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A Linear Regression Approach for Determining Option Pricing for Currency-Rate Diffusion Model with Dependent Stochastic Volatility, Stochastic Interest Rate, and Return Processes

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Abstract

A three-factor exchange-rate diffusion model that includes three stochastically-dependent Brownian motion processes, namely, the domestic interest rate process, volatility process and return process is considered. A linear regression approach that derives explicit expressions for the distribution function of log return of foreign exchange rate is derived. Subsequently, a closed form workable formula for the call option price that has an algebraic expression similar to a Black-Scholes model, which facilitates easier study, is discussed.

Keywords

Option Pricing, Interest-Rate Parity Condition, Black-Scholes Model, Linear Regression Approach, Spot Option, Ito Calculus

1. Introduction

A foreign exchange rate depends on the supply and demand dynamics of a currency. The exchange rate is a function of trade balance, the interest rate differential and differential inflation expectations between the two countries [1] [2].

Let $S(u)$, $u \geq 0$ = exchange rate process over the time interval: $u \geq 0$, where u = number of domestic currency units, e.g., \$, per unit of foreign currency = \$-price of foreign currency.

As interest rate $r_D(u)$ increases, \$ appreciates because investors prefer \$-denominated bonds. Assuming a frictionless, arbitrage-free continuous-time economy in [1], we define a diffusion process model for $S(u)$. In addition, using interest-rate parity condition we have

$$S(h)=S_0 e^{\int_0^h (r_D(u)-r_F(u))\mathrm{d}u}, \text{ see [1]}.$$

In the following section, the formula for valuations of currency spot options is considered, where we obtain a closed form formula for the call option price that has a simple algebraic expression, which is similar to the call option price expression of a Black-Scholes model, making it much easier to compute its value and study. As in [2], we can define an implied volatility function and derive its skewness property.

Subsequently, the proposed three-factor exchange-rate diffusion model is discussed, such that the stochastic volatility process and the stochastic domestic interest rate process each have a stochastically dependent Brownian motion return process.

In the next section, a linear regression approach that derives explicit expressions for the distribution function of $\ln S(s)$ is treated.

Foreign exchange rate option modeling is the subject of several well-known papers and in chapters within [3] [4] [5] [6]. Leveraging Heston's model [4] for this application would introduce complexity due to the need to numerically integrate conditional characteristic functions obtained as solutions of nonlinear pdf to derive the call option prices. An equivalent two-factor Black-Derman-Toy model [2] can be formulated with introduction of $H(u)$.

The method suggested in this paper results in Black-Scholes type formula for call option pricing, which is easily computable.

Finally, we provide concluding remarks and suggestions for future direction.

2. Currency Spot Option

Given the spot rate $S(0)=s_0$, consider the present value of option

$$C^*(s_0,K,*)=E_0\left[\exp\left(-\int_0^s r_D(u)\mathrm{d}u\right)*\left(S(s)-K^+\right)\right] \quad (1)$$

where K is the known strike price and $(r_D(u),u\ge 0)$ is a mean-reverting stochastic process given in (2) below. $S(s)$ is the value of the exchange rate at the option's maturity price. The option to purchase foreign currency over the counter can be exercised when $S(s)$ > the strike price exchange rate K.

3. A Diffusion Process Model

A continuous-time risk-adjusted and risk-neutral exchange rate model, under a Martingale Measure Q, is defined below as a diffusion process (2), mean-reverting stochastic processes: Volatility $\{H(u),u\ge 0\}$ (3) and domestic interest rate $r_D(t)$ process (4), and foreign interest rate r_F is a known constant.

$$\frac{\mathrm{d}S(u)}{S(u)}=\left(\left(r_D(u)-r_F\right)+\frac{H^2(u)}{2}\right)\mathrm{d}u+\left(H(u)+\upsilon\right)\mathrm{d}B_1(u) \quad (2)$$

$$\mathrm{d}H(u) = \alpha\left(H(u) - \theta\right)\mathrm{d}u + \eta \mathrm{d}V_1(u),\ \theta, \eta \ge 0,\ \alpha > 0 \tag{3}$$

$$\mathrm{d}r_D(u) = \beta\left(r_D(u) - \lambda\right)\mathrm{d}u + \zeta \mathrm{d}V_2(u),\ \lambda, \zeta \ge 0,\ \beta > 0 \tag{4}$$

$$\therefore \mathrm{d}\ln S(u) = \left(r_D(u) - r_F - \frac{\upsilon^2}{2}\right)\mathrm{d}u + \left(H(u) + \upsilon\right)\mathrm{d}B_1(u) \tag{5}$$

Equation (5) is obtained from Equation (2) by the application of Ito calculus [7].

Assumption:

$V_1(t) = \rho B_1(t) + \delta C(t)$, where $\delta \triangleq \sqrt{1-\rho^2}$ and $B_1(t)$ and $C(t)$ are independent Brownian processes.

$$V_2(t) = \rho_1 B_1(t) + \delta_1 C_1(t),\ \text{where}\ \delta_1 = \sqrt{1-\rho_1^2}$$
$$\text{and } B_1(t) \text{ and } C_1(t) \text{ are independent Brownian processes.} \tag{6}$$

From the assumption above, the return processes $V_j(u), u \ge 0, j = 1, 2$ are correlated with $B_1(u)$ and that $V_j(t), j = 1, 2$ are standard Brownian motion processes.

Then it follows, see [2] [3], that the distributions of $H(u)$ and $r_D(u)$ are Gaussian processes.

Alternatively, $H(u)$ and $r_D(u)$ may be expressed as:

$$H(u) = q(u) + \int_0^u \psi(t)\mathrm{d}V_1(t).$$
$$\therefore H(u) = q(u) + \int_0^u \psi(t)\left[\rho \mathrm{d}B_1(t) + \delta \mathrm{d}C(t)\right]. \tag{7}$$

$$E\left(H(u)\right) = q(u) = \kappa \mathrm{e}^{-\alpha u} + \theta\left(1 - \mathrm{e}^{-\alpha u}\right)$$
$$= \theta + (\kappa - \theta)\mathrm{e}^{-\alpha u}, \alpha > 0,$$

where θ is the long-term mean and where $H(0) = \kappa$.

$$\operatorname{var}\left(H(u)\right) \triangleq \sigma_H^2 = \int_0^u \psi^2(t)\mathrm{d}t = \int_0^u \eta^2 \mathrm{e}^{-2\alpha t}\mathrm{d}t = \eta^2\left[\frac{1-\mathrm{e}^{-2\alpha u}}{2\alpha}\right]$$
$$\text{So } P\left(H(u) < 0\right) = \Phi\left(-\frac{\mu_H}{\sigma_H}\right) \tag{8}$$

Remark 1:

From (8), choosing $\theta > 0$ and that is small in value, we can make $P\left(H(u) < 0\right)$ negligible.

If, alternatively, we assume that $\{H(u), u \ge 0\}$ has a square root process [8], then the random variable $H(u)$ distribution is non-central $\chi^2(.)$. For simplicity we chose the mean-reverting process model (3).

$$E\left(r_D(u)\right) = q_1(u) = \kappa_1 \mathrm{e}^{-\beta u} + \lambda\left(1 - \mathrm{e}^{-\beta u}\right) = \lambda + (\kappa_1 - \lambda)\mathrm{e}^{-\beta u}, \beta > 0$$

$$\operatorname{var}\left(r_D(u)\right) = \left[\int_0^u \psi_1^2(t)\mathrm{d}u\right]$$

where $\psi_1(u) = \zeta \mathrm{e}^{-\beta u}, u > 0$ and $r_D(0) = \kappa_1$

$$\therefore r_D(u) = q_1(u) + \int_0^u \psi_1(t)\left[\rho_1 \mathrm{d}B_1(t) + \delta_1 \mathrm{d}C_1(t)\right] \tag{9}$$

Assuming $r_D(0)=\kappa_1$, and

$$\begin{aligned}&\psi_1(t)=\theta e^{-\beta t},\ \theta,\beta\ge 0,\ 0\le t\le u\\&q_1(u)=\kappa_1 e^{-\beta u}+\lambda\left(1-e^{-\beta u}\right)=\lambda+(\kappa_1-\lambda)e^{-\beta u}\\&Q_1(s)=\int_0^s q_1(u)\mathrm{d}u=(\kappa_1-\lambda)\frac{1-e^{-\beta s}}{\beta}+\lambda s\end{aligned}\tag{10}$$

The Brownian motion processes $V_1(u)$ and $B_1(u)$ are as follows:

$$V_1(u)=\rho B_1(u)+\delta C_1(u),\ \text{where}\ \ \delta=\sqrt{1-\rho^2}\tag{11}$$

In addition, the Brownian motion processes $B_1(u)$ and $C_1(u)$ under Q are independent.

Remark 2:

$H(u), u\ge 0$: the volatility process.

It follows from [2] that the distribution of $H(u)$ is:

$$H(u)\sim N\left(q(u),\frac{\eta}{\sqrt{2\alpha}}\left(1-e^{-2\alpha u}\right)^{1/2}\right),\ 0\le u\le s.\tag{12}$$

Alternatively, $H(u)$ may be expressed as

$$\begin{aligned}H(u)&=q(u)+\int_0^u\psi(t)\mathrm{d}V_1(t)\\&=q(u)+\int_0^u\psi(t)\left[\rho\mathrm{d}B_1(t)+\delta\mathrm{d}C(t)\right]\end{aligned}\tag{13}$$

where $\delta\triangleq\sqrt{1-\rho^2}$ and $\psi(u)=\eta e^{-\alpha u},\ \eta,\alpha\ge 0$.

See [9] for a similar assumption. See also [2] and [3].

Note that $B_1(s)$ has a normal distribution with mean 0 and variance s, so $B_1(s)$ can be written as $B_1(s)=Z(s)\sqrt{s}$, where $Z(s)$ is a standard normal variable. Then $\ln X(s)$ can be written as a quadratic function of $Z(s)=\frac{B_1(s)}{\sqrt{s}}, s>0$ plus a residual term $\varepsilon(s)$. {See Proposition 1 below}.

For $\mathrm{d}\xi=\mathrm{d}C(t), 0\le t\le u, 0\le u\le s$, we define a volatility process

$$V(\xi)=\left\{V_u=\int_0^u\psi(t)\mathrm{d}C(t), 0\le u\le s\right\}.$$

Define $\sigma(V(s))\triangleq\left(\frac{1}{s}\int_0^s\left[\frac{\eta^2\left(1-e^{-2\alpha u}\right)}{2\alpha}\right]\mathrm{d}u\right)^{1/2}\triangleq\bar{V}$, as the average standard deviation in the case of uncorrelated Brownian motion process $\{C(u), 0\le u\le s\}\triangleq\xi$ [See [10], p. 182].

Proposition 1:

$$r_D(u)=q_1(u)+\int_0^u\psi_1(t)\left[\rho_1\mathrm{d}B_1(t)+\delta_1\mathrm{d}C_1(t)\right];$$

$$\begin{aligned}\therefore\int_0^s r_D(u)\mathrm{d}u&=\int_0^s q_1(u)\mathrm{d}u+\int_0^s\mathrm{d}u\int_0^u\psi_1(t)\left[\rho_1\mathrm{d}B_1(t)+\delta_1\mathrm{d}C_1(t)\right]\\&=Q_1(s)+\rho_1 B_1(s)\left(G_1(s)-\chi_1(s)\right)+\delta_1\bar{V}_1+e_3(s).\end{aligned}\tag{14}$$

where

$$Q_1(s) \triangleq \int_0^s q_1(u)\mathrm{d}u = (\kappa_1 - \lambda)\frac{(1-\mathrm{e}^{-\beta s})}{\beta} + \lambda s$$

$$G(s) \triangleq \int_0^s \frac{\zeta(1-\mathrm{e}^{-\beta t})}{\beta}\mathrm{d}t = \frac{\zeta}{\beta}\left(\sum_{j=1}^{\infty}\frac{(\beta s)^j}{j!\,j}(-1)^{j+1}\right) \tag{15}$$

$$\chi(s) = \frac{\frac{\zeta}{\beta}\left(\sum_{j=1}^{\infty}\frac{\beta^j s^{j+1}}{(j+1)!\,j}(-1)^{j+1}\right)}{s}$$

Proof: See **Appendix B**.

We consider a mean-reverting Gaussian process model (2), the volatility stochastic processes $\{H(u), 0 \le u \le s\}$ and the processes, $V_1(u)$ and $B_1(u)$ in (3) to be correlated; where $V_1(u)$ is a standard Brownian motion return process. In addition, in (3), we define the volatility $H(u)\varepsilon\mathrm{R}$ as a mean reverting Gaussian process with θ as its long-term mean.

Assumption 1:

$$V_1(t) = \rho_1 B_1(t) + \delta_1 C(t),\ \text{where } \delta_1 = \sqrt{1-\rho_1^2} \tag{16}$$

In (4), we define the domestic interest rate process $r_D(u)$ as a mean reverting Gaussian process with λ as its long-term mean. The process $\{r_D(u), 0 \le u \le s\}$ is such that the return process $V_2(u)$ is a correlated standard Brownian motion process to $B_1(u)$. The foreign interest rate $r_F(u)$ is a constant r_f

Assumption 2:

It follows from [2] that the distribution of $r_D(u)$:

$$r_D(u) \sim N\left(q_1(u), \frac{\zeta}{\sqrt{2\beta}}\left(1-\mathrm{e}^{-2\beta u}\right)^{1/2}\right), 0 \le u \le s. \tag{17}$$

Now we use the results obtained in Proposition 1 to derive an explicit expression for

$$\int_0^s \mathrm{d}\ln S(u) = \ln S(s) - \ln S_0$$

Proposition 2:

$$\begin{aligned}&\int_0^s \mathrm{d}\ln S(u)\\ &= \int_0^s \left(r_D(u) - r_F\right)\mathrm{d}u - \frac{\upsilon^2}{2}s + \int_0^s\left[\upsilon + \theta + q_0(u)\right]\mathrm{d}B_1(u)\\ &+ \int_0^s\left[\int_0^u \psi(t)\left(\rho\mathrm{d}B_1(t) + \delta\mathrm{d}C(t)\right)\right]\mathrm{d}B_1(u);\end{aligned} \tag{18}$$

Remark 3:

From the expression for

$\int_0^s r_D(u)\mathrm{d}u = Q_1(s) + \rho_1 B_1(s)\left(G_1(s) - \chi_1(s)\right) + \delta_1\overline{V}_1$. the stochastic terms

$\rho_1 B_1(s)(G_1(s)-\chi_1(s))$ modifies $n(s)$ and the constant term $Q_1(s)+\delta_1\bar{V}_1$ modifies $p^*(s,h)$ with the addition of $\rho_1 B_1(s)(G_1(s)-\chi_1(s))$ and the constant terms $Q_1(s)+\delta_1\bar{V}_1$ modifies $p^*(s,h)$ with the addition of $Q_1(s)+\delta_1\bar{V}_1$.

Then, using the results in [2], Proposition 1 and those in **Appendix A** and **Appendix B** we have:

$$\begin{aligned}\ln S(s) &= \ln s_0 - \frac{\upsilon^2}{2}s - r_F s + (\upsilon+\theta)B_1(s) + \left[\gamma_1(s)B_1(s)+e_1(s)\right] \\ &\quad + \left[\gamma_2(s)\rho\left(\frac{B_1^2(s)}{2}-\frac{s}{2}\right)+e_2(s)\right]+Q_1(s) \\ &\quad + \rho_1 B_1(s)(G_1(s)-\chi_1(s)) + \delta_1 V_1 \bar{Y}_1 + e_3(s) \\ &\triangleq Z^2(s)\frac{m(s)}{2} + Z(s)n(s) + p(s) + \varepsilon(s) \triangleq U(s,\xi)+\varepsilon(s)\end{aligned} \tag{19}$$

Therefore,

$$U(s,\xi) = Z^2(s)\frac{m(s)}{2} + Z(s)n(s) + p(s),$$

$$m(s) = \rho\gamma_2(s)s;$$

$$n(s) = \left[(\upsilon+\theta+\delta\bar{V})+\gamma_1(s)+\gamma_3(s)+\left(\rho_1(G_1(s)-\chi_1(s))+\delta_1\bar{V}_1\right)\right]\sqrt{s};$$

Remark 4:

Note that $n(s)$ in this paper is an updated version from the $n(s)$ in [2], due to our treatment of a stochastic interest rate: $\int_0^s r_D(u)\,\mathrm{d}u$

$$p(s) = \ln s_0 - \frac{1}{2}\upsilon^2 s - \rho\gamma_2(s)\frac{s}{2} + \varepsilon(s) + Q_1(s) - r_F s;$$

$$\begin{aligned}\sigma^2(V(s)) &= \int_0^s \left[\frac{\eta^2\left(1-\mathrm{e}^{-2\alpha u}\right)}{2\alpha}\right]\mathrm{d}u \\ &= \frac{1}{s}\int_0^s \left(\frac{\eta^2\left(1-\mathrm{e}^{-2\alpha u}\right)}{2\alpha}\right)^2 \mathrm{d}u \\ &= \frac{1}{s}\frac{\eta^2\left(2\alpha s-\left(1-\mathrm{e}^{-2\alpha s}\right)\right)}{4\alpha^2} \triangleq \bar{V}^2,\end{aligned}$$

In the case of $C(u), u \ge 0$.

$$\sigma^2(V_1(s)) = \frac{1}{s}\frac{\zeta^2\left(2\beta s-\left(1-\mathrm{e}^{-2\beta s}\right)\right)}{4\beta^2} \triangleq \bar{V}_1^2, \text{ where } C_1(u), u \ge 0 \tag{20}$$

$$m(s) = \rho\gamma_2(s)s;$$

$$n(s) = \left[(\upsilon+\theta+\delta\bar{V})+\gamma_2(s)+\lambda+\gamma_3(s)+\omega_2+\rho_1(G_1(s)-\chi_1(s))\right]\sqrt{s} \tag{21}$$

$$p(s) = \ln S_0 - \frac{1}{2}\upsilon^2 s - \rho\gamma_2(s)\frac{s}{2} + \varepsilon(s) + Q_1(s) - r_F s + \delta_1\bar{V}_1$$

$\varepsilon(s) = e_1(s) + e_2(s) + e_3(s)$ where

$$\begin{aligned}
Q_1(s) &= \int_0^s q_1(u)\,\mathrm{d}u = \frac{(\kappa_1-\lambda)\left(1-\mathrm{e}^{-\beta_1 s}\right)}{\beta_1} + \lambda s; \\
\gamma_1(s) &= \frac{1}{s}\int_0^s q_0(u)\,\mathrm{d}u = \frac{(\kappa-\theta)\left(1-\mathrm{e}^{-\alpha s}\right)}{\alpha s}; \\
\gamma_2(s) &= \frac{2\eta}{s^2}\int_0^s\int_0^u \psi(t)\,\mathrm{d}t\mathrm{d}u = \frac{2\eta}{s^2}\int_0^s \frac{\left(1-\mathrm{e}^{-\alpha u}\right)}{\alpha}\mathrm{d}u \\
&= \frac{2\eta}{s^2}\frac{\left(\alpha s-\left(1-\mathrm{e}^{-\alpha s}\right)\right)}{\alpha^2}.
\end{aligned} \tag{22}$$

$$Var\left(e_1(s)\right) = (\kappa-\theta)^2\left[\left(\frac{1-\mathrm{e}^{-2\alpha s}}{2\alpha}\right)-\left(\frac{1-\mathrm{e}^{-\alpha}}{\alpha s}\right)^2\right];$$

$$Var\left(e_2(s)\right) = \left\{\frac{\eta^2 s}{2\alpha}-\frac{\eta^2}{4\alpha^2}\left(1-\mathrm{e}^{-2\alpha s}\right)\right\}-\left\{\frac{\eta s}{\alpha}-\frac{\eta}{\alpha^2}\left(1-\mathrm{e}^{-\alpha s}\right)\right\}^2$$

$Var\left(e_3(s)\right)$ is provided in (B1)

$$Cov\left(e_i(s), e_j(s)\right) = 0;\ i, j = 1,2,3,\ i \neq j$$

Case 1: $\rho > 0$;

$$\begin{aligned}
&\int_0^s \mathrm{d}\ln S(u) \\
&= \int_0^s \left(r_D(u)-r_F\right)\mathrm{d}u - \frac{\upsilon^2}{2}s + \int_0^s \left(\lambda_1+q_1(u)\right)\mathrm{d}u + \int_0^s \left[\upsilon+\theta+q_0(u)\right]\mathrm{d}B_1(u) \\
&\quad + \int_0^s\left[\int_0^u \psi(t)\left(\rho\mathrm{d}B_1(t)+\delta C(t)\right)\right]\mathrm{d}B_1(u) + \int_0^s\left[\psi_1(t)\left(\rho_1\mathrm{d}B_1(u)+\delta_1\mathrm{d}C_1(u)\right)\right];
\end{aligned}$$

Let $S(0) = s_0$

$$\begin{aligned}
\ln S(s) = \ln S_0 &- \frac{\upsilon^2}{2}s - r_F s + \left(\upsilon+\theta+\delta\bar{V}\right)B_1(s) \\
&+ \left[\gamma_1(s)B_1(s)+e_1(s)\right] + \left[\gamma_2(s)\rho\left(\frac{B_1^2(s)}{2}-\frac{s}{2}\right)+e_2(s)\right] \\
&+ Q_1(s) + \rho_1 B_1(s)\left(G_1(s)-\chi_1(s)\right) + \delta_1\bar{V}_1 + e_3(s).
\end{aligned}$$

Let $\ln S(s) \triangleq Z^2(s)\frac{m(s)}{2} + Z(s)n(s) + p(s) + \varepsilon(s) \triangleq U(s,\xi) + \varepsilon(s)$.

Assumption 3: $U(s,\xi)$ and $\varepsilon(s)$ are independent random variables.

Assumption 4: $n^2(s,\xi) - 2m(s)\left(p(s,h)-\omega\right) \geq 0$.

Assumption 5: $m(s) \neq 0$ and $m(s) < 1$.

If Assumptions (4) and (5) hold, then the conditional risk-neutral distribution of $\ln S(s)$ is:

Proposition 3:

$$\begin{aligned}
&1 - F_{\ln S(s)}(\omega, h) \\
&= P\left(\ln S(s) \geq \omega \mid \varepsilon(s) = h\right) = \begin{cases} 1-\Phi\left(z_1(h)\right)+\Phi\left(z_2(h)\right), \omega \geq \omega^*(\rho,h,\xi) \\ 1, \omega \leq \omega^*(\rho,h,\xi) \end{cases}
\end{aligned}$$

$$\therefore F_{\ln S(s)}(\omega,h) = P(\ln S(s) \le \omega \mid \varepsilon(s) = h) = \begin{cases} \Phi(z_1(h)) - \Phi(z_2(h)), \omega \ge \omega^*(\rho,h,\xi) \\ 0, \omega \le \omega^*(\rho,h.\xi) \end{cases} \quad (23)$$

where

$$z = z_1(h,\xi), z_2(h,\xi) = \frac{-n(s) \pm \sqrt{n^2(s,\xi) - 2m(s)(p(s)+h-\omega)}}{m(s)}$$
$$\omega^*(\rho,h) = h + p(s) - \frac{n^2(s)}{2m(s)} \quad (24)$$

If $n^2(s,\xi) - 2m(s)(p(s)-h-\omega^*) = 0$, then the roots of the equation defined in (24) are equal so that $z_1 = z_2 = z^*(\xi) = -\frac{n(s,\xi)}{m(s)}$, then there exists a value $\omega^*(\rho,h,\xi)$ such that

$$P(\ln S(s) \ge \omega^*(\rho,h) \mid \varepsilon(s) = h, \tilde{V}^2(s) = \xi) = 1.$$

In other words, $\omega^*(\rho,h,\xi)$ is the lowest value for the conditional random variable $\ln(S(s))$.

Remark 5:

Since we know the CDF of ln*S*(*s*) we can estimate the parameters of the underlying model (2)-(5).

Case 2: Conditional Risk-neutral Distribution function of $(\ln S(s) \mid \varepsilon(s) = h, \bar{V}(s) = \xi)$, $m(s) < 0$. Suppose $\rho < 0$, Conditional risk-neutral distribution of $\ln S(s) \mid (\varepsilon(s) = h, \bar{V}^2(s) = \xi)$ is as follows:

$$\begin{aligned} &1 - F_{\ln S(s)|\xi}(\omega,h) \\ &= P(\ln S(s) \ge \omega \mid \varepsilon(s) = h, \bar{V}(s) = \xi) \\ &= \begin{cases} \Phi(z_2(h)) - \Phi(z_1(h)), \omega \le \omega^*(\rho,h,\xi) \\ 0, \omega \ge \omega^*(\rho,h,\xi) \end{cases} \end{aligned}$$

$$\therefore F_{\ln S(s)|\xi}(\omega,h) = \begin{cases} 1 - \Phi(z_2(h)) + \Phi(z_1(h)), \omega \le \omega^*(\rho,h,\xi) \\ 1, \omega \ge \omega^*(\rho,h) \end{cases}$$

where

$$z = z_1(h,\xi), z_2(h,\xi) = \frac{-n(s,\xi) \pm \sqrt{n^2(s,\xi) - 2m(s)(p(s)+h-\omega)}}{m(s)}$$

Example 1

$$\begin{aligned} &\mathrm{d}r_D(u) = \beta(r_D(u) - \lambda)\mathrm{d}u + \zeta \mathrm{d}V_2(u), \lambda, \zeta \ge 0, \beta > 0, r_D(0) = \kappa_1; \\ &\mathrm{d}r_D(u) = 0.3(r_D(u) - 0.02)\mathrm{d}u + 0.04\mathrm{d}V_2(u), r_D(0) = 0.03; \\ &\mathrm{d}H(u) = \alpha(H(u) - \theta)\mathrm{d}u + \eta V_1(u), H(0) = \kappa; \\ &\mathrm{d}H(u) = (H(u) - 0.1)\mathrm{d}u + 0.3V_1(u), H(0) = 0.6 \end{aligned}$$

$r_D(u)$:

β	λ	ζ	$r_D(0)=\kappa_1$	ρ_1
0.3	0.2	0.4	0.03	0.6

$H(u)$:

α	θ	η	ρ	κ	r_f
1	0.1	0.3	−0.8	0.6	0.8

Then

$$p^*(s,h) \triangleq p(s)+h-Q_1(s)-\ln s(0)+\ln K-\rho_1\overline{Y}_1$$

Remark 6:

From the expression for

$$\int_0^s r_D(u)\mathrm{d}u = Q_1(s)+\rho_1 B_1(s)\left(G_1(s)-\chi_1(s)\right)+\delta_1\overline{V}_1.$$

the stochastic terms $\rho_1 B_1(s)\left(G_1(s)-\chi_1(s)\right)$ modify the term $n(s)$ and the constant terms $Q_1(s)+\delta_1\overline{V}_1$ modifies $p^*(s,h)$ with the addition of $Q_1(s)+\delta_1\overline{V}_1$.

Proof:

Apply a proof similar to the one in **Appendix A** of [2] using the result for $\int_0^s r_D(u)\mathrm{d}u$ in **Appendix B** of the current paper. See also Proposition 4.

Remark 7:

Assume $\rho>0$, which implies that $m(s)>0$.

If Assumption (3) holds then the conditional risk-neutral distribution of $\left\{\ln S(s)\,|\,\varepsilon(s)=h,\overline{V}(s)=\xi\right\}$ is:

$$1-F_{\ln S(s)}(\omega,h)$$

$$=P\left(\ln S(s)\ge\omega\,|\,\varepsilon(s)=h\right)=\begin{cases}1-\Phi\left(z_1(h)\right)+\Phi\left(z_2(h)\right),\omega\ge\omega^*(\rho,h,\xi)\\ 1,\omega\le\omega^*(\rho,h,\xi)\end{cases}$$

$$\therefore F_{\ln S(s)|\xi}(\omega,h)=P\left(\ln S(s)\le\omega\,|\,\varepsilon(s)=h\right)=\begin{cases}\Phi\left(z_1(h)\right)-\Phi\left(z_2(h)\right),\omega\ge\omega^*(\rho,h,\xi)\\ 0,\omega\le\omega^*(\rho,h,\xi)\end{cases} \tag{25}$$

where

$$z=z_1(h,\xi),z_2(h,\xi)=\frac{-n(s)\pm\sqrt{n^2(s,\xi)-2m(s)\left(p(s)+h-\omega\right)}}{m(s)}$$
$$\omega^*(\rho,h)=h+p(s)-\frac{n^2(s)}{2m(s)} \tag{26}$$

If $n^2(s,\xi)-2m(s)\left(p(s)-h-\omega^*\right)=0$, then the roots of the equation defined in (26) are equal so that $z_1=z_2=z^*(\xi)=-\frac{n(s,\xi)}{m(s)}$, then there exists a value

$\omega^*(\rho,h,\xi)$ such that $P\left(\ln S(s)\geq\omega^*(\rho,h)\,|\,\varepsilon(s)=h,\tilde{V}^2(s)=\xi\right)=1$.

In other words, $\omega^*(\rho,h,\xi)$ is the lowest value of the conditional random variable $\left\{\ln S(s)\,|\,\varepsilon(s)=h\sigma_{\varepsilon(s)}\right\}$.

Call option price:

$$c^*\left(s,S(s),K,S(0)\right)=E_0^Q\exp\left[-\int_0^s r_D(u)\,\mathrm{d}u\right]+\left[\left|\ln S(s)-\ln K\right|^+\right]$$

Proposition 4:

$$C^*\left(s_0,K,r,\eta,h,\rho<0\right)$$
$$=E_0^Q\left(\mathrm{e}^{-\int_0^s r_D(u)\mathrm{d}u}\left(S(s)-K\right)^+\,|\,\varepsilon(s)=h\right)$$

$$\begin{cases}0,\text{ for }\ln K>\omega^*(\rho,h);\\ =\left(s_0\dfrac{1}{\sqrt{1-m(s)}}\left\{\Phi\left(\dfrac{z_2(h)\left(1-m(s)\right)-n(s)}{\left(1-m(s)\right)^{1/2}}\right)-\Phi\left(\dfrac{z_1(h)\left(1-m(s)\right)-n(s)}{\left(1-m(s)\right)^{1/2}}\right)\right\}\right)\\ *\left\{\exp\left(p^*(s,h)+\dfrac{n^2(s)}{2\left(1-m(s)\right)}\right)\right\}-KE\left[\mathrm{e}^{-\int_0^s r_d(u)\mathrm{d}u}\,|\ln S(s)\geq\ln K\right]\end{cases}$$

where from Proposition 1

$$\int_0^s r_D(u)\,\mathrm{d}u=Q_1(s)+\rho_1B_1(s)\left(G_1(s)-\chi_1(s)\right)+\delta_1\bar{V}_1.$$

See **Appendix B**.

$$p^*(s,h)=\left\{p(s)\,|\,\varepsilon(s)=h\right\}+\ln s_0-\ln K+\left[Q_1(s)+\delta_1\bar{V}_1\right]$$

Remark 8:

Given the formula for

$\int_0^s r_D(u)\,\mathrm{d}u=Q_1(s)+\rho_1B_1(s)\left(G_1(s)-\chi_1(s)\right)+\delta_1\bar{V}_1$, the stochastic expression $\rho_1B_1(s)\left(G_1(s)-\chi_1(s)\right)$ modifies the function $n(s)$ and the constant terms $Q_1(s)+\delta_1\bar{V}_1$, modifies $p^*(s,h)$ with the addition of $Q_1(s)+\delta_1\bar{V}_1$.

$$K\left[\mathrm{e}^{-\int_0^s r_D(u)\mathrm{d}u}\,|\ln S(s)\geq\ln K\right]$$
$$=\int_{z_{1h}}^{z_{2h}}K\exp\left\{\left(Q_1(s)+\delta_1\bar{V}_1\right)+\left(G_1(s)-\chi(s)\rho_1z\sqrt{s}\right)\right\}\mathrm{e}^{-z^2/2}\mathrm{d}z \qquad (27)$$
$$=K\int_{z_{1h}}^{z_{2h}}\mathrm{e}^{A(s)+A_1(s)z-z^2/2}\mathrm{d}z;$$

Let

$$A_1(s)z\triangleq\rho_1\left(G_1(s)-\chi_1(s)\right)B_1(s)=\rho_1\left(G_1(s)-\chi_1(s)\right)z\sqrt{s};$$
$$A(s)=Q_1(s)+\delta_1\bar{V}_1$$

$$A(s)+A_1(s)z-\frac{z^2}{2}=-\left[z-A_1(s)/2\right]^2/2+A(s)-\frac{A_1^2(s)}{8}; \qquad (28)$$

Then

$$K\int_{z_1}^{z_{2h}} e^{A(s)+A_1(s)z-z^2/2}dz$$
$$=\left[\Phi\left[z_{2h}-A_1(s)/2\right]-\Phi\left[z_{1h}-A_1(s)/2\right]\right]\exp\left(A(s)-\frac{A_1^2(s)}{8}\right);$$

Hedge Ratio:

$$\Delta=\frac{\partial C^*(.)}{\partial s_0}$$
$$=\begin{cases}\frac{1}{\sqrt{1-m(s)}}\left\{\Phi\left(\frac{z_2(h)(1-m(s))-n(s)}{(1-m(s))^{1/2}}\right)-\Phi\left(\frac{z_1(h)(1-m(s))-n(s)}{(1-m(s))^{1/2}}\right)\right\}\\ *\exp\left(p^*(s,h)+\frac{n^2(s)}{2(1-m(s))}\right),\ln K<\omega^*(\rho,h);\\ 0,\ln K\geq\omega^*(\rho,h);\end{cases}$$

Δ-Neutral Portfolio

Delta-Neutral Portfolio

Consider the following portfolio that includes a short position of one European call and a long position of delta units of the domestic currency.

The portfolio of delta-neutral positions is defined as:

$$P=s_0\Delta-c^*\Rightarrow \text{Hedge ratio}(P)=0$$

We obtain below Conditional Risk-neutral Distribution function of

$$\left(\ln S(s)\,|\,\varepsilon(s)=h\sigma_\varepsilon,\bar{V}(s)=\xi\right),\ m(s)<0 \tag{29}$$

by considering the cases of: $h = 1$, 0 and −1

We use a discrete approximation (see [2], (28)).

Suppose $\rho<0$, which implies $m(s)\leq 0$.

Again, we consider the Equations (1)-(4) to define Example 1 below.

$$\frac{dS(u)}{S(u)}=\left(r_D(u)-r_F+\frac{H^2(u)}{2}\right)du+\left(H(u)+\upsilon\right)dB_1(u); \tag{30}$$

$$dH(u)=\alpha\left(H(u)-\theta\right)du+\eta dV_1(u),\theta,\eta\geq 0,\alpha>0; \tag{31}$$

$$dr_D(u)=\beta\left(r_D(u)-\lambda\right)du+\zeta dV_2(u),\lambda,\zeta\geq 0,\beta>0; \tag{32}$$

$$\therefore d\ln S(u)=\left(r_D(u)-r_F-\frac{\upsilon^2}{2}\right)du+\left(H(u)+\upsilon\right)dB_1(u); \tag{33}$$

Let

$$dr_D(u)=\beta\left(r_D(u)-\lambda\right)du+\zeta dV_2(u),\lambda,\varsigma\geq 0,\beta>0,r_D(0)=\kappa_1;$$
$$dr_D(u)=0.3\left(r_D(u)-0.02\right)du+0.04dV_2(u),r_D(0)=0.03;$$
$$dH(u)=\alpha\left(H(u)-\theta\right)du+\eta V_1(u),H(0)=\kappa;$$
$$dH(u)=\left(H(u)-0.1\right)du+0.3V_1(u),H(0)=0.6$$

Then, $H(u)$:

υ	α	η	θ	ρ	κ	δ	s	$H(0)$
0.08	1	0.2	0.1	−0.8	0.1	0.6	0.5	0.06

And $r_D(u)$:

β	θ_1	λ	δ_1	ζ	ρ_1	r_f
0.3	0.2	0.1	0.5	0.04	$\sqrt{3/4}$	0.06

If Assumption (2) holds then the unconditional risk-neutral distribution of $\ln S(s) | (\varepsilon(s) = h, \bar{V}^2(s) = \xi)$ and $U(s, \xi)$ are independent random variables.

Then Figure 1 depicts the unconditional risk-neutral distribution of

$$P\{\ln S(s) \leq u | h\} = F_{\ln(S(s))}(u | h).$$

Remark 9:

Future movement of values of risk-free interest rate and volatility are uncertain and as they increase, they affect call option values as depicted in the above Figure 2, Figure 3 ([5], p. 204). Sudden changes in their values may occur because of economic shock. See the models suggested in [11] [12].

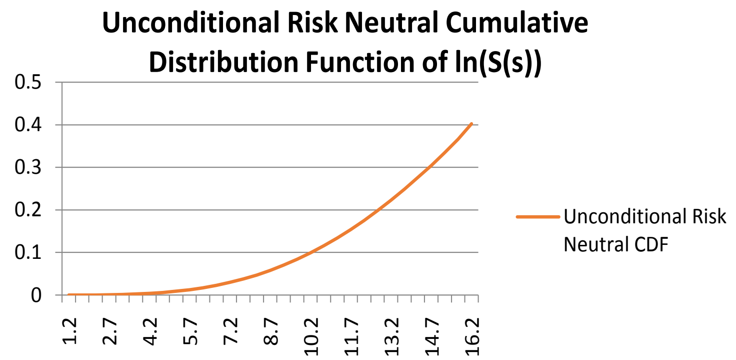

Figure 1. Unconditional risk-neutral CDF of ln*S*(*s*), strike price (cents) *k* from 1.1 to 16.2.

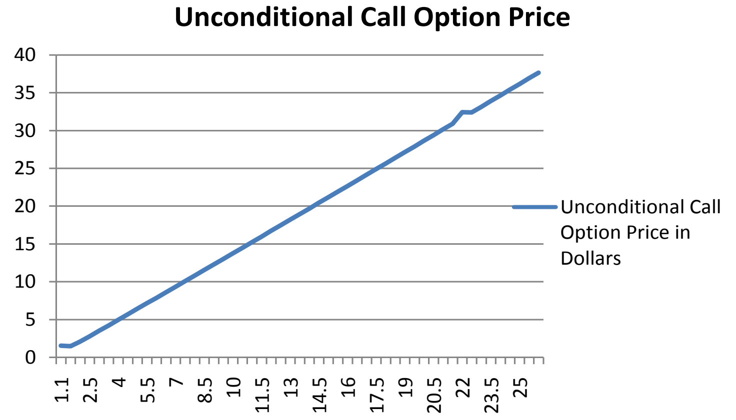

Figure 2. Unconditional call option price with strike price *k* (cents) from 1.1 to 26.

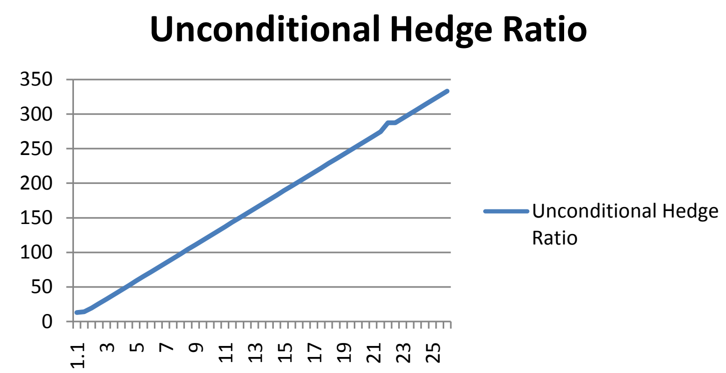

Figure 3. Unconditional hedge ratio with strike price k (cents) from 1.1 to 26.

4. Conclusion

We define a three-factor exchange-rate diffusion model with 1) stochastic volatility process, 2) stochastic domestic interest rate process, and 3) return process which are Brownian motion return processes that are stochastically dependent. Further generalization is possible with the assumption of domestic and foreign stochastic interest rate processes which are subject to economic shocks [11] [12]. The results are applicable to bond option models ([5], p. 783).

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Appendix A

$$\gamma_2(s)=\frac{2}{s^2}\int_0^s \mathrm{d}u\int_0^u \psi(t)\mathrm{d}t=\frac{2}{s^2}\eta\int_0^s\frac{1-\mathrm{e}^{-\alpha u}}{\alpha}\mathrm{d}u=\frac{2}{s^2}\eta\frac{\left(s-\left(1-\mathrm{e}^{\alpha s}\right)/\alpha\right)}{\alpha}$$
$$=\frac{2}{s^2}\eta\frac{\left(\alpha s-\left(1-\mathrm{e}^{\alpha s}\right)\right)}{\alpha^2}.$$
$$\gamma_2(s)=\frac{2}{s^2}\int_0^s \mathrm{d}u\int_0^u \psi(t)\mathrm{d}t$$
is the regression coefficient.

$$e_2(s)=\frac{2}{s^2}\int_0^s\int_0^u\left(\psi(t)-\gamma_2(s)\right)\mathrm{d}B_1(t)\mathrm{d}B_1(u).$$

$$E\left(e_2(s)\right)=0; E\left(e_1(s)\right)=0, Cov\left(e_1(s),e_2(s)\right)=0 \tag{2A1}$$

$$Var\left(e_2(s)\right)=\int_0^s \mathrm{d}u\int_0^u\left(\psi(t)-\gamma_2(s)\right)^2\mathrm{d}t=\frac{2}{s^2}\int_0^s\int_0^u\psi^2(t)\mathrm{d}t\mathrm{d}u-\gamma_2^2(s)$$

Then the regression equation is

$$\psi^{**}(s)=\gamma_2(s)\left(B_1^2(s)/2-\frac{s}{2}\right)+e_2(s); \tag{2A2}$$

Assumption 6:

$$e_2(s)\approx N\left(0,Var^{1/2}\left(e_2(s)\right)\right)\quad \text{(Approximately)} \tag{2A3}$$

Note that $Cov\left(e_1(s),e_2(s)\right)=0$ and

$$e_1(s)\approx N\left(0,Var^{1/2}\left(e_2(s)\right)\right).$$

$$\therefore\left(Var\left(e_1(s)\right)+Var\left(e_2(s)\right)^{1/2}\right)=Var\left(e_1(s)\right)+Var\left(e_2(s)\right)$$

Assumption 7:

$$\varepsilon(s)=e_1(s)+e_2(s)\sim N\left(0,\left(Var\left(e_1(s)\right)+Var\left(e_2(s)\right)\right)^{1/2}\right)\quad \text{(Approximately)} \tag{2A4}$$

Proof of Proposition 1:

$$\int_0^s \mathrm{d}\left[\ln S(u)\right]=\int_0^s \mathrm{d}u\left(\left\{r_d(u)-r_f\right\}-\frac{\upsilon^2}{2}\right)$$
$$+\int_0^u\left(\upsilon+\theta+q_0(u)+\psi(t)\left(\rho\mathrm{d}B_1(t)+\delta\mathrm{d}C_1(t)\right)\right)\mathrm{d}B_1(u);$$

$$\ln S_s=\ln s_0+\int\left\{r_d(u)-r_f\right\}\mathrm{d}u-\frac{1}{2}\upsilon^2 s+\left[(\upsilon+\theta)B_1(s)\right.$$
$$+\left[\gamma_1(s)B_1(s)+e_1(s)\right]+\left[\gamma_2(s)\rho\left(\frac{B_1^2(s)}{2}-\frac{s}{2}\right)+e_2(s)\right]+\left.\delta\bar{V}B_1(s)\right]$$
$$=\ln s_0+\int_0^s r_d(u)\mathrm{d}u-\frac{1}{2}\upsilon^2 s+(\upsilon+\theta)\rho\left(\frac{B_1^2(s)}{2}-\frac{s}{2}\right)$$
$$+\left[\gamma_1(s)B_1(s)+e_1(s)\right]+\delta\bar{V}B_1(s)+\rho\left[\gamma_2(s)\left(\frac{B_1^2(s)}{2}-\frac{s}{2}\right)+e_2(s)\right].$$

Appendix A from [2]

$$\int_0^s \mathrm{d}B_1(u)\int_0^u \psi(t)\mathrm{d}B_1(t) = \gamma_2(s)\int_0^s B_1(u)\mathrm{d}B_1(u) + e_2(s)$$
$$= \gamma_2(s)\left(B_1^2(s) - \frac{s}{2}\right) + e_2(s)$$

where

$$\gamma_2(s) = \frac{2}{s^2}\int_0^s \mathrm{d}u\int_0^u \psi(t)\mathrm{d}t = \frac{2}{s^2}\eta\int_0^s \frac{1-\mathrm{e}^{-\alpha u}}{\alpha}\mathrm{d}u$$
$$= \frac{2}{s^2}\eta\frac{\left(s-\left(1-\mathrm{e}^{\alpha s}\right)/\alpha\right)}{\alpha} = \frac{2}{s^2}\eta\frac{\left(\alpha s-\left(1-\mathrm{e}^{\alpha s}\right)\right)}{\alpha^2}$$
$$e_2(s) = \int_0^s \left(\psi^*(u) - \gamma_2(s)\right)\mathrm{d}u,$$

where

$$\psi^*(u) = \int_0^u \psi(t)\mathrm{d}t$$
$$Var\left(e_2(s)\right) = \int_0^s \left(\psi^*(u)\right)^2 \mathrm{d}t\mathrm{d}u - \gamma_2^2(s)$$
$$= \int_0^s \left(\eta^2\frac{1-\mathrm{e}^{-2\alpha u}}{2\alpha}\right)\mathrm{d}u - \left[\frac{2}{s}\eta\frac{\left(s-\left(1-\mathrm{e}^{\alpha s}\right)/\alpha\right)}{\alpha}\right]^2$$

Appendix B

$$r_D(u) = q_1(u) + \int_0^u \psi_1(t)\mathrm{d}V_2(t)$$
$$= q_1(u) + \int_0^u \psi_1(t)\left[\rho_1\mathrm{d}B_1(t) + \delta_1\mathrm{d}C_1(t)\right];$$
$$\therefore \int_0^s r_D(u)\mathrm{d}u = \int_0^s q_1(u)\mathrm{d}u + \int_0^s \mathrm{d}u\int_0^u \psi_1(t)\left[\rho_1\mathrm{d}B_1(t) + \delta_1\mathrm{d}C_1(t)\right]$$
$$= Q_1(s) + \rho_1 B_1(s)\left(G_1(s) - \chi_1(s)\right) + \delta_1\bar{V}_1 + e_3(s)$$
$$Q_1(s) = \int_0^s q_1(u)\mathrm{d}u;$$

See [13].

$$G(s) = \frac{\vartheta}{\beta s}\left(\frac{s^2}{2!} - \frac{s^3}{3!} + \cdots\right)$$

because

$$G(s) = \frac{\vartheta}{\beta s}\int_0^s \left(u - \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right)\mathrm{d}u$$

Let $\int_0^u \psi_1(t)\mathrm{d}B_1(t) \triangleq \gamma(u)B_1(u) + e_4(u)$

where $\sigma_{e_4}^2 = \int_0^u \psi_1^2(u)\mathrm{d}u - \gamma^2(u).$

Let

$$G(\upsilon) \triangleq \int_0^v \gamma(u)\,\mathrm{d}u = \int_0^v \frac{\vartheta\left(1-\mathrm{e}^{-\beta u}\right)}{\beta u}\mathrm{d}u = \frac{\vartheta}{\beta}\left(v - \frac{v^2}{2!} + \frac{v^3}{3!} + \cdots\right)$$

$$\rho_1 \int_0^s \gamma(u) B(u)\,\mathrm{d}u = \rho_1 \int_0^s B(u)\,\mathrm{d}G(u) = \rho_1\left[B(s)G(s) - \int_0^s G(u)\,\mathrm{d}B(u)\right]$$
$$= \rho_1 B(s)\left(G(s) - \chi(s)\right)$$

Let

$$\int_0^s G(u)\,\mathrm{d}B(u) \triangleq \chi(s)\int_0^s \mathrm{d}B(u) + e_3(s);$$

where applying Wilk's linear regression [14], we get

$$\sigma_{e_3}^2(s) = \int_0^s G^2(u)\,\mathrm{d}u - \chi^2(s)$$
$$\chi(s) = \frac{\int_0^s G(u)\,\mathrm{d}u}{s} = \frac{\int_0^s \mathrm{d}u \int_0^u \frac{\vartheta\left(1-\mathrm{e}^{-\beta t}\right)}{\beta t}\mathrm{d}t}{s} = \frac{\vartheta}{\beta s}\int_0^s \left(u - \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right)\mathrm{d}u \quad \text{(B1)}$$