

Conditioning the Information in Portfolio Optimization

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Abstract

This paper proposes a theoretical analysis of the impact of a suboptimal information set on the two main components used in asset pricing, namely the physical and neutral probability measures and the pricing kernel they define. The analysis is carried out by means of a portfolio optimization problem for a small and rational investor. Solving for the maximal expected utility of terminal wealth, it proves the existence of an information premium between what is required by the theory, that is a complete information set—thus a fully conditional measure—and what is instead achievable by an econometrician using real data. Searching for the best bounds, it then studies the impact of the premium on the pricing kernel. Finally, exploiting the strong interconnection between the pricing kernel and its densities, the impact of the premium on the risk-neutral measure is analyzed.

Keywords

Information Premium, Portfolio Optimization Problem, Suboptimal Filtration Set, Optimal Bounds

1. Introduction

Decision making under uncertainty is an important problem in financial economics. The role of probability is to provide the tools to try to evaluate, as much accurately as possible, which are the possible future scenarios that an investor can face. Knowing that no purely stochastic events can be perfectly forecasted, the degree of accuracy of the outcomes strongly depends upon the quality and quantity of knowledge one has about the outcome of the actions. Unfortunately, despite what is too often assumed by many models in financial economics and more specifically by most of the asset pricing

models, the investors' knowledges about future events are far from being complete.

Starting from a very common finance stochastic problem, a portfolio optimization, this paper shows how this naive assumption may impact on the final performances of a small and rational investor that trades in risky and riskless assets. The distance between a theoretically optimal portfolio and the forecasted one is defined as the information premium. Section 2 shows how this premium proxies the amount of information lost in the so called real world measure. As it will be shown, most of this information is due to the forward looking beliefs of investors that play a crucial role in any investment decision. They are very difficult to estimate using backward looking data only. How this premium may flow from the suboptimally estimated real world measure to the risk-neutral one through the pricing kernel is also analyzed. Aside from the portfolio optimization problem, the contribution of this paper to the literature has a broader impact. It is in fact the scarcity and not the abundance of information the norm for an investors that deals with daily risky decisions using real data. It follows that the same information premium concept and how it may spread on different quantities can be applied to many other risky decisions under uncertainty.

The risk-neutral measure is a risk-adjusted real-world probability. Defined in an arbitrage-free, complete and efficient market, the equivalence of the risk-neutral measure, also known as the equivalent martingale measure (EMM), follows from the three fundamental theorems of asset pricing (FTAP) (Delbaen and Schachermayer, (1994), (1998) [1] [2], Harrison and Pliska, (1981), (1983) [3] [4], Jarrow, (2012) [5], Jarrow and Larsson, (2012) [6]).

Mathematically, the change of measure is made possible by a kernel, an operator. In finance, once properly discounted, this operator takes the name of pricing kernel (henceforth: PK). The tight interconnection existing between the two measures and the PK thus follows by construction and is at the base of most of the philosophy that underpins all of the actual asset pricing theory.

Exploiting this interrelationship, the knowledge of any of these two random variables implies uniquely the third. As a drawback, any possible misestimation may spread and affect more variables. Although too often not considered, this is particularly true in terms of the quality and quantity of the information used in pricing. While it is well known from a theoretical viewpoint, it is in fact still unclear econometrically which information should be transferred among the two measures (and how) to assure the required homogeneity.

A main goal of this paper is to build a bridge between what is required from a mathematical viewpoint and what is instead achievable on a daily basis by an econometrician or an investor that use real data. It follows that, although analyzed from a theoretical viewpoint, most of the main results of the paper have also strong empirical implications.

Dealing with the investors' behavior and defined under the so called real-world probability, it is a well known fact that empirically both the PK as well as the real-world measure are non linear, unobservables and complex quantities to estimate.

Among the other advantages, the non trivial determination of these quantities allowed for the creation of the risk-neutral measure, one of the greatest innovations of all times in financial economics. In fact, the risk-adjustment applied to the artificially created risk-neutral measure, getting rid of the difficulties relative to the estimation of the subjective beliefs, makes its overall determination much less problematic and straightforward.

However, if from one side the creation of the risk-neutral measure solved most of the above problems, from the other side it left the theory silent on the best way to properly estimate both the real-world measure and the pricing kernel. A consequence of that is that there is still not a cut and clear agreement on the best way to properly estimate these random variables. Despite that it is by now conventional wisdom, both in academia and in the business community, to use option data and the relative underlying to estimate the risk-neutral and the real-world measures separately. The extraction of the relative PK thus follows from their discounted ratio.

Here is where most of the problems lie. The joint use of a properly estimated risk-neutral measure with a not so well estimated real-world one may have strong impact not only on the real-world measures and its connected pricing techniques, but also on the relative pricing kernel. At the heart of the problems there are the remarkably different features that characterize the two measures and that lead to very different final outputs. This is especially true from the viewpoint of the market information captured by the two measures, a characteristic too often assumed as negligible in most of the empirical works. In fact, if thanks to the information provided by the implied volatility and its higher moments, the risk-neutral measure extracted from option surfaces is, from an informative viewpoint, a naturally unbiased measure¹, the real-world measure extracted from the underlying stream of past returns is a systematically biased sub-optimal measure. This is always true whenever the measure is estimated by means of a stream of past returns. Being a single value of the stock not enough to extract a density, this is always a necessary step. Options data, on the other side, provide for each day a complete set of strikes and time to maturities, thus making the inference process immediate and correct with no need of further past data.

This backward bias, often ignored, is the cause of possible mispricing and illusory arbitrages. This bias is the key element analyzed by the paper.

At the root of this backward bias there is the inability of capturing (by using historical stock data only) the investors' subjective beliefs which play a crucial role in any investment decision and thus must be part of the information set of any investor.

We define it as a backward bias because, by construction, today investors' beliefs of a future decision are naturally forward looking. An investor decides if and how to trade depending, among the others, on her personal beliefs. Therefore, the evaluation of any risky investment must take into account the forward nature of the investors' subjective beliefs. But what is crystal clear theoretically under a mathematical viewpoint is not the same achievable in practice by an econometrician. For example, counterintuitively with

¹This is always true, by construction, as long as there is no option mispricing.

respect to its nature, the market subjective distribution of future returns is estimated by means of backward looking data (*i.e.*: Äit-Sahalia and Lo (2000) [7], Jackwerth (2000) [8], Brown and Jackwerth (2004) [9], Engle and Rosenberg (2002) [10], Barone Adesi, Engle and Mancini (2008) [11], Yatchev and Härdle (2006) [12]). Relying on these data, an important fraction of the investor's risk and preferences is systematically lost. As a consequence, a discrepancy between what is empirically obtainable and what is theoretically required by the neoclassical asset pricing literature arises. While the above are just a fraction of the huge literature that deals with the estimation of the real-world measure and the relative pricing kernel, it is well known in literature that the estimation of a complete, hence conditional, real-world filtration set is econometrically a non trivial problem. In fact, it not only requires an econometrically advanced model in its estimation, but also (and probably more importantly) the availability of particular data. There is in fact no econometric model that can make backward looking data forward looking. The larger the forward looking information bias (usually due to higher market volatility), the larger is the subsequent mispricing.

The remaining of the paper analyzes this problem and is so organized: Section 2 recalls the above presented backward bias problem from a more rigorous mathematical viewpoint. The problem is then investigated through a portfolio optimization analysis for a small and rational investor under different utility functions. To show the existence and impact of a possible information premium the optimization problem is carried out from the point of view of an investor both with a complete and an incomplete information set. Section 6 analyzes the same problem showing how the existence of a backward bias may impact the area of existence of the pricing kernel. Finally, the connection with the risk-neutral measure and its impacts on the risk neutral pricing closes the paper.

2. Theoretical Motivation

From a probabilistic viewpoint the above presented backward bias, caused by the different assets used in estimation, can be represented by two filtration sets, with one of which is by construction smaller:

$$\mathcal{H}_t \subset \mathcal{F}_t \quad \forall t > 0 \quad (1)$$

Both sets are increasing² in time and contain all available and potentially usable information:

$$\mathcal{H}_t = \{x_{-\infty}, \dots, x_{t-1}, x_{t-\Delta_t}\} \quad \text{and} \quad \mathcal{F}_t = \{x_{-\infty}, \dots, x_{t-1}, x_t\} \quad (2)$$

What makes \mathcal{H}_t smaller is Δ_t : the fraction of missing forward looking information that involves any risky decision to undertake from today with respect to a future time. From here on, \mathcal{F}_t represents the theoretical information set and \mathcal{H}_t the suboptimal one. In connection with the motivation presented above \mathcal{H}_t would be the filtration set obtainable from a stream of past stock returns while \mathcal{F}_t the one obtainable from a

²Two assumptions underpin this statement: the first is that information is time-varying and the second is that decision makers keep memory of all the past data.

cross section of options data.

Being the information set the driver of any forecasting model, a suboptimal filtration set has a sure impact on the estimation of any random variable. It is in fact well known that future stochastic events are usually modelled by means of conditional expectations under the real-world measure³:

$$\mathbb{E}^P (X | \mathcal{F}_t) = \int_{\mathcal{F}} x dP \quad \text{for each } F \in \mathcal{F} \tag{5}$$

where the conditionality is with respect to what is known at time t : the information set. Let us apply the generic case to a finance related problem: the today prediction of the unknown price $P_{t,t+1}$ given I_t , the information available at time t :

$$P_{t,t+1} = \mathbb{E}^P (P_{t,t+1} | I_t) = \int_{-\infty}^{+\infty} P_{t,t+1} \cdot f(P_{t,t+1} | I_t) dP_{t,t+1} \quad \text{for } t < t+1 \tag{6}$$

Given the time horizon of the prediction, here $t, t+1$, the above integral⁴ represents the weighted sum of the averaged possible future values, $P_{t,t+1}$, under the physical probability. From (6) emerges clearly the importance of the right use of the information set. The average is in fact computed using the I_t conditional probabilities, $f(P_{t,t+1} | I_t) dP_{t,t+1}$. All information available thus enter into the $P_{t,t+1}$ forecast. The poorer the information set, the worse the final outcome.

Given the different informative content of the inputs it follows that while theoretically, under some technical assumptions, the same expectation can be estimated indifferently using either the physical \mathbb{E}^P or the risk-neutral \mathbb{E}^Q measure, in reality this equivalence is often violated.

For example, applying and extending under the two measures the above fundamental asset pricing Equation (6) to compute the today price, $G_{t,T}$, of a generic contingent claim that expires at time T :

$$G_{t,T} = e^{-r_t T} \mathbb{E}^Q [\varphi_{t,T}(S_T) | \mathcal{F}_t] \tag{7}$$

$$= e^{-r_t T} \int_0^{\infty} M_{t,T}(S_T) \cdot \varphi_{t,T}(S_T) p_{t,T}(S_T | S_t) dS_T \tag{8}$$

$$= e^{-r_t T} \mathbb{E}^P [M_{t,T}(S_T) \cdot \varphi_{t,T}(S_T) | \mathcal{F}_t] \tag{9}$$

$$= e^{-r_t T} \mathbb{E}^P [M_{t,T}(S_T) \cdot \varphi_{t,T}(S_T)] \tag{10}$$

$$\neq e^{-r_t T} \mathbb{E}^{\hat{P}} [\hat{M}_{t-\Delta_t, T}(S_T) \cdot \varphi_{t,T}(S_T)] \tag{11}$$

³Let us assume that at time t we want to forecast the tomorrow's value of a random variable X_{t+1} given the set of available information I_t . This is an optimization problem; more precisely we pick the best predictor among all possible predictors by choosing the one that minimizes the expected quadratic prediction error:

$$\mathbb{E}^P [(x_{t+1} - \hat{x}_{t+1|t})^2 | I_t] \tag{3}$$

Given the problem, the best (minimum mean squared error: MMSE) predictor is the conditional expectation given the information set:

$$\hat{x}_{t+1|t} \equiv \mathbb{E}^P (x_{t+1} | I_t) \tag{4}$$

⁴The extremes of the integral may also be defined *i.e.*: put and call options are bounded by their strike prices either above or below.

$$= e^{-r_t T} \mathbb{E}^{\hat{P}} \left[M_{t-\Delta_t, T}(S_T) \cdot \varphi_{t, T}(S_T) \mid \mathcal{H}_t \right] \quad (12)$$

$$= e^{-r_t T} \int_0^\infty \hat{M}_{t-\Delta_t, T}(S_T) \cdot \varphi_{t, T}(S_T) \hat{p}_{t-\Delta_t, T}(S_T \mid S_t) dS_T \quad (13)$$

$$= e^{-r_t T} \mathbb{E}^{\hat{Q}} \left[\varphi_{t, T}(S_T) \mid \mathcal{H}_t \right] \quad (14)$$

$$= \hat{G}_{t-\Delta_t, T} \quad (15)$$

where, for each $t \in T$, we use:

$$x = \mathbb{E}^P(x \mid \mathcal{F}_t) \quad \text{and} \quad \hat{x} = \mathbb{E}^{\hat{P}}(x \mid \mathcal{H}_t) \quad (16)$$

to represent the conditional expectation under the optimal (left) and suboptimal (right) information set. For all equations, $\tau = T - t$ represents the time-to-maturity and $\varphi_{t, T}(S_T)$ the terminal payoff of the product given the value of the underlying S_T . The inequality in (11) is due to the missing but necessary information (relative to T) of the today (time t) risk-physical measure:

$$p_{t, T}(S_T \mid S_t) \neq p_{t-\Delta_t, T}(S_T \mid S_t). \quad (17)$$

Although the missing information is referred to the future, its impact is absorbed today and propagated onto tomorrow price forecast. As a consequence, while the left hand side is fully conditional on all values known today, the right hand side is not. Furthermore, if the PK is extracted from the discounted ratio of the pricing over the physical measure, it is immediate that an improper use of the information set propagates a.s. from the real-world to the risk-neutral pricing equation through the PK, $\hat{M}_{t-\Delta_t, T}$. The difference in (17) is then the focus of the paper.

It follows naturally that a suboptimal information set has also an impact on the relative probability measures:

$$\mathcal{F} \Rightarrow P \quad \text{and} \quad \mathcal{H} \Rightarrow \hat{P} \quad (18)$$

$$\mathcal{F} \Rightarrow Q \quad \text{and} \quad \mathcal{H} \Rightarrow \hat{Q} \quad (19)$$

where P (resp. \hat{P}) represents the physical probability measure for a the optimal \mathcal{F} (resp. suboptimal \mathcal{H}) information set. The same logic applies for the risk-neutral measures. The difference between $G_{t, T}$ and $\hat{G}_{t-\Delta_t, T}$ thus mirrors the distance between an empirically biased asset pricing with respect to the theoretical one. This distance represents the sub-optimality of the information set: what we will call as the information premium.

In a nutshell: conditional expectations reflect the change in unconditional probabilities given some auxiliary information; the definition and use of this information is then of fundamental importance to evaluate this change correctly. If the filtration used is missing of relevant information, hence is suboptimal, the projection onto the smaller set leads a.s. to an inequality:

$$\mathbb{E}^P(X \mid \mathcal{F}_t) \neq \mathbb{E}^{\hat{P}}(X \mid \mathcal{H}_t) = \int_{\mathcal{H}} x dP \quad \text{for each } H \in \mathcal{H} \quad (20)$$

Although from an economic perspective consumption-based models would be the most comprehensive answer to all of the asset pricing questions, it is well known that

they are not fully reliable. At the origin of their biases there are the high difficulties concerning a proper estimation of the investors' consumption. To overcome the issue, a natural alternative approach to the classical Lucas asset pricing model is the following: given an economy where one can observe the prices of the available assets and try to model the subjective distribution of their final payoffs, what is the optimal portfolio for a small and rational investor⁵? Although the problem is very well-known in literature, not much emphasis has been put on the frequent biases that can arise whenever the investors' subjective beliefs that compose her distribution are suboptimally estimated.

Extending Øksendal (2005) [13] this paper analyzes this problem starting from a simple economy described by a Levy-Itô mixed model and it analyzes how the obtained outputs may impact on the total profitability of the investors.

3. The Optimization Problem

To analyze the above problem we model a simple economy made of two assets and we study the impact of a smaller information set onto a rational investor that, endowed with a positive initial capital, wants to maximize her final welfare choosing among the set of admissible portfolios:

$$\Lambda(x_t) = \sup_{\xi_t \in \mathcal{A}_{\mathcal{I}_t}} E^P \left[U \left(X^{\xi_t}(T) \right) \right] \tag{21}$$

Under this framework, the conditionality of the expectation in (21) is represented by the set of feasible portfolios $\mathcal{A}_{\mathcal{I}_t}$ where \mathcal{A} represents the set of admissible portfolios and its subscript \mathcal{I}_t restricts this set to the information available at time t . We will compute and compare the same optimization problem conditional on the complete and a suboptimal information set. Given a fixed time window, $[0, T]$, and for each $t \in T$, the optimal portfolio weight of the investor conditional to the information set (if it exists), is represented by ξ_t . It follows that $\Lambda_t < \infty$ is the optimal value of the problem and both random variables (the complete and the suboptimal) are then \mathcal{I}_t -adapted stochastic processes.

Equation (21) is nothing but a classical consumer problem (CP) where the optimization is with respect to some financial asset class. As a consequence x_t is a financial instrument that proxies consumption: *i.e.* wealth. The resolution of the optimization problem is linked to the arbitrary choice of a regular utility function $U(\cdot): [0, \infty] \rightarrow [-\infty, \infty]$ ⁶. For simplicity the optimization is first analyzed assuming a generic logarithmic utility function⁷:

$$U(x_t) = \ln x_t \quad x_t > 0 \tag{22}$$

⁵For small investor we define an unsophisticated investor that cannot affect the market prices with her trading. Rationality is defined under the neo-classical theory as a risk averse investor that always prefers more to less.

⁶A utility function is assumed to be regular if it is differentiable, concave and non-negative.

⁷With no loss of generality results applying to different types of utility functions. The main goal of the paper is to provide an intuitive explanation of the problem more than just a set of mathematical solutions of the optimization. Therefore the model is one-dimensional and only the cases of log and power utility functions are studied. Although much less explicit, similar results can be achieved with other utility functions and in a multidimensional framework.

The resolution of the asset pricing problem (21) leads to the optimal expected logarithmic utility of the investor terminal wealth $X^{\xi_T}(T)$.

The entire optimization is performed under the physical measure, P , and conditional to the filtration set of the investor, \mathcal{I}_t . This leads naturally to the use of the partial information approach presented in the previous section. It assumed a fixed time frame $0 \leq t \leq T \leq \infty$ such that, for each time $t \geq 0$:

$$\mathcal{I}_t = \mathcal{H}_t \subset \mathcal{F}_t \subset \mathcal{F} \quad (23)$$

It follows that, depending on the filtration in use, the degree of adeptness (measurability) of the stochastic process may change, thus impacting in various form on the final outcomes of the optimization. As a technical remark: both filtrations lie in filtered probability spaces:

$$\left(\Omega, (\mathcal{H}_t)_{0 \leq t \leq T}, P, \mathbb{H}\right) \subset \left(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, P, \mathbb{F}\right) \quad (24)$$

and satisfy the usual hypotheses.

The goal of the presented framework is to model the economy of a single investor with a suboptimal information set, where the stochastic processes that affect the optimal portfolio choice are \mathcal{H}_t and not \mathcal{F}_t -adapted, thus reflecting the poorer decisional power of the investor, due to the lack of forward looking information.

4. The Lévy-Itô Model and the Portfolio Optimization Problem

Probably as a direct consequence of the much larger theoretical literature relative to the enlargement of filtration with respect to the one relative to the shrinkage of filtration, the same degree of richness is reflected on their different applications, *i.e.*: portfolio optimization. Equation (21) mirrors the well-known stochastic control problems related to the insider information (*i.e.* see Biagini and Øksendal (2005) [14]). While the insider information cases are characterized by an informed trader that has a larger information set with respect to the one of the “honest” trader, here things go on the opposite directions. In both cases extra assumptions and computational tools are required. Differently than the insider case, the literature for these problems is far more scarce.

Following Øksendal (2005) [13] our starting point is a Lévy-Itô market model composed by two assets: one risky and one risk-free. Given this simple economy⁸, the investor implements her portfolio through a dynamic trading of the two assets. The choice of modelling the stochastic part of the risky investment by means of a Brownian motion (Itô process) and a pure jump process (Lévy process) instead of just using the classical diffusion-Itô model is justified by the higher descriptive power of the former⁹.

More in detail, the two assets that compose the economy are:

- A riskless asset, whose rate of return is allowed to fluctuate but that is otherwise

⁸Extending the problem to a model with 1 risk-free asset and n , $1 < n < \infty$ risky assets is surely more realistic but the obtained results would be much less immediate to interpret. As a main difficulty there is the delicate calibration of the possible correlation among the assets.

⁹For modelling details we refer to Barndorff-Nielsen (1998) [15] and Cont and Tankov (2004) [16].

risk-free, which is represented by a risk-free bond P_0 , with unit price $P_0(t)$ at time t is:

$$dP_0(t) = r(t)P_0(t)dt \quad 0 \leq t \leq T \tag{25}$$

$$P_0(0) = 1 \tag{26}$$

where $r(t) = r(t, \omega) > 0$ is \mathcal{I}_t -adapted and represents the time invariant risk-free rate in the market¹⁰.

- A risky asset, represented by a stock P driven by a one dimensional Brownian motion and a pure jump process (Lévy-Itô process), with unit price $P(t)$ at time t is:

$$dP(t) = P(t^-) \left[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \theta(t, z) \tilde{N}(dt, dz) \right] \tag{27}$$

$$P(0) > 0 \quad 0 \leq t \leq T \tag{28}$$

Technical assumptions (T.A.):

For each $t \in [0, T]$, $\omega \in \Omega$ and $z \in \mathbb{R} \setminus \{0\}$ it is assumed that the parameters of the continuous part of the process $\mu(t, \omega)$, $\sigma(t, \omega)$ and $B(t, \omega)$ satisfy the following technical assumptions (T.A.):

T.A.1 \mathcal{I}_t -progressively measurable, hence time dependent and non-anticipating.

T.A.2 bounded on $[0, T] \times \Omega$.

T.A.3 parameters $\mu(t) = \mu(t, \omega)$ and $\sigma(t) = \sigma(t, \omega)$ represent respectively the investors' expected returns form $P(t)$ and the volatility of $P(t)$.

T.A.4 $B(t) = B(t, \omega)$ is an \mathcal{I}_t -adapted one dimension Brownian Motion.

While the parameters governing the jump part of the process are such that:

T.A.5 $\theta(t, z, \omega)$ is \mathcal{I}_t -adapted hence time dependent, and non-anticipating.

T.A.6 $\tilde{N}(dt, dz) = N(dt, dz) - \nu_{\mathcal{I}_t}(dz)dt$ is the \mathcal{I}_t -compensated Poisson random measure of $\eta(t) = \eta(t, \omega): [0, T] \times \Omega \rightarrow \mathbb{R}$ where:

$$\eta(t) = \int_0^t \int_{\mathbb{R}} \theta(t, z) \tilde{N}(dt, dz) \tag{29}$$

T.A.7 to prevent the process to be ≤ 0 , we set: $\theta(t, z) > -1$ for a.a. t, z with respect to $dt \times \nu_{\mathcal{I}_t}(dz)$.

T.A.8 $\mathbb{E}^P[\eta^2(t)] < \infty$ for all $t \geq 0$.

Further details on the main characteristics that govern the jump part of (27) are in Appendix (9.1) and we refer the reader to Tankov and Cont (2000) [16] for more details about Lévy processes in finance.

The application of the Itô formula for semimartingales plays a key role for the resolution of the problem. It turns out that also its application is fully dependent to a proper assessment of the filtration set. Let us quickly recall it:

Theorem 1. *If $A(t)$ is an Itô-Lévy process.*

$$dA(t) = at + bB(t) + \int_{\mathbb{R}} \alpha(t, z) \tilde{N}(dt, dz) \tag{30}$$

¹⁰Equation (25) could be further refined by using a time varying risk-free rate as proposed by *i.e.*: Korn and Kraft (2001) [17]. The usual precision/tractability trade-off arises. To guarantee a higher tractability in the model we leave it deterministic.

such that:

$$\int_0^t |a(s)| ds + \int_0^t b^2(s) ds + \int_0^t \int_{\mathbb{R}} \alpha^2(s, z) \nu(dz) ds < \infty \tag{31}$$

and $D(t) = f(t, A(t))$, where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $f \in C^{1,2}$, then, by the Itô formula for semimartingales:

$$\begin{aligned} dD(t) = & \left\{ \frac{df}{da}(t, A(t)) + \frac{df}{da}(t, A(t))a(t) + \frac{1}{2} \frac{d^2f}{da^2}(t, A(t))b^2(t) \right\} dt \\ & + \frac{df}{da}(t, A(t))b(t)dB(t) + \int_{\mathbb{R}} \left\{ f(t, A(t) + \alpha(t, z)) - f(t, A(t)) \right\} \tilde{N}(dt, dz) \tag{32} \\ & + \int_{\mathbb{R}} \left\{ f(t, A(t) + \alpha(t, z)) - f(t, A(t)) - \frac{df}{da}(t, A(t))\alpha(a, z) \right\} \nu(dz) dt \end{aligned}$$

the process $D(t)$ is an Itô-Lévy process as well.

Let us apply theorem (1) to the portfolio problem. If the presented T.A. are satisfied and if the t and z dependent parameters of Equations (25) and (27) are \mathcal{I}_t -adapted such that:

$$\mathbb{E}^P \left\{ \int_0^T \left[|r(s)| + |\mu(s)| + \sigma^2(s) + \int_{\mathbb{R}} \theta^2(s, z) \nu_{\mathcal{I}_t}(dz) \right] ds \right\} < \infty \tag{33}$$

then, by the Itô formula for semimartingales, the solution of (27) is:

$$\begin{aligned} P(t) = & P(0) \exp \left\{ \int_0^t \left[\mu(s) - \frac{1}{2} \sigma^2(s) - \int_{\mathbb{R}} (\theta(s, z) - \ln(1 + \theta(s, z))) \nu_{\mathcal{I}_t}(dz) \right] ds \right. \\ & \left. + \int_0^t \sigma(s) dB(s) + \int_0^t \int_{\mathbb{R}} \ln(1 + \theta(s, z)) \tilde{N}(dt, dz) \right\}; \quad 0 \leq t \leq T \end{aligned} \tag{34}$$

Now, suppose that $\xi(t) = \xi(t, \omega) : [0, T] \times \Omega \rightarrow \mathbb{R}, \omega \in \Omega$ is an \mathcal{I}_t -measurable stochastic process representing the fraction of wealth $X(t)$ invested the investor in the risky asset and $(1 - \xi(t))$ is invested in the risk-free asset. The evolution of the total wealth process, $X(t) = X^{\xi}(t)$ ¹¹, of the investor is then:

$$dX(t) = (1 - \xi(t))X(t)r(t)dt + \xi(t)X(t) \left\{ \mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \theta(t, z) \tilde{N}(dt, dz) \right\} \tag{35}$$

$$X(0) = x > 0 \quad (\text{Initial capital}) \tag{36}$$

or, collecting terms:

$$\begin{aligned} dX(t) = & X(t^-) \left\{ \left[r(t) + (\mu(t) - r(t))\xi(t) \right] dt + \sigma(t)\xi(t)dB(t) \right. \\ & \left. + \xi(t) \int_{\mathbb{R}} \theta(t, z) \tilde{N}(dt, dz) \right\}; \quad 0 \leq t \leq T \end{aligned} \tag{37}$$

$$X(0) = x > 0 \quad (\text{Initial capital}) \tag{38}$$

A key element for the analysis of the paper is how to define an admissible portfolio under the different filtrations in use.

Definition 4.1. *Given a small and rational investor, a portfolio process $\xi(t)$ is assumed to be \mathcal{I}_t -admissible for each $t \in T$ if:*

- is \mathcal{I}_t -adapted for each t , where $0 < t < T < \infty$.

¹¹The parameter ξ is time-varying and its time dependency is equivalently represented with ξ_t or $\xi(t)$.

- $\mathbb{E}^P \left\{ \int_0^T \left[|\mu(t) - r(t)| \cdot |\xi(t)| + \sigma^2(t) \xi^2(t) + \int_{\mathbb{R}} \xi^2(t) \theta^2(t, z) \nu_{\mathcal{I}}(dz) \right] dt \right\} < \infty.$
- $\xi(t)\theta(t, z) \geq -1$ a.s. for $dt \times \nu_{\mathcal{I}_t} dz$ for a.a. t and z .

Given this framework, we analyze the stochastic control problem of a small investor whose goal is to maximize $\mathbb{E}^P(U^{\xi_t}(T))$ over a finite time window $[0, T]$ and over the class of all possible time (t) admissible portfolios $\mathcal{A}_{\mathcal{I}_t}(t)$. Formally:

Proposition 2. *We solve a finite horizon $T < \infty$ stochastic control problem of a rational and small investor endowed with a positive initial capital $X(0) = x > 0$ and described by a generic utility function $U(\cdot)$, here a logarithmic utility function, whose goal is to maximize her expected utility from terminal wealth, i.e.: $\mathbb{E}^P[U(\cdot)(T)]$ by investing continuously in a risky, P , and in a risk-free asset, P_0 .*

The optimal value of the problem, denoted by, $\Lambda_{\mathcal{I}_t}^{\text{Log}}$, is valid only if, for each $t \in [0, T]$ and $\omega \in \Omega$, there exists an optimal portfolio process, $\xi(t, \omega) = \xi_{\mathcal{I}_t}^*(t, \omega)$, which belongs to the set of admissible portfolios, $\mathcal{A}_{\mathcal{I}_t}$, s.t.:

$$\Lambda_{\mathcal{I}_t}^{\text{Log}}(x_t) = \sup_{\xi_t \in \mathcal{A}_{\mathcal{I}_t}} \mathbb{E}^P \left[\ln X^{\xi_t}(T) \right] = \mathbb{E}^P \left[\ln X^{\xi_t^*}(T) \right] \tag{39}$$

where the value function of the problem is assumed to be:

$$\Lambda_{\mathcal{I}_t}^{\text{Log}}(x_t) < \infty \quad \forall x_t \in [0, \infty) \tag{40}$$

Given proposition (2) it emerges clearly that the solution of the problem is strongly related to the admissibility of the portfolio with respect to the information set, \mathcal{I}_t . To better underline its importance, the same portfolio problem will be analyzed under two different information sets:

- 1) Theoretical case: $\mathcal{I}_t = \mathcal{F}_t$: the information set is complete.
- 2) Real-world case: $\mathcal{I}_t = \mathcal{H}_t \subset \mathcal{F}_t$: some relevant information is missing, hence cannot be reflected in the final asset price.

The former case pertains to an investor which is able to set up an asset pricing model that fully capture *all* past, present and future relevant pricing information, hence also and above all her forward looking beliefs with respect to the future outcome. The latter pertains instead to a more realistic suboptimal case. Illusory arbitrages may be the naive consequence of an investor not being fully aware of pertaining to the former or the latter group.

While is at least since Merton (1969) [18] and Samuelson (1969) [19] that the stochastic control problem for $(\mathcal{F}_t)_{t \geq 0} = \mathbb{F}$ -adapted portfolios is well-known in literature (a good review of the subject is, among the others, Cvitanic and Karatzas (1992) [20]), the one for an Itô-Lévy market model with a suboptimal information set is not. We extend the literature answering this problem and analyzing its effect in asset pricing under different viewpoints.

4.1. The Theoretical Optimal Choice

This subsection solves the portfolio optimization problem for an investor with logarithmic utility function under the theoretical case: when the filtration set is the complete one:

$$\text{Theoretical Case : } \mathcal{I}_t = \mathcal{F}_t \quad (41)$$

Assuming for all $t \in T$, that $\xi_t \in \mathcal{A}_{\mathcal{I}_t}$ and applying the Itô theorem for semi-martingales to (37) the admissible investor's terminal wealth is:

$$\begin{aligned} X^{\xi_{t \in \mathcal{F}_t}}(T) = x_t \exp \left\{ \int_0^T \left[r(s) + (\mu(s) - r(s)) \xi_{\in \mathcal{F}_s}(s) - \frac{1}{2} \sigma^2(s) \xi_{\in \mathcal{F}_s}^2(s) \right] ds \right. \\ \left. + \int_0^T \xi_{\in \mathcal{F}_s}(s) \sigma(s) dB(s) + \int_0^T \int_{\mathbb{R}} \ln(1 + \xi_{\in \mathcal{F}_s}(s) \theta(s, z)) \tilde{N}(ds, dz) \right. \\ \left. - \int_0^T \int_{\mathbb{R}} \left(\xi_{\in \mathcal{F}_s}(s) \theta(s, z) - \ln(1 + \xi_{\in \mathcal{F}_s}(s) \theta(s, z)) \right) \nu_{\mathcal{F}_s}(dz, ds) \right\} \end{aligned} \quad (42)$$

Given the evolution of the total wealth of the investor, and assuming that:

$$\mathbb{E}^P \left[\ln X^{\xi_{t \in \mathcal{F}_t}}(T) \right] < \infty \quad (43)$$

the expected value of the problem is:

$$\begin{aligned} \mathbb{E}^P \left[\ln X^{\xi_{t \in \mathcal{F}_t}}(T) \right] - \ln(x_t) = \mathbb{E}^P \left\{ \int_0^T \left[r(s) + (\mu(s) - r(s)) \xi_{\in \mathcal{F}_s}(s) - \frac{1}{2} \sigma^2(s) \xi_{\in \mathcal{F}_s}^2(s) \right] ds \right. \\ \left. - \int_{\mathbb{R}} \left(\xi_{\in \mathcal{F}_s}(s) \theta(s, z) - \ln(1 + \xi_{\in \mathcal{F}_s}(s) \theta(s, z)) \right) \nu_{\mathcal{F}_s}(dz) \right\} \end{aligned} \quad (44)$$

Fixing s and ω , the objective function to maximize is:

$$\begin{aligned} h(\xi_{\in \mathcal{F}_s}(s)) = (\mu(s) - r(s)) \xi_{\in \mathcal{F}_s}(s) - \frac{1}{2} \sigma^2(s) \xi_{\in \mathcal{F}_s}^2(s) \\ - \int_{\mathbb{R}} \left(\xi_{\in \mathcal{F}_s}(s) \theta(s, z) - \ln(1 + \xi_{\in \mathcal{F}_s}(s) \theta(s, z)) \right) \nu_{\mathcal{F}_s}(dz) \end{aligned} \quad (45)$$

where h is a positive and concave function. Taking the first order condition with respect to $\xi_{\in \mathcal{F}_s}(s)$ and equating the result to zero we obtain the solution

$$\xi_{\in \mathcal{F}_s}(s) = \xi_{\in \mathcal{F}_s}(s, \omega):$$

$$0 = h'(\xi_{\in \mathcal{F}_s}(s)) = \mu(s) - r(s) - \sigma^2(s) \xi_{\in \mathcal{F}_s}(s) - \int_{\mathbb{R}} \left(\theta(s, z) - \frac{\theta(s, z)}{1 + \xi_{\in \mathcal{F}_s}(s) \theta(s, z)} \right) \nu_{\mathcal{F}_s}(dz) \quad (46)$$

Collecting terms, the extra return from the portfolio is:

$$\mu(s) - r(s) = \sigma^2(s) \xi_{\in \mathcal{F}_s}(s) + \int_{\mathbb{R}} \frac{\xi_{\in \mathcal{F}_s}(s) \theta^2(s, z)}{1 + \xi_{\in \mathcal{F}_s}(s) \theta(s, z)} \nu_{\mathcal{F}_s}(dz) \quad (47)$$

which, for the continuous case $\theta(s, z) = 0$:

$$\xi_{\in \mathcal{F}_s}(s) = \frac{\mu(s) - r(s)}{\sigma^2(s)} \quad (48)$$

whose validity depends on $\sigma(s) \neq 0$ a.s. for a.a. (s, ω) , $s \in [0, T]$ and $\omega \in \Omega$. $\xi_{\in \mathcal{F}_s}(s)$ is the optimal portfolio process $\xi_{\in \mathcal{F}_s}^*(s)$ only if is in the set of admissible portfolios given the information set:

$$\xi \in \mathcal{A}_{\mathcal{F}_s} \rightarrow \xi_{\in \mathcal{F}_s}(s) = \xi_{\in \mathcal{F}_s}^*(s) \quad \text{for all } s \in [0, T] \quad (49)$$

It follows that the optimal portfolio value for the case of a theoretical (or full) information set \mathcal{F}_t is:

$$\Lambda_{\mathcal{F}_T}^{\text{Log}} = \mathbb{E}^P \left[\ln X^{\xi \in \mathcal{F}_T} (T) \right] = \ln x_t + \mathbb{E}^P \left[\int_0^T \left(r(s) + \frac{(\mu(s) - r(s))^2}{2\sigma^2(s)} \right) ds \right] \quad (50)$$

With no loss of generality the investor initial capital can be set as $X(0) = 1$ s.t.:

$$\Lambda_{\mathcal{F}_T}^{\text{Log}} = \mathbb{E}^P \left[\ln X^{\xi \in \mathcal{F}_T} (T) \right] = \mathbb{E}^P \left[\int_0^T \left(r(s) + \frac{(\mu(s) - r(s))^2}{2\sigma^2(s)} \right) ds \right] \quad (51)$$

In conclusion: for a finite-time complete market model with one riskless and one risky asset, the maximal expected logarithmic utility of the terminal wealth for a small and rational investor with a complete set of information is the integrated sum of the risk-free rate and a fraction of the Sharpe ratio squared.

4.2. The Real-World Optimal Choice

This subsection analyzes again the same problem but for the viewpoint of an econometrician that uses real data:

$$\text{Real-world case : } \mathcal{I}_t = \mathcal{H}_t \subset \mathcal{F}_t \quad (52)$$

It thus focus the attention, both theoretically and conceptually, on how the missing information of the filtration set propagates and affect onto the final profit of the investor.

Except for $\xi_{t \in \mathcal{H}_t}$ and $\nu_{\mathcal{H}_t}(dz, dt)$, which are affected by the scarcer informative content given by the suboptimal filtration set, the starting point for the restricted case is the same as the complete one:

$$\begin{aligned} X^{\xi \in \mathcal{H}_t} (T) = x_t \exp \left\{ \int_0^T \left[r(s) + (\mu(s) - r(s)) \xi_{\in \mathcal{H}_s}(s) - \frac{1}{2} \sigma^2(s) \xi_{\in \mathcal{H}_s}^2(s) \right] ds \right. \\ \left. + \int_0^T \int_{\mathbb{R}} \xi_{\in \mathcal{H}_s}(s) \sigma(s) dB(s) + \int_0^T \int_{\mathbb{R}} \ln(1 + \xi_{\in \mathcal{H}_s}(s) \theta(s, z)) \tilde{N}(ds, dz) \right. \\ \left. - \int_0^T \int_{\mathbb{R}} \left(\xi_{\in \mathcal{H}_s}(s) \theta(s, z) - \ln(1 + \xi_{\in \mathcal{H}_s}(s) \theta(s, z)) \right) \nu_{\mathcal{H}_s}(dz, ds) \right\} \quad (53) \end{aligned}$$

Given the evolution of the total wealth of the investor and assuming that:

$$\mathbb{E}^{\hat{P}} \left[\ln X^{\xi_{t \in \mathcal{H}_t}} (T) \right] < \infty \quad (54)$$

the expected value of the problem is:

$$\begin{aligned} & \mathbb{E}^{\hat{P}} \left[\ln X^{\xi_{t \in \mathcal{H}_t}} (T) \right] - \ln(x_t) \\ &= \mathbb{E}^{\hat{P}} \left\{ \int_0^T \left[r(s) + (\mu(s) - r(s)) \xi_{\in \mathcal{H}_s}(s) - \frac{1}{2} \sigma^2(s) \xi_{\in \mathcal{H}_s}^2(s) \right. \right. \\ & \quad \left. \left. - \int_{\mathbb{R}} \left(\xi_{\in \mathcal{H}_s}(s) \theta(s, z) - \ln(1 + \xi_{\in \mathcal{H}_s}(s) \theta(s, z)) \right) \nu_{\mathcal{H}_s}(dz) \right] ds \right\} \quad (55) \end{aligned}$$

To account for the smaller amount of information for the optimization of the restricted optimal portfolio $\xi_t \in \mathcal{A}_{\mathcal{H}_t}$, we need to insert, for all $t \in T$, an extra conditioning of the expectation with respect to \mathcal{H}_t :

$$\begin{aligned} & \mathbb{E}^{\hat{P}} \left[\ln X^{\xi_{t \in \mathcal{H}_t}}(T) \right] - \ln(x_t) \\ &= \mathbb{E}^{\hat{P}} \left\{ \int_0^T \mathbb{E}^P [r(s) + (\mu(s) - r(s)) \xi_{\in \mathcal{H}_s}(s) - \frac{1}{2} \sigma^2(s) \xi_{\in \mathcal{H}_s}^2(s) \right. \\ & \quad \left. - \int_{\mathbb{R}} (\xi_{\in \mathcal{H}_s}(s) \theta(s, z) - \ln(1 + \xi_{\in \mathcal{H}_s}(s) \theta(s, z))) \nu_{\mathcal{H}_s}(dz) \mid \mathcal{H}_s] ds \right\} \end{aligned} \quad (56)$$

or, applying the same convention as before:

$$\hat{x}(t) = \mathbb{E}^{\hat{P}} [x(t) \mid \mathcal{H}_t] \quad (57)$$

then:

$$\begin{aligned} &= \mathbb{E}^{\hat{P}} \left\{ \int_0^T \left[r(s) + (\hat{\mu}(s) - r(s)) \xi_{\in \mathcal{H}_s}(s) - \frac{1}{2} \hat{\sigma}^2(s) \xi_{\in \mathcal{H}_s}^2(s) \right. \right. \\ & \quad \left. \left. - \int_{\mathbb{R}} (\xi_{\in \mathcal{H}_s}(s) \hat{\theta}(s, z) - \mathbb{E}^P (\ln(1 + \xi_{\in \mathcal{H}_s}(s) \theta(s, z)) \mid \mathcal{H}_s)) \nu_{\mathcal{H}_s}(dz) \right] ds \right\} \end{aligned} \quad (58)$$

where $(\hat{\cdot})$ identifies the value of the parameters under the \mathcal{H}_t restriction.

Dealing with the subjective beliefs that affect the investor's investment decisions, the consequences of a suboptimal filtration set have an impact only on the risky assets of the investor's portfolio. Although allowed to fluctuate, the risk-free asset is in fact assumed to be free of any subjective and objective risks thus not affecting the overall investor's subjective behaviour with respect to risky decision to undertake.

This can be easily "demonstrated" starting from the definition of the classical physical pricing equation:

$$f_{t,T}(B_{t,T}) = \mathbb{E}^P (m_{t,T} \cdot B_{t,T} \mid \mathcal{F}_t) \quad (59)$$

where $f_{t,T}(B_{t,T})$ represents the time t pricing functional of a risk-free bond, B , which matures at time T and $m_{t,T}$ its relative PK. By construction, the role of the PK in asset pricing is to adjust the future payoff accounting for all possible risky future outcomes that can affect the underlying. A risk-free product is, by definition, without risk¹². Not having to consider any risk or time preference of the investor, its expected final payoff is at any point in time equal to one, no matter what happens in the market. Given a finite time window, $t = 0, \dots, T < \infty$, this is translated in a PK:

$$\mathbb{E}^P (m_{t,T} \mid \mathcal{F}_0) = 1 \quad \text{for all } t \in T \quad (60)$$

As a consequence, at any Δt , the risk-free bond is immune to any risk and Equation (59) can be iterated-back in time such that its time 0 price is:

$$f_{0,T}(B_{0,T}) = \mathbb{E}^P (B_{0,T} \mid \mathcal{F}_0) \quad (61)$$

Given the role of the PK, the absence of risk for the underlying in question, and differently than (57), Equation (61) is not affected by any possible missing forward looking information:

¹²This is not entirely true in reality since all assets may be affected by some risk. Nevertheless, although no products may have an a.s. probability of being totally risk-free, some of them have negligible risk potential so that they can be assumed as risk-free assets.

$$\mathbb{E}^P (m_{t,T} | \mathcal{F}_0) = \mathbb{E}^{\hat{P}} (m_{t,T} | \mathcal{H}_0) = 1 \quad \text{for all } t \in T \tag{62}$$

It follows that the riskless bond has the same value, no matter under which filtration:

$$f_{0,T} (B_{0,T}) = \mathbb{E}^P (B_{0,T} | \mathcal{F}_0) \tag{63}$$

$$= \mathbb{E}^{\hat{P}} (B_{0,T} | \mathcal{H}_0) \tag{64}$$

It follows naturally that no hat is needed on any of the riskless parameters present in the portfolio problems *i.e.*: $r(t)$.

As for the complete case, let's fix s, ω such that the suboptimal objective function is:

$$\begin{aligned} \hat{h}(\xi_{\in \mathcal{H}_s}(s)) &= (\hat{\mu}(s) - r(s)) \xi_{\in \mathcal{H}_s}(s) - \frac{1}{2} \hat{\sigma}^2(s) \xi_{\in \mathcal{H}_s}^2(s) \\ &\quad - \int_{\mathbb{R}} \left[\xi_{\in \mathcal{H}_s}(s) \hat{\theta}(s, z) - \mathbb{E}^{\hat{P}} \left(\ln(1 + \xi_{\in \mathcal{H}_s}(s) \theta(s, z)) | \mathcal{H}_s \right) \right] \nu_{\mathcal{H}_s}(dz) \end{aligned} \tag{65}$$

which is again positive and concave hence solvable:

$$\begin{aligned} 0 = \hat{h}'(\xi_{\in \mathcal{H}_s}(s)) &= \hat{\mu}(s) - r(s) - \hat{\sigma}^2(s) \xi_{\in \mathcal{H}_s}(s) \\ &\quad - \int_{\mathbb{R}} \left[\hat{\theta}(s, z) - \mathbb{E}^{\hat{P}} \left(\frac{\theta(s, z)}{1 + \xi_{\in \mathcal{H}_s}(s) \theta(s, z)} | \mathcal{H}_s \right) \right] \nu_{\mathcal{H}_s}(dz) \end{aligned} \tag{66}$$

It follows that the excess return under the suboptimal filtration set is:

$$\hat{\mu}(s) - r(s) = \hat{\sigma}^2(s) \xi_{\in \mathcal{H}_s}(s) + \int_{\mathbb{R}} \left[\hat{\theta}(s, z) - \mathbb{E}^{\hat{P}} \left(\frac{\theta(s, z)}{1 + \xi_{\in \mathcal{H}_s}(s) \theta(s, z)} | \mathcal{H}_s \right) \right] \nu_{\mathcal{H}_s}(dz) \tag{67}$$

Remark 4.1. Equation (67) is valid only if $\xi_{\in \mathcal{H}_s}(s) \theta(s, z) \geq -1$ and if the a.s. uniform integrability with respect to $\nu_{\mathcal{H}_s}(dz)$ of:

$$\mathbb{E}^{\hat{P}} \left(\frac{\xi_{\in \mathcal{H}_s}(s) \theta^2(s, z)}{1 + \theta(s, z)} | \mathcal{H}_s \right) \tag{68}$$

applies for almost all s, ω .

Once more, for each $s \in [0, T]$, Equation (68) is a consequence of:

$$\hat{\theta}(s, z) = \mathbb{E}^{\hat{P}} (\theta(s, z) | \mathcal{H}_s) \tag{69}$$

In case of no jumps $\theta(s, z) = 0$:

$$\xi_{\in \mathcal{H}_s}(s) = \frac{\hat{\mu}(s) - r(s)}{\hat{\sigma}^2(s)} \tag{70}$$

where the above equation is valid only if $\hat{\sigma}(s) \neq 0$ a.s. for a.a. (s, ω) , $s \in [0, T]$ and $\omega \in \Omega$.

Once more, $\xi_{\in \mathcal{H}_s}(s)$ is the optimal portfolio process $\xi_{\in \mathcal{H}_s}^*(s)$ only if is in the set of admissible portfolios given the information set:

$$\xi \in \mathcal{A}_{\mathcal{H}_s} \rightarrow \xi_{\in \mathcal{H}_s}(s) = \xi_{\in \mathcal{H}_s}^*(s) \quad \text{for all } s \in [0, T] \tag{71}$$

It follows that the optimal restricted portfolio value is:

$$\Lambda_{\mathcal{H}_t}^{\text{Log}} = \mathbb{E}^{\hat{P}} \left[\ln X^{\xi_{e^{\mathcal{H}_t}}^*}(T) \right] = \ln x_t + \mathbb{E}^{\hat{P}} \left[\int_0^T \left(r(s) + \frac{(\hat{\mu}(s) - r(s))^2}{2\hat{\sigma}^2(s)} \right) ds \right] \quad (72)$$

With no loss of generality the initial capital of the investor is set to $X(0) = 1$ s.t.:

$$\Lambda_{\mathcal{H}_t}^{\text{Log}} = \mathbb{E}^{\hat{P}} \left[\ln X^{\xi_{e^{\mathcal{H}_t}}^*}(T) \right] = \mathbb{E}^{\hat{P}} \left[\int_0^T \left(r(s) + \frac{(\hat{\mu}(s) - r(s))^2}{2\hat{\sigma}^2(s)} \right) ds \right] \quad (73)$$

Now, combining (51) and (73) it follows that the use of a suboptimal information set leads to:

$$\Lambda_{\mathcal{F}_t}^{\text{Log}} - \Lambda_{\mathcal{H}_t}^{\text{Log}} = \frac{1}{2} \int_0^T E^P \left[\frac{(\mu(s) - r(s))^2}{\sigma^2(s)} \right] ds - \frac{1}{2} \int_0^T E^{\hat{P}} \left[\frac{(\hat{\mu}(s) - r(s))^2}{\hat{\sigma}^2(s)} \right] ds \quad (74)$$

$$= \Delta_t^{\text{Log}} \quad (75)$$

where Δ_t^{Log} represents the time t information premium.

Following the neoclassical literature, the optimal (or theoretical) case assumes the investor to be a natural and fully rational optimizer capable to obtain the highest possible reward given the market scenario. On the contrary, the more realistic real-world case assumes and implies some deficiencies in optimization. It follows naturally for the above obtained information premium:

$$\Delta_t > 0 \quad (76)$$

At the same time, the amount of premium cannot be determined *ex ante* and is highly dependent to the market scenario (*i.e.*: level of volatility in the market).

Empirically, to have at each point in time a *full* information set as required by the neoclassical theory is usually a mere illusion. Therefore, the goal of a good financial modeller should be, for each $t \in T$, to minimize Δ_t as much as possible

$$\min \Delta_t \Rightarrow \max \mathcal{I}_t \quad (77)$$

It follows immediately that to minimize Δ_t means to collect and model in the best way possible all the information that is relevant for pricing the assets. A proper estimation of the real-world probabilities is then of key importance for many day-by-day operations (*i.e.*: trading, risk management, asset management).

5. The Power Utility Case

A common drawback that pertains to the literature of the stochastic optimization problems (and not only) is its high reliance on the standardization of the parameters that govern most of models. In this paper the proposed model is highly dependent on the choice of the utility function used to describe the investor decisions. Given their widespread use in literature and as an alternative to broaden the analysis, the power utility function is proposed as a possible extension of the logarithmic utility one. Being aware that no investor can be properly and fully described by a parametric utility function, the proposed alternative is not a solution of the utility-problem but just a

possible alternative. It follows that all results must then be considered as approximations.

The problem remains the same:

Proposition 3. *We like to solve a finite horizon $T < \infty$ stochastic control problem of a rational and small investor endowed with a positive initial capital $X(0) = x > 0$ and described by a generic utility function $U(\cdot)$, here a power utility function, whose goal is to maximize her expected subjective utility from terminal wealth, i.e.:*

$\mathbb{E}^P [U(\cdot)(T)]$ *by investing continuously in a risky, P , and in a risk-free asset, P_0 .*

The optimal value of the problem, denoted by, $\Lambda_{\mathcal{I}_t}^{\text{Pow}}$, is valid only if, for each $t \in [0, T]$ and $\omega \in \Omega$, there exists an optimal portfolio process, $\xi(t, \omega) = \xi_{\mathcal{I}_t}^*(t, \omega)$, which belongs to the set of admissible portfolios, $\mathcal{A}_{\mathcal{I}_t}$, s.t.:

$$\Lambda_{\mathcal{I}_t}^{\text{Pow}}(x) = \sup_{\xi \in \mathcal{A}_{\mathcal{I}_t}} \mathbb{E}^P \left[\frac{1}{1-\lambda} X^{1-\lambda, \xi_{\mathcal{I}_t}}(T) \right] = \mathbb{E}^P \left[\frac{1}{1-\lambda} X^{1-\lambda, \xi_{\mathcal{I}_t}^*}(T) \right] \tag{78}$$

where the the value function of the problem is assumed to be:

$$\Lambda_{\mathcal{I}_t}^{\text{Pow}}(x) < \infty \quad \forall x_t \in [0, \infty) \tag{79}$$

As stated in proposition (3) and differently than (2) the investor is modelled through a Constant Relative Risk Aversion (CRRA) utility function:

$$U(x_t) = \frac{1}{1-\lambda} x_t^{1-\lambda} \quad x > 0 \quad \lambda \neq 1, \lambda > 0 \tag{80}$$

where $\lambda \in \mathbb{R}^+$ is a constant and measures the degree of relative risk aversion implicit in the utility function. Going into the limit (see **Appendix** (9)):

$$\lim_{\lambda \rightarrow 1} \frac{x_t^{1-\lambda} - 1}{1-\lambda} x_t^{1-\lambda} = \ln x_t \quad x_t > 0 \tag{81}$$

Given the new setup we follow the same procedure as before to solve the stochastic optimization problem under the two scenarios.

For a full information set: $\mathcal{I}_t = \mathcal{F}_t$ and with $\theta(s, z) = 0$:

$$\xi_{\mathcal{F}_s}^*(s) = \frac{1}{\lambda} \frac{\mu(s) - r(s)}{\sigma^2(s)} \tag{82}$$

where $\xi_{\mathcal{F}_s}^*(s)$ is the optimal portfolio process $\xi_{\mathcal{F}_s}^*(s)$ only if is in the set of admissible portfolios given the information set:

$$\xi \in \mathcal{A}_{\mathcal{F}_s} \rightarrow \xi_{\mathcal{F}_s}(s) = \xi_{\mathcal{F}_s}^*(s) \quad \text{for all } s \in [0, T] \tag{83}$$

For (82), to be valid, we need $\lambda \neq 0$ and $\sigma(s) \neq 0$ a.s. for a.a. (s, ω) , $s \in [0, T]$ and $\omega \in \Omega$ to hold.

Setting the investor initial capital to $X(0) = 1$:

$$\Lambda_{\mathcal{F}_T}^{\text{Pow}} = \mathbb{E}^P \left[\frac{1}{1-\lambda} X^{1-\lambda, \xi_{\mathcal{F}_T}^*}(T) \right] = \mathbb{E}^P \left[\int_0^T \left(r(s) + \frac{1}{\lambda} \frac{(\mu(s) - r(s))^2}{2\sigma^2(s)} \right) ds \right] \tag{84}$$

In conclusion, for a finite-time complete market with one riskless and and one risky asset, the maximal expected power utility of the terminal wealth for a small and rational

investor with a complete set of information, is the integrated sum of the risk-free rate and a fraction of the Sharpe ratio squared normalized by λ .

The same logic applies for the partial information set $\mathcal{I}_t = \mathcal{H}_t \subset \mathcal{F}_t$. In case of no jumps $\theta(s, z) = 0$:

$$\xi_{\in \mathcal{H}_s}(s) = \frac{1}{\hat{\lambda}} \frac{\hat{\mu}(s) - r(s)}{\hat{\sigma}^2(s)} \quad (85)$$

where λ , the indicator of the subjective degree of risk aversion of an investor, is fully impacted by the coarser filtration set so that:

$$\hat{\lambda} = \mathbb{E}^{\hat{P}}[\lambda | \mathcal{H}_s] \quad (86)$$

is a valid equivalence.

$\tilde{\xi}_{\in \mathcal{H}_s}(s)$ is the optimal portfolio process $\xi_{\in \mathcal{H}_s}^*(s)$ only if it is in the set of admissible portfolios given the information set:

$$\xi \in \mathcal{A}_{\mathcal{H}_s} \rightarrow \xi_{\in \mathcal{H}_s}(s) = \xi_{\in \mathcal{H}_s}^*(s) \quad \text{for all } s \in [0, T] \quad (87)$$

Also for the restricted Sharpe ratio, to be valid, we need $\hat{\lambda} \neq 0$ and $\hat{\sigma}(s) \neq 0$ a.s. for a.a. (s, ω) , $s \in [0, T]$ and $\omega \in \Omega$ to hold.

Setting the initial capital of the investor to $X(0) = 1$ the optimal restricted portfolio value is

$$\Lambda_{\mathcal{H}_t}^{\text{Pow}} = \mathbb{E}^{\hat{P}} \left[\frac{1}{1 - \lambda} X^{1 - \lambda, \xi_{\in \mathcal{H}_t}^*}(T) \right] = \mathbb{E}^{\hat{P}} \left[\int_0^T \left(r(s) + \frac{1}{\hat{\lambda}} \frac{(\hat{\mu}(s) - r(s))^2}{2\hat{\sigma}^2(s)} \right) ds \right] \quad (88)$$

Now, combining (84) and (88) it follows that the difference in filtration is equal to:

$$\Lambda_{\mathcal{F}_t}^{\text{Pow}} - \Lambda_{\mathcal{H}_t}^{\text{Pow}} = \frac{1}{2} \int_0^T \mathbb{E}^P \left[\frac{1}{\lambda} \frac{(\mu(s) - r(s))^2}{\sigma^2(s)} \right] ds - \frac{1}{2} \int_0^T \mathbb{E}^{\hat{P}} \left[\frac{1}{\hat{\lambda}} \frac{(\hat{\mu}(s) - r(s))^2}{\hat{\sigma}^2(s)} \right] ds \quad (89)$$

$$= \Delta_t^{\text{Pow}} \quad (90)$$

which, as $\lambda \rightarrow 1$ ¹³:

$$\Lambda_{\mathcal{F}_t}^{\text{Log}} - \Lambda_{\mathcal{H}_t}^{\text{Log}} = \frac{1}{2} \int_0^T \mathbb{E}^P \left[\frac{(\mu(s) - r(s))^2}{\sigma^2(s)} \right] ds - \frac{1}{2} \int_0^T \mathbb{E}^{\hat{P}} \left[\frac{(\hat{\mu}(s) - r(s))^2}{\hat{\sigma}^2(s)} \right] ds \quad (91)$$

$$= \Delta_t^{\text{Log}} \quad (92)$$

6. Connection with the PK

There is a well known and strong interconnection between optimal portfolios and the PK. The latter is in fact bounded by the highest feasible Sharpe ratio and vice-versa. The simplicity and generality of the rules that govern these bounds justify their widespread use in many aspects of financial economics (see Bekaert and Liu (2001) [21] for a review.). These bounds, also known as the Hansen-Jagannathan bounds (1991) [22], pose upper and lower limits on both random variables. Henrotte (2002) [23] inves-

¹³Which implies that also $\hat{\lambda} \rightarrow 1$.

tigates the tight but non-trivial relationship between the PK, its variance and the optimal portfolio choice. He defines that:

Definition 6.1. *The square of the Sharpe ratio of every portfolio is smaller than the variance of every normalized PK.*

From the definition, two are the main consequences in asset pricing:

- The square of the Sharpe ratio of any portfolio bounds the variance of any normalized PK:

$$SR^2 = PK \text{ lower bound}$$

- No Sharpe ratios squared can be greater than the variance of the normalized PK:

$$PK = SR^2 \text{ upper bound}$$

where the variance is normalized through the mean of the PK itself and SR^2 states for Sharpe Ratio squared.

In this section, I extend and generalize the above definition for the case of a suboptimal PK. Due to qualitative and quantitative importance of the missing information from the suboptimal filtration set it follows from Equation (74) that, for each $t \in T$, if \mathcal{F}_t and \mathcal{H}_t differs significantly, then:

$$\Delta_t^{Log} > 0 \tag{93}$$

which implies an a.s. inequality for the Sharpe ratios squared of the two portfolios:

$$\frac{(\mu(t) - r(t))^2}{\sigma^2(t)} > \frac{(\hat{\mu}(t) - r(t))^2}{\hat{\sigma}^2(t)} \tag{94}$$

All else equal, a better informed investor can aim to higher returns and lower volatility, which justify also conceptually the above inequality. These results show clearly that a suboptimal information set impacts directly on the total profitability of the investor’s portfolio and also indirectly on the quality of their bounds.

Different papers refine and extend the Hansen-Jagannathan bounds in several directions¹⁴. One of the main difficulties, common to many papers, is the time-varying estimation of the elements that compose the PK. Some authors propose unconditional PKs thus lowering the effectiveness of the findings. A time independent PK would be in fact of little usefulness for many day-by-day operations (*i.e.*: asset and risk management). Working on the insights of Gallant, Hansen and Tauchen (1990) [25] a more recent paper of Bekaert and Liu (2004) [21] extends the theory on the optimal bounds putting emphasis on the optimal use of the conditioning of the information. As a main result they show how, given some technical conditions¹⁵ the best bounds are the ones that maximizes the squared Sharpe Ratio:

$$S.R.(t)_{Max}^2 := \text{Best Bounds}(t) \leq PK \text{ lower bound}(t) \tag{95}$$

Applying the theorem to our findings emerges how a more informed investors can not only benefit from superior returns, but also from sharper, thus better, PK bounds.

¹⁴See Ferson and Siegel (2002) [24] and references therein.

¹⁵See [21], Section 1.4 pag 345 and 346 for the main theorem and its proof.

Given this result and from definition (6.1) it follows that:

$$\frac{\text{Var PK}_t(\mathcal{F})}{\text{Mean PK}_t(\mathcal{F})} > \frac{\text{Var PK}_t(\mathcal{H})}{\text{Mean PK}_t(\mathcal{H})} \quad (96)$$

7. Expressing the Information Premium as the Kullback-Leibler Divergence

This section shows how, through the PK, a suboptimal filtration set may propagate onto the risk-neutral pricing. Results can be so summarized:

- The information premium is nothing but the difference between the optimal and the suboptimal Kullback-Leibler divergences
- A suboptimal information set may affect the risk-neutral pricing by means of the restricted market price of risk that enters into the Girsanov theorem

Let us start defining the time-dependent theoretical and suboptimal market prices of risk¹⁶ as:

$$\text{Optimal} = \theta_t^{\mathcal{F}} = \theta_t = \frac{\mu(t) - r(t)}{\sigma(t)} \quad (97)$$

$$\text{Suboptimal} = \theta_t^{\mathcal{H}} = \hat{\theta}_t = \frac{\hat{\mu}(t) - r(t)}{\hat{\sigma}(t)} \quad (98)$$

Then, if θ_t and $\hat{\theta}_t$ are locally square integrable and if:

$$\mathbb{E}^P \left[\exp \left(\frac{1}{2} \int_0^t \theta_s^2 ds \right) \middle| \mathcal{F}_t \right] < \infty \quad (99)$$

$$\mathbb{E}^P \left[\exp \left(\frac{1}{2} \int_0^t \hat{\theta}_s^2 ds \right) \middle| \mathcal{H}_t \right] < \infty \quad (100)$$

then, for all $t \in [0, T]$ and $x_t \in \mathcal{F}_t$, $x_t \in \mathcal{H}_t$:

$$Q_t(X) = \mathbb{E}^P \left[M(t) 1_{x_t} \middle| \mathcal{F}_t \right] \quad (101)$$

$$\hat{Q}_t(X) = \mathbb{E}^{\hat{P}} \left[\hat{M}(t) 1_{x_t} \middle| \mathcal{H}_t \right] \quad (102)$$

are the optimal and suboptimal probability measures defined respectively on \mathcal{F}_t and \mathcal{H}_t where:

$$M(t) = \exp \left(\int_0^t \theta_s dW_s + \frac{1}{2} \int_0^t \theta_s^2 ds \right) \quad (103)$$

$$\hat{M}(t) = \exp \left(\int_0^t \hat{\theta}_s d\hat{W}_s + \frac{1}{2} \int_0^t \hat{\theta}_s^2 ds \right) \quad (104)$$

are the respective optimal and suboptimal pricing kernels. For both pricing kernels, W_t and \hat{W}_t are the respective optimal and suboptimal Brownian Motion defined on (Ω, \mathcal{F}, P) and (Ω, \mathcal{H}, P) so that θ_t and $\hat{\theta}_t$ are \mathcal{F}_t - and \mathcal{H}_t -adapted.

It follows that:

¹⁶With no loss of generality, the market price of risk of the previous sections, being scaled by its variance, is a proportional version of the theoretical one.

$$\mathbb{E}^P (M_t | \mathcal{F}_t) = 1 \tag{105}$$

such that M_t is a positive true martingale¹⁷. Given that some families of semi-martingales—namely: local and strict local martingales—may change the nature of their process (from absolutely continuous to mixed) once projected onto smaller filtration sets (Protter (2015) [26], Sala and Barone Adesi (2015) [27]), the same may not be true under the suboptimal scenario. From the previous sections and justified by the rational behaviours of the investors under the neoclassical theory, it follows that the optimal quantities are always larger with respect to the suboptimal ones.

Now, let's recall the information premium:

$$\Delta_t^{\text{Log}} = \frac{1}{2} \int_0^T \mathbb{E}^P \left[\frac{(\mu(s) - r(s))^2}{\sigma^2(s)} - \frac{(\hat{\mu}(s) - r(s))^2}{\hat{\sigma}^2(s)} \right] ds \tag{107}$$

Since the theory in object applies independently of the utility function used, we omit to specify it.

Assuming that the usual technical assumption holds (square local integrability and Novikov condition) the Kullback-Leibler divergence among the optimal densities is:

$$D_t^{KL} (P(t) | Q(t)) = \mathbb{E}^P \left(\log \frac{dP(t)}{dQ(t)} \right) \tag{108}$$

$$= \mathbb{E}^P \left(\log \frac{1}{M(t)} \right) \tag{109}$$

$$= \mathbb{E}^P \left(\int_0^t \theta ds + \frac{1}{2} \int_0^t \theta_s^2 ds \right) \tag{110}$$

$$= \frac{1}{2} \int_0^t \theta_s^2 \tag{111}$$

$$= \theta_t \tag{112}$$

The same hold for the suboptimal case:

$$\hat{D}_t^{KL} (\hat{P}(t) | Q(t)) = \mathbb{E}^{\hat{P}} \left(\frac{d\hat{Q}(t)}{d\hat{P}(t)} \right) \tag{113}$$

$$= \mathbb{E}^{\hat{P}} \left(\log \frac{1}{\hat{M}(t)} \right) \tag{114}$$

$$= \mathbb{E}^{\hat{P}} \left(\int_0^t \hat{\theta} d\hat{W}_s + \frac{1}{2} \int_0^t \hat{\theta}_s^2 ds \right) \tag{115}$$

$$= \frac{1}{2} \int_0^t \hat{\theta}_s^2 \tag{116}$$

$$= \hat{\theta}_t \tag{117}$$

¹⁷The Brownian motion under the new measure is:

$$W_t^Q = W_t - \int_0^t \theta_s ds \tag{106}$$

where:

$$D_t^{KL}(P(t)|Q(t)) = \begin{cases} < \infty, & \text{If } Q_t \ll P_t, \\ \infty, & \text{otherwise} \end{cases} \quad (118)$$

Condition (118) determines the highly remote, but theoretically possible, extreme case of an unbounded value due to the lack of absolute continuity of the measures¹⁸.

Now, taking the difference among the two distances:

$$D_t^{KL}(P(t)|Q(t)) - \hat{D}_t^{KL}(\hat{P}(t)|Q(t)) = \theta_t - \hat{\theta}_t \quad (119)$$

$$= \Delta_t \quad (120)$$

$$= \mathcal{F}_t - \mathcal{H}_t \quad (121)$$

In finance, the EMM has the advantage of being fully neutral and unaffected by subjective beliefs. Among the others, one of the main feature is that it prevents the problem of picking a parametric utility function to describe the fully non-parametric investors behavior. This paper shows how, in presence of information premiums, both the risk-neutral and the real-world measure and the relative pricing kernel can be strongly affected. Overall, the asset pricing bias can be so summarized:

Suboptimal information: Δ_t

↓

Suboptimal filtration set: $\mathcal{H}_t \subset \mathcal{F}_t$

↓

Impact on the risk physical measure: $\hat{P}_t \neq P_t$

↓

Impact on the market price of risk: $\hat{\theta}_t \neq \theta_t$

↓

Impact on the pricing kernel: $\hat{M}_t \neq M_t$

↓

Impact on the risk-neutral measure: $\hat{Q}_t \neq Q_t$

↓

Mispricing: $X_t \neq \hat{X}_t$

8. Conclusions

This paper investigates how possible misestimations of the investor's future beliefs may lead to mispricing.

Econometrically, it is conventional wisdom for most of the (in not the entire) literature to use backward-looking data to estimate forward-looking beliefs. This approach leads almost surely to biased estimations due to the use of a coarser filtration set. Starting from this well-known bias, this paper documents how the use of a suboptimal information set may affect the asset pricing in different ways. Due to the strong

¹⁸Under some conditions, *i.e.*: the existence of a measure under which both P_t and Q_t are absolutely continuous, it is possible to have a finite value even if Q_t is not absolutely continuous with respect to P_t . As well as we can have non finite values if the two measures are mutually absolutely continuous.

interconnection between the measures and the pricing kernel, possible misestimations of the physical measure load naturally onto the other two quantities.

Starting from a portfolio optimization problem for a small and rational investor, it shows how a partially-informed trader that wants to maximize her final wealth through dynamic trading may end up with a smaller profit with respect to a fully informed one. Performing the same optimization technique for a theoretical (full information set) and a real-world (partial information set) case, emerges the existence of an information premium. To minimize this premium, hence the bias, means to maximize the information estimation.

The existence of an information premium, impacting on the optimality of the pricing kernel bounds, has a direct effect on its usability. As a consequence, a less informed investor, having access to a smaller range of supports of the pricing kernel, has a reduced spectrum of possible scenario analysis. This has a direct consequence on the quality and the effectiveness of many day-by-day operations *i.e.*: hedging or trading strategies.

Focusing on the premium, it shows that the information premium arising from two investors is nothing but the Kullback-Leibler distance among the two set of densities.

To conclude, through the Girsanov's theorem, it shows how starting from a suboptimal information set, the bias propagates naturally from the physical to the risk-neutral pricing.

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Appendix

A1. Properties of Lévy Processes

Given a filtered probability space which satisfies the usual hypothesis $(\Omega, \mathcal{F}_t, \mathbb{F}, P)$ and a fixed time period $0 \leq t \leq T < \infty$, a generic stochastic process $(X_t)_{t \geq 0}$ whose values are in \mathbb{R}^d and $X_0 = 0$ is a Lévy process:

$$X_t = X(t, \omega): [0, \infty) \times \Omega \rightarrow \mathbb{R} \quad (122)$$

if satisfies the following properties:

- Independent increments: for an increasing sequence of times $t_0, t_1, t_2, \dots, t_n$ with $t_0 < t_1 < t_2 < \dots < t_n$, the random variables are time independent: $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$.
- Stationary increments: the law of $(X_{t+h} - X_t)$ is independent to t .
- $(X_t)_{t \geq 0}$ is a right continuous with left limits-cadlag-process s.t. the paths $t \rightarrow X_t$ are non-anticipating.
- Stochastic continuity: $\forall \delta > 0, \lim_{h \rightarrow 0} \mathbb{P}(|X_{t+h} - X_t| \geq \delta) = 0$.

where the last property assures that jumps occur at random times. If the probability of having a jump is not a.s. equal to 0, we would have the so called “calendar effect”.

Given a Lévy process and a measurable subset A , its measure ν on \mathbb{R}^d is called the Lévy measure $X : \nu(A)$ and is defined as:

$$\nu(A) = \mathbb{E} \left[\# \{t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in A\} \right] \quad A \in \mathcal{B}(\mathbb{R}) \quad (123)$$

The Lévy measure represents the expected number, per unit of time, of jumps whose size belongs to the measurable set $A \subset \mathbb{R}^d$.

Now, given a Lévy process $(X_t)_{t \geq 0}$ on \mathbb{R}^d and a Lévy measure ν , by the Lévy-Ito decomposition theorem any such a process can be decomposed as:

$$X_t = \alpha t + \beta B_t + X_t^I + \lim_{\epsilon \downarrow 0} \hat{X}_t^{\{\epsilon\}} \quad (124)$$

where $\alpha, \beta \in \mathbb{R}$ are constants, B_t is an \mathcal{F}_t -adapted Brownian Motion. The first two terms on the right hand side compose the continuous Gaussian Lévy process.

The discontinuous part is composed by:

$$X_t^I = \int_{|x| \geq 1, s \in [0, t]} x J_X(ds \times dx) \quad (125)$$

and (126)

$$\hat{X}_t^\epsilon = \int_{\epsilon \leq |x| < 1, s \in [0, t]} x \{J_X(ds \times dx) - \nu(ds) dx\} \quad (127)$$

$$\equiv \int_{\epsilon \leq |x| < 1, s \in [0, t]} x \hat{J}_X(ds \times dx) \quad (128)$$

where the last term converges a.s. and uniformly in $t \in T$. The former process is a compound Poisson process while the latter indicates the compensated version of the process¹⁹. All terms in (124) are independent.

The process X_t is identified by means of the characteristic triplet (A, ν, γ) where

¹⁹The jump integral is X_t^I replaced by its compensated version to avoid singularities and assure convergence.

A is the covariance matrix of the Brownian motion, ν is the Lévy measure and γ is the drift of the continuous Gaussian Lévy process.

A2. From Power to Logarithmic Utility Function

In this appendix, I prove Equation (80):

$$\lim_{\lambda \rightarrow 1} \frac{x_t^{1-\lambda} - 1}{1-\lambda} x_t^{1-\lambda} = \ln x_t \quad x_t > 0 \tag{129}$$

where, with no loss of generality, the utility function $x_t^{1-\lambda} / (1-\lambda)$ can be rewritten as:

$$U(x_t) = \frac{x_t^{1-\lambda} - 1}{1-\lambda} \tag{130}$$

which, given the iso-elastic properties of the utility functions, is a valid statement since the investors' optimal decisions are not affected by additive constant terms in the objective function so that:

$$\lim_{\lambda \rightarrow 1} \frac{x_t^{1-\lambda} - 1}{1-\lambda} = \ln x_t \quad x_t > 0 \tag{131}$$

With no loss of generality and to improve the readability of the proof, I omit visually the time dependence.

To prove the convergence in (131), I make use of the l'Hôpital's rule which I report here:

Theorem 4 (l'Hôpital's rule). *Let f and g be differentiable functions with $g'(x) \neq 0$ on an open interval O around a , except possibly at a .*

If:

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0 \text{ or } = \pm\infty \tag{132}$$

and if:

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} < \infty \text{ for } g'(x) \neq 0 \forall x \in O, \text{ with } x \neq a \tag{134}$$

then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \tag{136}$$

Which, applied to (130):

$$\lim_{\lambda \rightarrow 1} U(x) = \lim_{\lambda \rightarrow 1} \frac{1}{1-\lambda} (x^{1-\lambda} - 1) \tag{137}$$

$$= \lim_{\lambda \rightarrow 1} \frac{-\exp^{(1-\lambda)\ln x} \ln x}{-1} \tag{138}$$

$$= \lim_{\lambda \rightarrow 1} x^{1-\lambda} \ln x \tag{139}$$

$$= \ln x \lim_{\lambda \rightarrow 1} x^{1-\lambda} \tag{140}$$

$$= \ln x \tag{141}$$

By the same token:

$$\lim_{\lambda \rightarrow 0} U(x) = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} x^\lambda \quad (142)$$

$$= \lim_{\lambda \rightarrow 0} \frac{x^\lambda \ln x}{1} \quad (143)$$

$$= \ln x \quad x > 0 \quad (144)$$



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