance, however, it has two drawbacks in financial industry. One is that the capital requirements can be underestimated and the underestimated may be aggravated when heavy tail losses are incorrectly modeled by mild tail distribution. The second one is that the VaR may fail the subadditivity. Though TVaR has no these two disadvantages of VaR, it has not been widely accepted by practitioners in finance and insurance. In order to overcome this weakness, [24] proposed a new family of risk measures, namely GlueVaR distortion risk measures. We take different definitions of VaR from [24], therefore, a new definition of GlueVaR has been given in this paper.

Optimal reinsurance from an insurer's viewpoint or from a reinsurer's viewpoint has been studied for a long time in the literatures. However, as two parties of a reinsurance contract, there has a conflict of interests between an insurer and a reinsurer. The optimal reinsurance policy from one party's perspective may not be optimal for another party. Therefore, we consider a reciprocal reinsurance. Motivated by [21] and [24], we want to study the optimal reciprocal reinsurance strategy under GlueVaR distortion risk measures with MIF formula.

The rest of this paper is organized as follows. In Section 2, we give some notations and proposal a reciprocal reinsurance model. In Section 3, we derive the sufficient conditions that guarantee the existence of a reinsurance contract. In Section 4, we obtain the specific expression of optimal reinsurance. Section 5 concludes this paper.

2. The Model

2.1. Preliminaries and Notations

Definition 2.1. (*Distortion risk measure or distorted expectation*) A distortion function is a non-decreasing function $g:[0,1]\to[0,1]$ such that $g(0)=0$ and $g(1)=1$. The distortion risk measure or distorted expectation of the random variable *X* associated with distortion function *g*, notation $\varrho_g(X)$, is defined as

$$\varrho_g(X)=\int_{-\infty}^{0}\left[g\left(S_X(x)\right)-1\right]\mathrm{d}x+\int_0^{\infty}g\left(S_X(x)\right)\mathrm{d}x. \tag{2.1}$$

The most well-known examples of distortion risk measures are the VaR and TVaR, if we define the distortion functions, respectively, as follows

$$g_\alpha(x)=\mathbb{I}_{\{x>\alpha\}} \tag{2.2}$$

and

$$g_\beta(x)=\frac{x}{\beta}\mathbb{I}_{\{x\le\beta\}}+\mathbb{I}_{\{x>\beta\}}, \tag{2.3}$$

then the distorted expectation $\varrho_g(X)$ can be equivalently expressed as

$$\mathrm{VaR}_\alpha(X)=\inf\left\{x:P(X>x)\le\alpha\right\}=S_X^{-1}(\alpha) \tag{2.4}$$

and

$$\mathrm{TVaR}_\alpha(X)=\frac{1}{\alpha}\int_0^{\alpha}\mathrm{VaR}_q(X)\mathrm{d}q=\frac{1}{\alpha}\int_0^{\alpha}S_X^{-1}(q)\mathrm{d}q. \tag{2.5}$$

Definition 2.2. (GlueVaR *distortion risk measure*) Given the confidence levels $1-\alpha$ and $1-\beta$, when the distortion function for GlueVaR is specified to the following function

$$g_{\beta,\alpha}^{h_1,h_2}(x)=\begin{cases}\dfrac{h_1}{\beta}\times x, & x\in[0,\beta],\\ h_1+\dfrac{h_2-h_1}{\alpha-\beta}\times(x-\beta), & x\in[\beta,\alpha],\\ 1, & x\in[\alpha,1],\end{cases} \tag{2.6}$$

with $\alpha,\beta\in[0,1]$, $\alpha>\beta$, $h_1\in[0,1]$, and $h_2\in[h_1,1]$, then the corresponding distortion risk measure ϱ_g is the GlueVaR distortion risk measure, which is denoted by $\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X)$.

Remark 2.1. If the following notation is used,

$$\begin{cases} \omega_1 = h_1 - \dfrac{h_2 - h_1}{\alpha - \beta} \times \beta, \\ \omega_2 = \dfrac{h_2 - h_1}{\alpha - \beta} \times \alpha, \\ \omega_3 = 1 - h_2, \end{cases} \tag{2.7}$$

then the distortion function $g_{\beta,\alpha}^{h_1,h_2}(x)$ in (2.6) may be rewritten as

$$g_{\beta,\alpha}^{h_1,h_2}(x) = \omega_1 g_{T,\beta}(x) + \omega_2 g_{T,\alpha}(x) + \omega_3 g_{V,\alpha}(x), \tag{2.8}$$

where $g_{T,\beta}(x)$, $g_{T,\alpha}(x)$ and $g_{V,\alpha}(x)$ are the distortion functions corresponding to the $\mathrm{TVaR}_\beta(X)$, $\mathrm{TVaR}_\alpha(X)$ and $\mathrm{VaR}_\alpha(X)$, respectively. Therefore, GlueVaR is a risk measure that can be expressed as a linear combination of three risk measures as follows,

$$\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X) = \omega_1 \mathrm{TVaR}_\beta(X) + \omega_2 \mathrm{TVaR}_\alpha(X) + \omega_3 \mathrm{VaR}_\alpha(X), \tag{2.9}$$

where $\omega_i \in [0,1]$ for $i = 1,2,3$, and $\omega_1 + \omega_2 + \omega_3 = 1$.

Example 2.1. Assume that initial risk X follows an exponential distribution with parameter 0.001, then $\mathrm{VaR}_\alpha(X) = -1000\ln(\alpha)$, $\mathrm{TVaR}_\alpha(X) = -1000\ln(\alpha) + 1000$. When $\omega_1 = 0.2$, $\omega_2 = 0.3$ and $\omega_3 = 0.5$, the values of VaR, TVaR and GlueVaR at different confidence levels are calculated in Table 1.

Given α and β, the values in Table 1 indicate that GlueVaR is more conservative than VaR. Note that $\mathrm{VaR}_\alpha(X) \le \mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X)$, which means that GlueVaR may overcome the VaR's shortage of underestimating risks. On the other hand, GlueVaR is not, unlike TVaR, overly conservative. It seems clear that GlueVaR, a new risk measure based on distortion functions, can be valuable in the scope of finance and insurance.

Definition 2.3. (*Marginal indemnification function*) (See [[20], Definition 2]) For any indemnification function $f(X)$, the associated marginal indemnification is a function $h \in [0,1]$ such that

$$f(x) = \int_0^x h(t)\,\mathrm{d}t, \quad x \ge 0. \tag{2.10}$$

2.2. Model Set-Up

Based on the notations of the preceding subsection, we will introduce a reciprocal reinsurance model to study the optimal strategy which considers the interests of both an insurer and a reinsurer.

Problem 1 (*Optimization model of a reciprocal reinsurance*)

$$\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}\left(T_{f^*}(X)\right) = \min_{f \in \mathcal{F}} \mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}\left(T_f(X)\right), \tag{2.11}$$

where $\mathcal{F} = \{ f(x) :\ f(x)$ and $I_f(x)$ are non-decreasing and $f(x) = \int_0^x h(t)\,\mathrm{d}t,\ 0 \le h(t) \le 1 \}$.

Our objective is to find the optimal ceded loss function $f^*(X)$ and to characterize the corresponding $\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}\left(f^*(X)\right)$.

Table 1. VaR, TVaR and GlueVaR of initial risk X.

β	0.01	0.03	0.05	0.07	0.09
α	0.02	0.04	0.06	0.08	0.10
$\mathrm{TVaR}_{\beta}(X)$	5605.2	4506.6	3995.7	3659.3	3407.9
$\mathrm{TVaR}_{\alpha}(X)$	4912.0	4218.9	3813.4	3525.7	3302.6
$\mathrm{VaR}_{\alpha}(X)$	3912.0	3218.9	2813.4	2525.7	2302.6
$\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X)$	4550.6	3776.4	3349.9	3052.4	2823.7

3. Existence of Optimal Reinsurance Strategy

Lemma 3.1 *For any ceded loss functions* $f(X)$, $\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(f(X))$ *can be expressed as*

$$\begin{aligned}&\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(f(X))\\&=\int_0^{\infty}\left[\omega_1 g_{T,\beta}(S_X(x))+\omega_2 g_{T,\alpha}(S_X(x))+\omega_3 g_{V,\alpha}(S_X(x))\right]h(x)\mathrm{d}x,\end{aligned}\tag{3.1}$$

where $\omega_i\in[0,1]$ for $i=1,2,3$, and $\omega_1+\omega_2+\omega_3=1$.

Proof. As proved in Lemma 2.1 of Zhuang *et al.* (2016), for any distortion function *g*,

$$\varrho_g(f(X))=\int_0^{\infty}g[S_X(t)]\mathrm{d}f(t).$$

Obviously, $\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(f(X))$ may be rewritten as (3.1). ■

Lemma 3.2 *For any* $\lambda\in[0,1]$ *and ceded loss function* $f(X)$, *total risks* $T_f(X)$ *can be expressed as*

$$\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(T_f(X))=\lambda\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X)+(1-2\lambda)\int_0^{\infty}\varphi(S_X(x))h(x)\mathrm{d}x,\tag{3.2}$$

where

$$\varphi(S_X(x))=\omega_1 g_{T,\beta}(S_X(x))+\omega_2 g_{T,\alpha}(S_X(x))+\omega_3 g_{V,\alpha}(S_X(x))-(1+\rho)S_X(x).$$

Proof. From definitions of $T_{I_f}(X)$ and $T_{R_f}(X)$, $T_f(X)$ can be rewritten as

$$T_f(X)=\lambda X+(1-2\lambda)\left[f(X)-\delta_f(X)\right].\tag{3.3}$$

By the comonotonic additivity of the distortion risk measures, total risks $T_f(X)$ under the GlueVaR distortion risk measures can be expressed as

$$\begin{aligned}\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(T_f(X))=&\lambda\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X)+(1-2\lambda)\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(f(X))\\&-(1-2\lambda)\delta_f(X).\end{aligned}\tag{3.4}$$

Based on the fact that

$$\delta_f(X)=(1+\rho)E(f(X))=(1+\rho)\int_0^{\infty}S_X(x)h(x)dx,\tag{3.5}$$

with the expressions (3.1), (3.4) and (3.5), we get

$$\begin{aligned}&\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(T_f(X))\\&=\lambda\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}(X)+(1-2\lambda)\int_0^{\infty}\left[\omega_1 g_{T,\beta}(S_X(x))\right.\\&\quad\left.+\omega_2 g_{T,\alpha}(S_X(x))+\omega_3 g_{V,\alpha}(S_X(x))-(1+\rho)S_X(x)\right]h(x)\mathrm{d}x.\end{aligned}$$ ■

Lemma 3.3 *Let* h^* *be the optimal marginal indemnification function, then it satisfies*

$$\begin{aligned}&\min_{f\in\mathcal{F}}\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}\left(T_f\left(X\right)\right)\\&=\lambda\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}\left(X\right)+\left(1-2\lambda\right)\int_0^\infty\varphi\left(S_X\left(x\right)\right)h^*\left(x\right)\mathrm{d}x.\end{aligned}\tag{3.6}$$

Suppose that $f^*\left(x\right)=\int_0^x h^*\left(z\right)\mathrm{d}z$ for $x\in\left[0,\infty\right)$. Then h^* solves (3.6) if and only if f^* solves (2.11).

Proof. This follows from the same arguments used in the proof to Proposition 2.1 of Zhuang *et al.* (2016). ■

Theorem 3.1 *For* $\lambda\in\left[0,1\right]$, $h^*\left(x\right)$ *solves 3.6 if and only if it satisfies the followings.*

1). If $0\le\lambda<\frac{1}{2}$, then

$$h^*\left(x\right)=\begin{cases}1, & \varphi\left(S_X\left(x\right)\right)<0,\\ \xi\in\left[0,1\right], & \varphi\left(S_X\left(x\right)\right)=0,\\ 0, & \varphi\left(S_X\left(x\right)\right)>0.\end{cases}\tag{3.7}$$

2). If $\lambda=\frac{1}{2}$, then

$$h^*\left(x\right)=\xi\in\left[0,1\right].\tag{3.8}$$

3). If $\frac{1}{2}<\lambda\le1$, then

$$h^*\left(x\right)=\begin{cases}0, & \varphi\left(S_X\left(x\right)\right)<0,\\ \xi\in\left[0,1\right], & \varphi\left(S_X\left(x\right)\right)=0,\\ 1, & \varphi\left(S_X\left(x\right)\right)>0.\end{cases}\tag{3.9}$$

Proof. Note that minimizing $\mathrm{GlueVaR}_{\beta,\alpha}^{h_1,h_2}\left(T_f\left(X\right)\right)$ is equivalent to minimizing $\left(1-2\lambda\right)\int_0^\infty\varphi\left(S_X\left(x\right)\right)h\left(x\right)\mathrm{d}x$ of (3.2). In the next, we will prove the results from three cases.

1). For the cases $0\le\lambda<\frac{1}{2}$, $1-2\lambda>0$.

a) If $\varphi\left(S_X\left(x\right)\right)<0$, then the minimum $\left(1-2\lambda\right)\int_0^\infty\varphi\left(S_X\left(x\right)\right)h\left(x\right)\mathrm{d}x$ is attained at $h\left(x\right)=1$.

b) If $\varphi\left(S_X\left(x\right)\right)=0$, then $\left(1-2\lambda\right)\int_0^\infty\varphi\left(S_X\left(x\right)\right)h\left(x\right)\mathrm{d}x=0$ for any $h\left(x\right)=\xi\in\left[0,1\right]$.

c) If $\varphi\left(S_X\left(x\right)\right)>0$, then the minimum $\left(1-2\lambda\right)\int_0^\infty\varphi\left(S_X\left(x\right)\right)h\left(x\right)\mathrm{d}x$ is attained at $h\left(x\right)=0$.

2). For the cases $\lambda=\frac{1}{2}$, $\left(1-2\lambda\right)\int_0^\infty\varphi\left(S_X\left(x\right)\right)h\left(x\right)\mathrm{d}x=0$ for any $h\left(x\right)=\xi\in\left[0,1\right]$.

3). For the cases $\frac{1}{2}<\lambda\le1$, $1-2\lambda<0$.

a) If $\varphi(S_X(x))<0$, then the minimum $(1-2\lambda)\int_0^{\infty}\varphi(S_X(x))h(x)\mathrm{d}x$ is attained at $h(x)=0$.

b) If $\varphi(S_X(x))=0$, then $(1-2\lambda)\int_0^{\infty}\varphi(S_X(x))h(x)\mathrm{d}x=0$ for any $h(x)=\xi\in[0,1]$.

c) If $\varphi(S_X(x))>0$, then the minimum $(1-2\lambda)\int_0^{\infty}\varphi(S_X(x))h(x)\mathrm{d}x$ is attained at $h(x)=1$. ■

4. Explicit Solutions

In Section 3, we have derived the optimal marginal indemnification function h^*. It seems very concise but we can not obtain the optimal reinsurance strategy f^* directly. In this section, we want to derive the optimal reinsurance contract f^* bases on optimal marginal indemnification function h^*.

Let $t=S_X(x)$ and denote $\psi(t)=\varphi(S_X(x))$, we have

$$\psi(t)=\omega_1 g_{T,\beta}(t)+\omega_2 g_{T,\alpha}(t)+\omega_3 g_{V,\alpha}(t)-(1+\rho)t, \tag{4.1}$$

where

$$g_{T,\beta}(x)=\frac{x}{\beta}\mathbb{I}_{\{x\le\beta\}}+\mathbb{I}_{\{x>\beta\}}, \tag{4.2}$$

$$g_{T,\alpha}(x)=\frac{x}{\alpha}\mathbb{I}_{\{x\le\alpha\}}+\mathbb{I}_{\{x>\alpha\}}, \tag{4.3}$$

$$g_{V,\alpha}(x)=\mathbb{I}_{\{x>\alpha\}}. \tag{4.4}$$

With the expression (4.1)-(4.4), $\psi(t)$ may be reexpressed as

$$\psi(t)=\begin{cases} k_1 t, & [0,\beta], \\ k_2 t+\omega_1, & (\beta,\alpha], \\ k_3 t+1, & (\alpha,1], \end{cases} \tag{4.5}$$

which has two positive zeros,

$$t_1=\frac{\omega_1\alpha}{(1+\rho)\alpha-\omega_2}, \quad t_2=\frac{1}{1+\rho},$$

where

$$k_1=\frac{\omega_1}{\beta}+\frac{\omega_2}{\alpha}-(1+\rho), \tag{4.6}$$

$$k_2=\frac{\omega_2}{\alpha}-(1+\rho), \tag{4.7}$$

$$k_3=-(1+\rho). \tag{4.8}$$

Theorem 4.1 *For any ceded loss function* $f(x)\in\mathcal{F}$, *if* $\lambda=\frac{1}{2}$, *then*

$$f^*(x)=\xi x, \quad \xi\in[0,1].$$

Proof. From (2.10) and (3.8), we can derive above results easily. ■

Theorem 4.2 *For* $0\le\lambda<\frac{1}{2}$, *and any ceded loss function* $f(x)\in\mathcal{F}$, *optim-*

al reinsurance contracts f^* *to Problem* 1 *are given as follows*:

1). If $k_1 > 0$ and $k_2 \ge 0$, then $f^*(x) = x \wedge S_X^{-1}(t_2)$.

2). If $k_1 > 0$ and $k_2 < 0$, then

$$f^*(x) = \begin{cases} x \wedge S_X^{-1}(t_2), & \psi(\alpha) \ge 0, \\ x \wedge S_X^{-1}(t_2) + \left(x - S_X^{-1}(\alpha)\right)_+ \wedge \left(S_X^{-1}(t_1) - S_X^{-1}(\alpha)\right), & \psi(\alpha) < 0, \psi(\alpha+) > 0. \\ x \wedge S_X^{-1}(t_1), & \psi(\alpha) < 0, \psi(\alpha+) \le 0, \end{cases}$$

3). If $k_1 = 0$, then

$$f^*(x) = \begin{cases} x \wedge S_X^{-1}(t_2) + \left(x - S_X^{-1}(\alpha)\right)_+ \wedge \left(S_X^{-1}(\beta) - S_X^{-1}(\alpha)\right) + \xi\left(x - S_X^{-1}(\beta)\right)_+, & \psi(\alpha+) > 0, \\ x \wedge S_X^{-1}(\beta) + \xi\left(x - S_X^{-1}(\beta)\right)_+, & \psi(\alpha+) \le 0. \end{cases}$$

4). If $k_1 < 0$, then

$$f^*(x) = \begin{cases} x \wedge S_X^{-1}(t_2) + \left(x - S_X^{-1}(\alpha)\right)_+, & \psi(\alpha+) > 0, \\ x, & \psi(\alpha+) \le 0. \end{cases}$$

Proof. Analyse the optimal reinsurance contract with (3.7) for the case $0 \le \lambda < \frac{1}{2}$. From (4.5)-(4.8), clearly $k_1 > k_2 > k_3$ and $k_3 < 0$. Note that $\psi(\beta) = \psi(\beta+)$, but $\psi(\alpha) < \psi(\alpha+)$, which means that $\psi(t)$ is discontinuous at the point $t = \alpha$. Therefore, we consider the followings.

1). When $k_1 > 0$, there has three cases about k_2, which are $k_2 > 0$, $k_2 = 0$ and $k_2 < 0$.

a) If $k_2 > 0$, then $\psi(\alpha) > 0$. t_2 exists since $\psi(\alpha+) > \psi(\alpha) > 0$ and $\psi(1) = -\rho < 0$. Note that $\psi(t) > 0$ in $(0, t_2)$, $\psi(t) < 0$ in $(t_2, 1]$ as Figure 1. With the expression (3.7), we have that $h^*(x) = 1$ for $x \in \left(0, S_X^{-1}(t_2)\right)$, $h^*(x) = 0$ for $x \in \left(S_X^{-1}(t_2), \infty\right)$ as Figure 2, thus $f^*(x) = x \wedge S_X^{-1}(t_2)$.

b) If $k_2 = 0$, then $\psi(\alpha) > 0$. Similar to 1), $f^*(x) = x \wedge S_X^{-1}(t_2)$.

c) When $k_2 < 0$, $\psi(\alpha)$ has three cases $\psi(\alpha) > 0$, $\psi(\alpha) = 0$ and $\psi(\alpha) < 0$. Since $\psi(t)$ is discontinuous at the point $t = \alpha$, we have to consider the cases of $\psi(\alpha+)$.

i) If $\psi(\alpha) \ge 0$, then $\psi(\alpha+) > 0$. Therefore, t_2 exists. $\psi(t) > 0$ in $(0, t_2)$, $\psi(t) < 0$ in $(t_2, 1]$. Furthermore, $h^*(x) = 1$ for $x \in \left(0, S_X^{-1}(t_2)\right)$, $h^*(x) = 0$ for $x \in \left(S_X^{-1}(t_2), \infty\right)$, so $f^*(x) = x \wedge S_X^{-1}(t_2)$.

ii) If $\psi(\alpha) < 0$, then t_1 exists. If $\psi(\alpha+) > 0$, then t_2 exists since $\psi(1) < 0$. Note that $\psi(t) > 0$ in $(0, t_1)$ and $(\alpha+, t_2]$, $\psi(t) < 0$ in $(t_1, \alpha]$ and $(t_2, 1]$. Furthermore, $h^*(x) = 1$ for $x \in \left(0, S_X^{-1}(t_2)\right) \cup \left(S_X^{-1}(\alpha), S_X^{-1}(t_1)\right)$, $h^*(x) = 0$ for $x \in \left(S_X^{-1}(t_2), S_X^{-1}(\alpha)\right) \cup \left(S_X^{-1}(t_1), \infty\right)$, so $f^*(x) = x \wedge S_X^{-1}(t_2) + \left(x - S_X^{-1}(\alpha)\right)_+ \wedge \left(S_X^{-1}(t_1) - S_X^{-1}(\alpha)\right)$.

iii) If $\psi(\alpha) < 0$, then t_1 exists. When $\psi(\alpha+) \le 0$, $\psi(t) > 0$ in $(0, t_1)$, $\psi(t) < 0$ in $(t_1, 1]$. Furthermore, $h^*(x) = 1$ for $x \in \left(0, S_X^{-1}(t_1)\right)$, $h^*(x) = 0$ for $x \in \left(S_X^{-1}(t_1), \infty\right)$, so $f^*(x) = x \wedge S_X^{-1}(t_1)$.

2). When $k_1 = 0$, from (4.6) and (4.7), we obtain that $k_2 < 0$ and $\psi(\alpha) < 0$. Next, we consider the cases of $\psi(\alpha+)$.

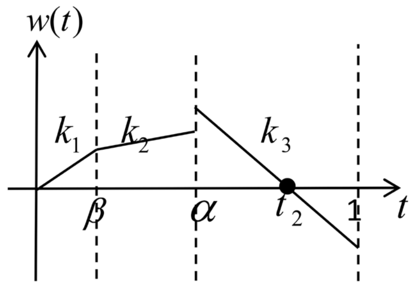

Figure 1. $k_1 > 0$, $k_2 > 0$.

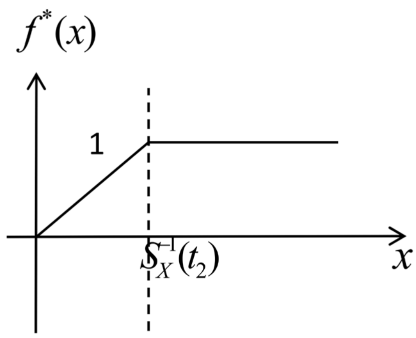

Figure 2. $0 \le \lambda < \frac{1}{2}$.

a) When $\psi(\alpha+) > 0$, we can derive that $\psi(t) > 0$ in (α, t_2), $\psi(t) = 0$ in $[0, \beta]$, and $\psi(t) < 0$ in $(\beta, \alpha]$ and $(t_2, 1]$. Furthermore, $h^*(x) = 1$ for $x \in [0, S_X^{-1}(t_2)) \cup (S_X^{-1}(\alpha), S_X^{-1}(\beta))$, $h^*(x) = 0$ for $x \in (S_X^{-1}(t_2), S_X^{-1}(\alpha))$, and $h^*(x) = \xi$ for $x \in [S_X^{-1}(\beta), \infty)$. Therefore,

$f^*(x) = x \wedge S_X^{-1}(t_2) + (x - S_X^{-1}(\alpha))_+ \wedge (S_X^{-1}(\beta) - S_X^{-1}(\alpha)) + \xi(x - S_X^{-1}(\beta))_+$.

b) When $\psi(\alpha+) \le 0$, $\psi(t) = 0$ in $[0, \beta]$ and $\psi(t) < 0$ in $(\beta, 1]$. Furthermore, $h^*(x) = 1$ for $x \in [0, S_X^{-1}(\beta))$, and $h^*(x) = \xi$ for $x \in [S_X^{-1}(\beta), \infty)$. Therefore, $f^*(x) = x \wedge S_X^{-1}(\beta) + \xi(x - S_X^{-1}(\beta))_+$.

3). When $k_1 < 0$, note that $k_2 < 0$ and $\psi(\alpha) < 0$. There has three cases for $\psi(\alpha+)$.

a) When $\psi(\alpha+) > 0$, $\psi(t) > 0$ in (α, t_2) and $\psi(t) < 0$ in other cases. Furthermore, $h^*(x) = 1$ for $x \in (0, S_X^{-1}(t_2)) \cup (S_X^{-1}(\alpha), \infty)$, $h^*(x) = 0$ for $x \in (S_X^{-1}(t_2), S_X^{-1}(\alpha))$. Therefore, $f^*(x) = x \wedge S_X^{-1}(t_2) + (x - S_X^{-1}(\alpha))_+$.

b) If $\psi(\alpha+) \le 0$, then $\psi(t) < 0$ in $(0,1)$. Therefore, $h^*(x) = 1$ when $x \in (0, \infty)$, $f^*(x) = x$. ■

Theorem 4.3 *For* $\frac{1}{2} < \lambda \le 1$, *and any ceded loss function* $f(x) \in F$, *optimal reinsurance contracts* f^* *to Problem* 1 *are given as follows:*

1). If $k_1 > 0$ and $k_2 \ge 0$, then $f^*(x) = (x - S_X^{-1}(t_2))_+$.

2). If $k_1 > 0$ and $k_2 < 0$, then

$$f^*(x)=\begin{cases}\left(x-S_X^{-1}(t_2)\right)_+, & \psi(\alpha)\geq 0,\\ \left(x-S_X^{-1}(t_2)\right)_+\wedge\left(S_X^{-1}(\alpha)-S_X^{-1}(t_2)\right)+\left(x-S_X^{-1}(t_1)\right)_+, & \psi(\alpha)<0,\psi(\alpha+)>0,\\ \left(x-S_X^{-1}(t_1)\right)_+, & \psi(\alpha)<0,\psi(\alpha+)\leq 0.\end{cases}$$

3). If $k_1=0$, then

$$f^*(x)=\begin{cases}\left(x-S_X^{-1}(t_2)\right)_+\wedge\left(S_X^{-1}(\alpha)-S_X^{-1}(t_2)\right)+\xi\left(x-S_X^{-1}(\beta)\right)_+, & \psi(\alpha+)>0,\\ \xi\left(x-S_X^{-1}(\beta)\right)_+, & \psi(\alpha+)\leq 0.\end{cases}$$

4). If $k_1<0$, then

$$f^*(x)=\begin{cases}\left(x-S_X^{-1}(t_2)\right)_+\wedge\left(S_X^{-1}(\alpha)-S_X^{-1}(t_2)\right), & \psi(\alpha+)>0,\\ 0, & \psi(\alpha+)\leq 0.\end{cases}$$

Proof. Analyse the optimal reinsurance contract with (3.9) for the case $\frac{1}{2}<\lambda\leq 1$.

1). When $k_1>0$, there has three cases about k_2.

a) If $k_2>0$, then $\psi(\alpha)>0$. Since $\psi(\alpha+)>\psi(\alpha)>0$ and $\psi(1)=-\rho<0$, then t_2 exists. Therefore, $\psi(t)>0$ in $(0,t_2)$ and $\psi(t)<0$ in $(t_2,1]$ as Figure 1. With the expression (3.9), we have that $h^*(x)=0$ for $x\in\left(0,S_X^{-1}(t_2)\right)$ and $h^*(x)=1$ for $x\in\left(S_X^{-1}(t_2),\infty\right)$ as Figure 3, so $f^*(x)=\left(x-S_X^{-1}(t_2)\right)_+$.

b) If $k_2=0$, then $\psi(\alpha)>0$. Similar to 1), $f^*(x)=\left(x-S_X^{-1}(t_2)\right)_+$.

c) When $k_2<0$, $\psi(\alpha)$ has three cases $\psi(\alpha)>0$, $\psi(\alpha)=0$ and $\psi(\alpha)<0$. Since $\psi(t)$ is discontinuous at the point $t=\alpha$, we have to consider the cases of $\psi(\alpha+)$.

i) If $\psi(\alpha)\geq 0$, then $\psi(\alpha+)>0$, t_2 exists since $\psi(1)=-\rho<0$. Note that $\psi(t)>0$ in $(0,t_2)$ and $\psi(t)<0$ in $(t_2,1]$. Furthermore, $h^*(x)=0$ for $x\in\left(0,S_X^{-1}(t_2)\right)$, $h^*(x)=1$ for $x\in\left(S_X^{-1}(t_2),\infty\right)$, so $f^*(x)=\left(x-S_X^{-1}(t_2)\right)_+$.

ii) If $\psi(\alpha)<0$ and $\psi(\alpha+)>0$, then t_1 and t_2 exists. Clearly $\psi(t)>0$ in $(0,t_1)$ and $(\alpha+,t_2]$, $\psi(t)<0$ in $(t_1,\alpha]$ and $(t_2,1]$. Furthermore, $h^*(x)=0$ for $x\in\left(0,S_X^{-1}(t_2)\right)\cup\left(S_X^{-1}(\alpha),S_X^{-1}(t_1)\right)$, $h^*(x)=1$ for $x\in\left(S_X^{-1}(t_2),S_X^{-1}(\alpha)\right)\cup\left(S_X^{-1}(t_1),\infty\right)$, so $f^*(x)=\left(x-S_X^{-1}(t_2)\right)_+\wedge\left(S_X^{-1}(\alpha)-S_X^{-1}(t_2)\right)+\left(x-S_X^{-1}(t_1)\right)_+$.

iii) If $\psi(\alpha)<0$ and $\psi(\alpha+)\leq 0$, then t_1 exists. Clearly, $\psi(t)>0$ in $(0,t_1)$ and $\psi(t)<0$ in $(t_1,1]$. Furthermore, $h^*(x)=0$ for $x\in\left(0,S_X^{-1}(t_1)\right)$, $h^*(x)=1$ for $x\in\left(S_X^{-1}(t_1),\infty\right)$, so $f^*(x)=\left(x-S_X^{-1}(t_1)\right)_+$.

2). When $k_1=0$, from (4.6) and (4.7), we obtain that $k_2<0$ and $\psi(\alpha)<0$. Next, we consider the cases of $\psi(\alpha+)>0$.

a) When $\psi(\alpha+)>0$, we can derive that $\psi(t)>0$ in (α,t_2), $\psi(t)=0$ in $[0,\beta]$, and $\psi(t)<0$ in $(\beta,\alpha]$ and $(t_2,1]$. Furthermore, $h^*(x)=0$ for $x\in\left[0,S_X^{-1}(t_2)\right)\cup\left(S_X^{-1}(\alpha),S_X^{-1}(\beta)\right)$, $h^*(x)=1$ for $x\in\left(S_X^{-1}(t_2),S_X^{-1}(\alpha)\right)$, and $h^*(x)=\xi$ when $x\in\left[S_X^{-1}(\beta),\infty\right)$. Therefore, $f^*(x)=\left(x-S_X^{-1}(t_2)\right)_+\wedge\left(S_X^{-1}(\alpha)-S_X^{-1}(t_2)\right)+\xi\left(x-S_X^{-1}(\beta)\right)_+$.

b) When $\psi(\alpha+)\le 0$, $\psi(t)=0$ in $[0,\beta]$ and $\psi(t)<0$ in $(\beta,1]$. Furthermore, $h^*(x)=0$ for $x\in\left[0,S_X^{-1}(\beta)\right)$, and $h^*(x)=\xi$ for $x\in\left[S_X^{-1}(\beta),\infty\right)$. Therefore, $f^*(x)=\xi\left(x-S_X^{-1}(\beta)\right)_+$.

3). When $k_1<0$, note that $k_2<0$ and $\psi(\alpha)<0$. There has three cases for $\psi(\alpha+)$.

a) When $\psi(\alpha+)>0$, $\psi(t)>0$ in (α,t_2) and $\psi(t)<0$ in other cases. Furthermore, $h^*(x)=0$ for $x\in\left(0,S_X^{-1}(t_2)\right)\cup\left(S_X^{-1}(\alpha),\infty\right)$, $h^*(x)=1$ for $x\in\left(S_X^{-1}(t_2),S_X^{-1}(\alpha)\right)$. Therefore, $f^*(x)=\left(x-S_X^{-1}(t_2)\right)_+\wedge\left(S_X^{-1}(\alpha)-S_X^{-1}(t_2)\right)$.

b) If $\psi(\alpha+)\le 0$, then $\psi(t)<0$ in $(0,1)$. Therefore, $h^*(x)=0$ when $x\in(0,\infty)$, $f^*(x)=0$. ■

Example 4.1. Similar to Example 2.1, we assume the risk is measured by the GlueVaR risk measures under the expectation premium principle, for $\lambda\in[0,1]$, $\xi\in[0,1]$, $\omega_i\in[0,1]$, $i=1,2,3$ and $\omega_1+\omega_2+\omega_3=1$, optimal reinsurance contracts are given as follows.

From the reinsurer's point of view, as Case 1 in Table 2, the optimal reinsurance strategy can be in form of limited quota-share, $f^*(x)=x\wedge 405.47$, which means that if initial loss X less than 405.47, the case that an insurer ceded all loss to a reinsurer is optimal, and if initial loss X more than 405.47, the case that an insurer ceded 405.47 to a reinsurer is optimal.

From the insurer's point of view, as Case 6 in Table 2, the optimal reinsurance strategy $f^*(x)=0$, which means that an insurer should retain all loss to achieve itself optimality.

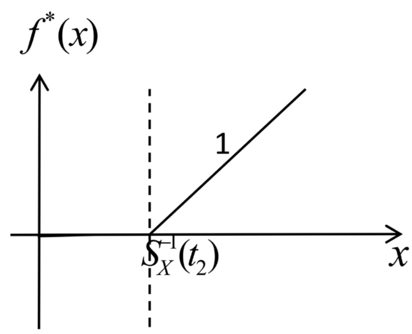

Figure 3. $\frac{1}{2}<\lambda\le 1$.

Table 2. Optimal ceded loss function.

Case	α	β	ω_1	ω_2	ω_3	λ	ρ	$f^*(x)$
1	0.05	0.01	0.20	0.30	0.50	0.00	0.50	$x\wedge 405.47$
2	0.10	0.05	0.10	0.05	0.85	0.20	1.00	$x\wedge 2995.73+\xi(x-2995.73)_+$
3	0.15	0.10	0.15	0.10	0.75	0.40	2.00	$x\wedge 1099.61+(x-1897.12)_+$
4	0.20	0.15	0.40	0.20	0.40	0.60	1.50	$(x-916.29)_+$
5	0.25	0.20	0.50	0.20	0.30	0.80	2.00	$(x-1099.61)_+\wedge 286.68+\xi(x-1609.44)_+$
6	0.30	0.25	0.60	0.10	0.30	1.00	3.00	0

From the perspectives of an insurer and a reinsurer, as Cases 2 - 5. Note that Cases 2 and 5 include the parameter $\xi \in [0,1]$, which means that reinsurance contracts can be different forms when the loss risk has been minimized. Case 3 means that the stop-loss after quota-share reinsurance (which is to say a stop-loss will be applied after a quota-share reinsurance) is optimal. Case 4 means that stop-loss reinsurance is optimal.

5. Conclusion

This article has studied the optimal reciprocal reinsurance with the GlueVaR distortion risk measures under the expected value premium principle. The GlueVaR distortion risk measure is a linear combination of two TVaR and one VaR with different confidence levels, which adds the difficulty than the case of only one VaR or the case of only one TVaR when we derive the optimal reinsurance contract. In this paper, we have expressed GlueVaR as a linear combination of three distortion risk measures with different distortion functions. Therefore, we can use MIF formula to deal with the complex optimization problems easily. The results indicate that depending on the risk measures's level of confidence (α and β), the safety loading (ρ) for the reinsurance premium, weight (λ) of an insurer in the reciprocal reinsurance model and the proportions (ω_1, ω_2 and ω_3) of the three risk measures in the definition of GlueVaR, the optimal reinsurance can be in the forms of quota-share, stop-loss, change-loss, or their combination, for example, stop-loss after quota-share. This paper has not considered the practical constraints, such as risk constraints or reinsurance premium constraints, which can be studied at a later time.

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Author Contributions

These authors contributed equally to this work.

Conflicts of Interest

The authors declare no conflict of interest.

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