Journal of Mathematical Finance, 2016, 6, 293-302
Published Online May 2016 in SciRes. http://www.scirp.org/journal/jmf
http://dx.doi.org/10.4236/jmf.2016.62025

Scientific Research Publishing

Production in General Equilibrium with Incomplete Financial Markets

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Received 17 February 2016; accepted 10 May 2016; published 13 May 2016

Abstract

This paper considers a general equilibrium model with incomplete financial markets where production sets depend on the financial decisions of the firms. In the short run, firms make financial choices in order to build up production capacity. Given production capacity firms make profit maximizing production decisions in period two. We provide the conditions of existence of equilibria.

Keywords

General Equilibrium, Incomplete Financial Markets, Production, Existence of Equilibria, Transversality

1. Introduction

Classical general equilibrium literature on production with incomplete markets has focused on variations of the Arrow's seminal two-period model with exogenous financial assets [1] [2]. In this framework, the firm's real sequential optimization structure is independent of its financial activities. Firms choose quantities of inputs of production in period one such that associated output choices in period two are optimal. This concept of the firm corresponds to the private ownership model of the firm introduced by Debreu [3], where the single argument of the firm's two period sequential optimization function is the real activity vector. In these recent models, influenced by Drèze [4] and Grossmann & Hart [5], optimality of the choice of a net real activity vector over two periods refers to the average utility of the group of owners of the firm, the stock holders. It is in that sense that the literature has assigned utilities to firms and that the firms' objective is to maximize some average utility of the share holders. The two concepts applied in most models, slightly differ in the choice of average utility utilized (average utility of initial/final share holders). For a sample of the huge literature applying these concepts see [4]-[8].

How to cite this paper: Stiefenhofer, P. (2016) Production in General Equilibrium with Incomplete Financial Markets. *Journal of Mathematical Finance*, **6**, 293-302. http://dx.doi.org/10.4236/jmf.2016.62025

This paper introduces a model of the firm, where its financial and real activities are independent of any average utility of the stock holders. It postulates that firms maximize long run profits and make financial and real decisions sequentially over two periods. The assumption of long run profit maximization is justified by the sequential optimization structure of the firm. Firms issue stocks in period one in order to acquire the cash needed to install production capacity. The optimal quantity of stocks issued by each firm is endogenously determined by the model. Once capacity is installed, after uncertain state of nature has occurred at the beginning of period two, firms produce real goods subject to capacity and technological constraints. The ownership structure introduced in this model eliminates the strategic choice problem of the firm present in the literature. Here, stock holders do not decide about the optimal input vector of the firm in period one. They invest in firms by purchasing stocks in order to transfer wealth across time and between uncertain states of nature. The total quantity of stocks demanded is equal to total quantity of stocks supplied by firms in the same period. The value of total stocks issued by a firm bounds the value of inputs a firm can purchase in period two. Real activities of the firm take place after uncertainty in period two has resolved. These production activities correspond to finding the optimal net activity vector at given prices and revealed state of the world such that profits are maximized at given production capacity.

The sine qua non of the model is then to show that equilibrium exists. It is shown that, for an endogenized price and technology dependent real asset structure, which is transverse to the reduced rank manifolds, equilibrium exists generically in the endowments by the application of Thom's parametric transversality theorem. Finally, the non-smooth convex production set case is considered, where the piecewise linear production manifolds are regularized by convolution. Existence then follows from the smooth case. Bottazzi [9] demonstrated generic existence of equilibrium for an exchange economy for price dependent smooth assets. Equilibria exist for more general asset structures.

The model is introduced in Section 2. Section 3 shows generic existence for convex smooth production manifolds.

2. The Model

We consider a two period $t \in \{0,1\}$ model with uncertainty in period 1 represented as states of nature. An element in the set of mutually exclusive and exhaustive uncertain events is denoted $s \in S = \{1,\cdots,S\}$, where by convention $s = 0$ represents the certain event in period 0, and S denotes the set of all mutually exclusive uncertain events. This set denotes the overall description of uncertainty in the model, which is characterized by idiosyncratic and aggregate risk. The general uncertainty space is described by the Cartesian product $S = \mathcal{S} \times \tilde{\mathcal{S}}$. For every production set Y_j, there exists a set of states of nature $S_j = \{1,\cdots,\mathbb{S}\}$, where $\mathbb{S} \geq 2$, for all S_j. Denote $\mathcal{S} = \{S_1,\cdots,S_j,\cdots,S_n\}$, where $S \subseteq \mathcal{S}$, the set of technological uncertain events. At aggregate level there are $\tilde{\mathcal{S}} = \{1,\cdots,\tilde{\mathcal{S}}\}$ states of nature. We count in total $(S+1)$ states of nature.

The economic agents are the $j \in \{1,\cdots,n\}$ producers and $i \in \{1,\cdots,m\}$ consumers which are characterized by sets of assumptions F and C bellow. There are $k \in \{1,\cdots,l\}$ physical commodities and $j \in \{1,\cdots,n\}$ financial assets, referred to as stocks. Physical goods are traded on each of the $(S+1)$ spot markets. Firms issue stocks which are traded at $s = 0$, yielding a payoff in the next period at uncertain state $s \in \{1,\cdots,S\}$. The quantity vector of stocks issued by firm j is denoted $z_j \in \mathbb{R}_-$. Other assets such as bonds or options can be introduced without any further difficulties. There are total $l(S+1)$ goods. The consumption bundle of agent i is denoted $x_i = (x_i(0), x_i(s),\cdots,x_i(S)) \in \mathbb{R}_{++}^{l(S+1)}$, with $x_i(s) = (x_i^1(s),\cdots,x_i^l(s)) \in \mathbb{R}_{++}^l$, and $\sum_{i=1}^m x_i = x$. The consumption space for each i is $X_i = \mathbb{R}_{++}^{l(S+1)}$, the strictly positive orthant. The associated price system is a collection of vectors represented by $p = (p(0), p(s),\cdots,p(S)) \in \mathbb{R}_{++}^{l(S+1)}$, with $p(s) = (p^1(s),\cdots,p^l(s)) \in \mathbb{R}_{++}^l$. There are n financial assets traded in period 0. Denote the quantity vector of stocks purchased by consumer i, $z_i = (z_i(1),\cdots,z_i(j),\cdots,z_i(n)) \in \mathbb{R}_+^n$, and denote $\sum_{i=1}^m z_i = z$, with associated spot price system

$q=\left(q(1),\cdots,q(j),\cdots,q(n)\right)\in\mathbb{R}^n_{++}$. We assume $l(S+1)$ complete commodity markets and model producers' sequential optimization behavior in an incomplete financial markets environment. Incomplete markets is shown to be a consequence of the technological uncertainty hypothesis. Denote producer *j*'s long run net activity vector $y_j=\left(y^m_j(s)\times y^n_j(s),\cdots,y^m_j(S)\times y^n_j(S)\right)\in\mathbb{R}^{lS}$, where $y^m_j(s)\in\mathbb{R}^m_-$ represents the long run input vector and $y^n_j(s)\in\mathbb{R}^n_+$ the associated feasible output vector. A state *s* net activity of the firm *j* is denoted $y_j(s)=\left(y^1_j(s),\cdots,y^l_j(s)\right)\in\mathbb{R}^l$, where by convention an element $y^k_j<0$ denotes a factor of production and $y^k_j\geq 0$ a good produced. Let $\sum_{j=1}^n y_j=y$ denote the long run net activity vectors.

Sequential behavior of the producers: Consider the sequential structure of the optimization problem of the firm. Firms build up long run production capacity in the first period, for that, they issue stocks. The value of total stocks issued in period one, denoted $qz_j=m_j$, where $m_j\in\mathbb{R}$ is a real number, bounds the quantity of goods a producer *j* can buy in state $s\in S$ at input prices $p_{INP}(s)$ in period two. Once money is received through financial markets, firms install production capacity, and production activities take place subject to constraint long run production sets in the second period. Uncertainty in production is introduced by a random variable $s\in S_j$ for every *j*. We assume that there are less uncertain states of the world *S* than financial assets *n* available for wealth transfer. Hence $n<S$ is out default asumption.

Assumption (T):

For every production set $Y_j(s)$, $s\in S_j\geq 2$.

Assumption (P):

Firms maximize long run profits.

Assumptions (F):

(i) For each *j*, $Y_j\big|_z\subset\mathbb{R}^{lS}$ is closed, convex, and $\left(\omega+\sum_{j=1}^n Y_j\big|_z\right)\cap\mathbb{R}^{lS}_+$ compact $\forall\omega_i\in\mathbb{R}^{lS}_{++}$. $0\in Y_j\big|_z \Leftarrow Y_j\big|_z\supset\mathbb{R}^{lS}_-$. $Y_j\big|_z\cap\mathbb{R}^{lS}_+=\{0\}$. (ii) For each *j*, denote $\partial Y_j\big|_z\subset\mathbb{R}^{nS}$ a C^∞ manifold for transformation maps (1) $\phi_j:\mathbb{R}^m_-\times\mathbb{R}^n_-\to\mathbb{R}^l_+$ non-linear for all $s\in S$ [1].

Production takes place in the second period, once capacity is installed and state $s\in S$ occurred. At $t=0$, firms choose z_j at price *q* such that long run profits are maximized in every state $s\in S$ subject to long run technological feasibility ϕ_j and capacity constraints m_j. Denote the long run production set $Y_j\big|_z$. This set is not independent of the firm's technology nor on its financial activities, denoted *Z*. More formally, the firm's sequential optimization problem is

$$(\bar{z},\bar{y})_j \arg\max\left\{\bar{p}(s)\Box y_j(s):\begin{vmatrix} y_j\in Y_j\big|_z \\ \bar{q}z_j=\left[p(s)\cdot y_j(s)\right]_{INP}\ \forall s\in S\end{vmatrix}\right\}. \tag{1}$$

Denote a long run equilibrium output vector associated with the production set boundary $\bar{y}_j\in\partial Y_{j,eff}\big|_z$. Each firm *j* is characterized by set of assumptions *F* (Debreu [3]). We modify Debreu's assumptions on production sets in order to allow the modeling of endogenous production capacity via financial assets. The $t=1$ maps implied by equation (1), $\pi_j:\mathbb{R}^l_{++}\times\mathbb{R}^l\times\mathbb{R}_+\to\mathbb{R}_+$, for each state $s\in S$ and all producers *j* define the $(S\times n)$ total long run payoff matrix, a collection of *n* vectors denoted

$$\Pi\left(p_1,\phi\big|_Z\right)=\begin{bmatrix} p(s)\cdot y_1(s) & \cdots & p(s)\cdot y_n(s) \\ \vdots & & \vdots \\ p(S)\cdot y_1(S) & \cdots & p(S)\cdot y_n(S)\end{bmatrix}, \tag{2}$$

[1]Here, C^∞ implies differentiability at any order required. The order depends on all transversality arguments employed. *M* denotes the inputs and *n* the output elements of the production set, and l = *m* + *n*.

where $\phi|_Z$ denotes the technology and capacity dependency of the payoff structure. We next introduce the consumer side of the economy.

The consumer: Each consumer $i \in \{1, \cdots, m\}$ is characterized by set of assumptions C of smooth economies (Debreu [10]).

Assumptions (C): a) $u_i : \mathbb{R}_+^{l(S+1)} \to \mathbb{R}$ is continuous on $\mathbb{R}_+^{l(S+1)}$, and C^∞ on $\mathbb{R}_{++}^{l(S+1)}$. $u_i(x_i) = \{x_i' \in \mathbb{R}_+^{l(S+1)} : u_i(x_i') \geq u_i(x_i)\} \subset \mathbb{R}_{++}^{l(S+1)}$, $\forall x_i \in \mathbb{R}_{++}^{l(S+1)}$. For each $x_i \in \mathbb{R}_{++}^{l(S+1)}$, $Du_i(x_i) \in \mathbb{R}_{++}^{l(S+1)}$, $\forall s$. For each $x_i \in \mathbb{R}_{++}^{l(S+1)}$, $h^{\mathrm{T}} D^2 u_i(x_i) h < 0$, for all nonzero hyperplane h such that $(Du_i(x_i))^{\mathrm{T}} h = 0$. b) Each i is endowed with $\omega_i \in \mathbb{R}_{++}^{l(S+1)}$.

Consumers want to transfer wealth between future spot markets. For that, they invest in firms in period $t = 0$, receiving a share of total dividend payoffs which are determined in the next period in return. Denote the sequence of $(S+1)$ budget constraints

$$B_{z_i} = \left\{ x_i \in \mathbb{R}_{++}^{l(S+1)}, z_i \in \mathbb{R}_+^n : \begin{array}{l} p(0) \cdot (x_i(0) - \omega_i(0)) = -q z_i \\ p(s) \square (x_i(s) - \omega_i(s)) = \Pi(p_1, \phi) \theta(z_i) \end{array} \right\}, \tag{3}$$

where[2] ownership structure is a $(n \times 1)$ vector defined by the mappings

$$\theta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+, \forall j, \tag{4}$$

where $z_i(j) \in \mathbb{R}_+$ is a positive real number for every $j = 1, \cdots, n$. $\theta_{ij} = z_i(j) \left[\sum_i z_i(j) \right]^{-1}$ is the proportion of total payoff of financial asset *j* hold by consumer $i \in I$. In compressed notation, we write

$$B_{z_i} = \left\{ x_i \in \mathbb{R}_{++}^{l(S+1)}, z_i \in \mathbb{R}_+^n : p(s) \square (x_i(s) - \omega_i(s)) \in \hat{\Pi}[z_i \mid \theta(z_i)] \right\} \tag{5}$$

where $\hat{\Pi}(p_1, q, y) = \begin{bmatrix} -q_1 & \cdots & -q_n \\ p(1) \cdot y_1(1) & \cdots & p(1) \cdot y_n(1) \\ \vdots & & \vdots \\ p(S) \cdot y_1(S) & \cdots & p_1(S) \cdot y_n(S) \end{bmatrix}$ represents the full payoff matrix of order $((S+1) \times n)$.

We introduce following prize normalization $\mathfrak{S} = \{ p \in \mathbb{R}_{++}^{l(S+1)} : \|p\| = \Delta \}$ such that the Euclidean norm vector of the spot price system $\|p\|$ is a strictly positive real number $\Delta \in \mathbb{R}_{++}$.

Definition 1. *A financial markets equilibrium with production* $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q}) \in \mathbb{R}_{++}^{l(S+1)m} \times \mathbb{R}_+^{l(S+1)n} \times \mathbb{R}^{nm} \times \mathfrak{S} \times \mathbb{R}_{++}^n$ *satisfies:*

a) $(\bar{x}_i; \bar{z}_i) \arg\max \{ u_i(x_i; z_i) : \bar{x}_i \in B_{\bar{z}_i}(\bar{p}, \bar{q}, \bar{y}; \omega_i) \}$ $\forall i$.

b) $(\bar{z}, \bar{y})_j \arg\max \left\{ \bar{p}(s) \square y_j(s) : \begin{array}{|l} y_j \in Y_j|_z \\ \bar{q} z_j = [\bar{p}(s) \cdot y_j(s)]_{INP} \ \forall s \in S \end{array} \right\}$ $\forall j$.

c) $\sum_i^m (\bar{x}_i - \omega_i) = \sum_j^n \bar{y}$.

d) $\sum_i^m \theta(\bar{z}_i)_j = 1$ $\forall j$, and $\sum_{j=1}^n \sum_{i=1}^m (\bar{z}_i)_j = 0$.

a) and b) are the optimization problems of the consumers and producers. c) and d) represent physical goods and financial markets clearance conditions. $\sum_{i=1}^m \theta(\bar{z}_i)_j = 1$ $\forall j$ states that each firm *j* is owned by the consumers. We now show that incomplete markets is a consequence of technological uncertainty and then move

[2] $\square$ denotes the box product. A "s by s" context dependent mathematical operation. For example the s by s inner product.

to the main section of the paper.

Proposition 1 $n < S \Leftarrow Y_j|_{\bar{z}}$ *for all j, and* $S_j \geq 2$.

Proof. Let $S_j = 1$ for every *j*. and $\sum_j S_j = \mathcal{S}$. Then long run profit prospects $\pi(p) > 0$ imply long run capacity adjustment and market entrance until $n = S$. Let $S > 1$ for every *j*, and $\sum_j S_j = \mathcal{S}$. Then, $\pi(p) > 0$ implies market entrance and the issue of new securities such that in the limit as $\pi(p) \to 0$ the number of firms $j \to n < S$ by assumption (*T*). Similar for negative long run profit prospects $\pi(p) < 0$, firms exit the market and $j > n \to n < S$. □

3. Generic Existence for Convex Smooth Production Manifolds

In this section, we show existence of equilibria. The strategy of the proof is to show that a pseudo equilibrium exists and that every pseudo equilibrium is also a financial markets equilibrium with production. It is known that pseudo equilibria exists for exchange economies. See Duffie, Shafer, Geanokopolos, Hirsh, Husseini, and others [9] [11]-[16]. Genakopolos *et al*. [8] showed that pseuedo equilibria exist for an economy with production for the case of exogenous financial markets. At variance with their model, where the firm's problem is to solve a Nash equilibrium, we show that a pseudo equilibrium for a more general price and technology dependent asset structure, permitting the modeling of production and its finance, exists.

Definition 2. *if* $\nexists z \in \mathbb{R}^n_{++}$ *s.t.*

$\hat{\Pi}(p_1, q, \phi)\left[z \mid \sum_{i=1}^m \theta(z_i)_{s=1}^S\right] \geq 0$, then $q \in \mathbb{R}^n_{++}$ is a no-arbitrage asset price relative to p_1.

Lemma 1. $\exists \beta \in \mathbb{R}^S_{++}$ *s.t.* $q = \sum_{s=1}^S \beta \square \Pi(p_1, \phi)$.

Proof. Immediate consequence of the separation theorem for $((S+1) \times n)$ matrices in Gale (1960). It asserts that either $\exists z \in \mathbb{R}^n_{++}$ such that $\hat{\Pi} z \geq 0$, or $\exists \beta \in \mathbb{R}^{S+1}_{++}$ such that $\beta \hat{\Pi} = 0$. □

We can now rescale equilibrium prices without affecting equilibrium allocations, let $P_1 = \beta \square \overline{p}_1$. The next step is to derive a normalized no arbitrage equilibrium definition [17]. Let $\beta \in \mathbb{R}^S_{++}$ be $\left(\frac{\lambda(s)}{\lambda}\right)_{i=1}$ the gradient vector from the optimization problem of agent 1, called the Arrow-Debreu agent. The Walrasian budget set for the Arrow-Debreu agent is a sequence of constraints denoted

$$B_1 = \left\{ x_1 \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} P \cdot (x_i - \tilde{\omega}_i) = 0 \\ P(s) \square (x_i(s) - \omega_i(s)) = \sum_j \theta_{ij} P(s) \square y_j(s) \end{array} \right\}. \tag{6}$$

For all consumers $i \geq 2$, the no arbitrage budget set consisting of a sequence of $(S+1)$ constraints is denoted

$$B_{i\geq 2} = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} P \cdot (x_i - \tilde{\omega}_i) = 0 \\ P(s) \square (x_i(s) - \omega_i(s)) \in \langle \Pi(P_1, \phi) \rangle \end{array} \right\}, \tag{7}$$

where $\langle \Pi(P_1, \phi) \rangle$ is the span of the income transfer space of period one. Replace $\langle \Pi(P_1, \phi) \rangle$ with L in $G^n(\mathbb{R}^S)$, where $G^n(\mathbb{R}^S)$ is the Grassmann manifold[3] with its known smooth $(S-n)n$ dimensional structure, and L an *n*-dimensional affine subspace of $G^n(\mathbb{R}^S)$.

Denote the pseudo opportunity set $B_i(P, L; \omega_i)$, for each *i*,

$$B_i = \left\{ x_i \in \mathbb{R}^{l(S+1)}_{++} : \begin{array}{l} P \cdot (x_i - \tilde{\omega}_i) = 0 \\ P(s) \square (x_i(s) - \omega_i(s)) \subset L \end{array} \right\}. \tag{8}$$

[3]See *i.e.* Dieudonnè [18] for properties of the Grassmann manifold. See Duffie and Shafer for an exposition of the Grassmann manifold in economics [11].

Let $\mathfrak{S}' = \left\{ p \in \mathbb{R}^{l(S+1)}_{++} : p^{0,1} = \Delta \right\}$ be the set of normalized prices, and let $\Delta \in \mathbb{R}_{++}$ be a fixed strictly positive real number. This convenient normalization singles out the first good at the spot $s = 0$ as the numeraire. We introduce following definitions for the long run payoff maps associated with sets $\mathfrak{S}$ and $\mathfrak{S}'$:

Definition 3. *For any* $p_1 \in \mathfrak{S}$, *such that* $\pi : \mathfrak{S} \times \mathbb{R}^l \times \mathbb{R}_+ \to \mathcal{A}$, *let* $\Gamma(P_1, \phi) = \beta \square \left[\left(proj_\Delta \left(\frac{1}{\beta} \right)^T \square P_1 \right) \square y \right]$, *where T denotes the transpose,* $proj_\Delta(z) = \Delta \left(\frac{z}{\|z\|} \right)$, $\frac{1}{\beta} = \left(\frac{1}{\beta(1)}, \cdots, \frac{1}{\beta(S)} \right) \in R^S_{++}$, *and* $\beta = (\beta(1), \cdots, \beta(S)) \in R^S_{++}$. *(ii) For any* $p_1 \in \mathfrak{S}'$, *such that* $\pi : \mathfrak{S}' \times \mathbb{R}^l \times \mathbb{R}_+ \to \mathcal{A}$, *let* $\Gamma(P_1, \phi) = \beta \square \left[\left(\left(\frac{1}{\beta} \right)^T \square P_1 \right) \square y \right]$, *where* $\mathcal{A}$ *is a set of* $(S \times n)$ *matrices A of order* $(S \times n)$.

We can now define the pseudo financial markets equilibrium with production. We then state the relational propositions between a full rank FE with production and a pseudo FE with production.

Definition 4. *A pseudo financial markets equilibrium with production* $(\bar{x}, \bar{y}), (\bar{P}, \bar{L}) \in \mathbb{R}^{l(S+1)m}_{++} \times \mathbb{R}^{l(S+1)n}_{+} \times \mathfrak{S}' \times G^n(\mathbb{R}^S)$ *satisfies*:

a) $(\bar{x}_1) \arg\max \left\{ u_1(x_1) \ s.t.\ x_1 \in B_1(\bar{P}, \omega_1) \right\} \quad i = 1.$

b) $(\bar{x}_i) \arg\max \left\{ u_i(x_i) \ s.t.\ x_i \in B_i(\bar{P}, \bar{L}, \omega_i) \right\} \quad \forall i \geq 2.$

c) $\langle \Gamma(\bar{P}_1, \bar{\phi}) \rangle \subset \bar{L}$, proper if $\langle \Gamma(\bar{P}_1, \bar{\phi}) \rangle = \bar{L}$.

e) $(\bar{y})_j \arg\max \left\{ \bar{p}(s) \square y_j(s) : \begin{vmatrix} y_j \in Y_j \big|_z \\ \bar{m}_j = [\bar{p}(s) \cdot y_j(s)]_{INP} \ \forall s \in S \end{vmatrix} \right\} \quad \forall j.$

e) $\bar{x}_1 + \sum_{i=2}^m \bar{x}_i = \sum_{i=1}^m \omega_i + \sum_{j=1}^n \bar{y}_j$.

Lemma 2. *Under assumptions C, demand mappings* $f_1(P, w_1)$ *and* $f_i(P, L, \omega)$ *for* $i = 2, \cdots, m$, *from argmax* a) *and* b) *are* C^∞. *Under assumptions F, supply mappings* $g_j(P)$ *for* $j = 1, \cdots, n$, *from argmax* d) *are* C^∞.

Proof. The details of this known result are omitted [11]. However, note that smoothness of demand and supply functions follows from the setup of the model for smooth economies. □

Proposition 2. *For every full rank FE with production* $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$, *there exists* $\beta \in \mathbb{R}^S_{++}$ *and a n-dimensional subspace* $L \in G^n(\mathbb{R}^S)$ *such that* $(\bar{x}, \bar{y}), (\bar{P}, \bar{L})$ *is a pseudo FE with production.*

Proof. By lemma 1, there exists $\beta \in \mathbb{R}^S_{++}$ such that (FE) spot prices at $\bar{p}$ can be rescaled such that $P = \beta \square \bar{p}$, then $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$ is a $(\bar{x}, \bar{y}), (\bar{P}, L)$ equilibrium. Since by definition $\beta \in \mathbb{R}^S_{++}$ is $\left(\frac{\lambda^s}{\lambda^0} \right)_{i=1}$ of agent 1 at $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$, agent 1's consumption bundle is $\bar{x}_1$, since $\nabla u_1(x_1) = P$, and $P \square (x - \omega) = 0$.

On the contrary, if have a $(\bar{x}, \bar{y}), (\bar{P}, L)$ equilibrium, and $\bar{z}_2, \cdots, \bar{z}_m$ such that a) $\sum_{i=1}^m (\bar{x}_i - \omega_i) = \sum_{j=1}^n \bar{y}$, b) $\left\{ \bar{x}_1 \in \mathbb{R}^{l(S+1)}_{++} : P(x - \omega) = 0 \right\}$, c) $(\bar{x}, \bar{y}, \bar{z})$ solves $i \geq 2$ maximization problem for constraints $B^{FM}_{z_i}$. Then by defining $\bar{z}_1 = -\sum_{i=2}^m \bar{z}_i$, every $(\bar{x}, \bar{y}), (\bar{P}, L)$ is a $(\bar{x}, \bar{y}, \bar{z}), (\bar{p}, \bar{q})$ equilibrium.

Remark: Since agent 1 faces only the Arrow-Debreu constraints, his behavior is identical in both models.

Observation (2): Suppose $\left(P,\tilde{L}\right)\in\Psi^{\rho}$ are elements of the (FE) pseudo equilibrium manifold, and conditions a) $\left[I \quad F\right]\hat{V}\left(P_1,\phi\right)=0,$ and (ii) $\begin{bmatrix} Q_{\rho\times(S-n)} \\ F^{Q}_{(n-\rho)\times(S-n)} \end{bmatrix}_{n\times(S-n)} -\left[L\right]_{n\times(S-n)}=0$ hold.

Under these conditions, a consumption bundle $\overline{x}_i \quad (i\geq 2)$ is feasible under the constraints b) in the ψ model if and only if $\overline{x}_i \quad (\forall i)$ is feasible under the constraints holding with equality in a) in the (FE) model.

The next step is then to show that $\tilde{L}=\left(L^{\perp}\subset\left\langle\Gamma^{\rho}\left(P_1,\phi\right)\right\rangle^{\perp}\right)$ exists. Recall that

$$\Psi^{\rho}=\left\{\left(P_1,\left\langle\Gamma^{\rho}\left(P_1,\phi\right)\right\rangle^{\perp},L^{\perp}\right)\in P^{\rho}\times G^{S-n+\rho}\left(\mathbb{R}^{S}\right)\times G^{S-n}\left(\mathbb{R}^{S}\right):L^{\perp}\subset\left\langle\Gamma^{\rho}\left(P_1,\phi\right)\right\rangle^{\perp}\right\}.$$

Let $e=\left\langle\Gamma^{\rho}\left(P_1,\phi\right)\right\rangle^{\perp}$ and $l=L^{\perp}\subset\left\langle\Gamma^{\rho}\left(P_1,\phi\right)\right\rangle^{\perp}$. Relabel an element $\left(\hat{P},\hat{e},\hat{l}\right)$ of Ψ^{ρ} in the orthogonal basis of $\mathbb{R}^{S}$ such that in the neighborhood of $\hat{e}$, the vector space e is spanned by the columns of a $S\times\left(S-n+\rho\right)$ matrix $\begin{bmatrix} 1_{(S-n+\rho)\times(S-n+\rho)} \\ E_{(n-\rho)\times(S-n+\rho)} \end{bmatrix}$. Similarly, in the neighborhood of $\hat{l}$, the vector space l in the same orthogonal basis of $\mathbb{R}^{S}$ is spanned by the columns of a $S\times\left(S-n\right)$ matrix $\begin{bmatrix} 1_{(s-n)\times(s-n)} \\ L_{n\times(s-n)} \end{bmatrix}$. We also rewrite the financial return matrix $V\left(.,.\right)$ in this basis, such that it becomes $\hat{V}\left(P_1,\phi\right)=\begin{bmatrix} {}_{(n-\rho)\times(n-\rho)} \\ {}_{(S-n+\rho)\times(n-\rho)} \end{bmatrix}, S\times\left(n-\rho\right)$.

Condition (1): $e=\left(span\left(\Gamma^{\rho}\left(P_1,\phi\right)\right)\right)^{\perp}$.

Translate $\begin{bmatrix} I_{(S-n+\rho)\times(S-n+\rho)} \\ E_{(n-\rho)\times(S-n+\rho)} \end{bmatrix}_{S\times(S-n+\rho)}\rightarrow\left[I \quad E\right]$ then condition (1) becomes

$$\left[I \quad E\right]\hat{V}\left(P_1,\phi\right)=0. \tag{9}$$

Condition (2): $l\in G^{S-n}\left(\mathbb{R}^{S}\right)\subset e\in G^{S-n+\rho}\left(\mathbb{R}^{S}\right)$.

Need to find a matrix Q such that $\begin{bmatrix} I \\ E \end{bmatrix}-\begin{bmatrix} I \\ L \end{bmatrix}=0.$ We first partition $\begin{bmatrix} I \\ E \end{bmatrix}$ such that it becomes $\begin{bmatrix} 1_{(S-n)\times(S-n)} & 0_{(S-n)\times\rho} \\ 0_{\rho\times(S-n)} & 1_{\rho\times\rho} \\ E'_{(n-\rho)\times(S-n)} & E''_{(n-\rho)\times\rho} \end{bmatrix}$, then

$$\begin{bmatrix} 1_{(S-n)\times(S-n)} & 0_{(S-n)\times\rho} \\ 0_{\rho\times(S-n)} & 1_{\rho\times\rho}Q_{\rho\times(S-n)} \\ E'_{(n-\rho)\times(S-n)} & E''_{(n-\rho)\times\rho}Q_{\rho\times(S-n)} \end{bmatrix}=\begin{bmatrix} 1_{(S-n)\times(S-n)} \\ 1_{\rho\times\rho}Q_{\rho\times(S-n)} \\ E'_{(n-\rho)\times(S-n)}+E''_{(n-\rho)\times\rho}Q_{\rho\times(S-n)} \end{bmatrix}=\begin{bmatrix} 1_{(S-n)\times(S-n)} \\ Q_{\rho\times(S-n)} \\ E^{Q}_{(n-\rho)\times(S-n)} \end{bmatrix}.$$

Q is a $\left(\rho\times S-n\right)$ matrix. Condition (2) can then be written in terms of Q and E:

$$\begin{bmatrix} Q_{\rho\times(S-n)} \\ E^{Q}_{(n-\rho)\times(S-n)} \end{bmatrix}_{n\times(S-n)}-\left[L\right]_{n\times(S-n)}=0. \tag{10}$$

The final step is then to show that the pseudo equilibrium manifold Ψ^{ρ}, parameterized by P and Q is locally identified by a diffeomorphism $\Lambda\left(P,\tilde{L}\right)$, defined by $(9)\times(9)\mapsto\Psi^{\rho}$. The partial derivative $D^{-1}_{P,Q}\Lambda\left(P,\tilde{L}(Q)\right)$ exists, moreover, the map is bijective. □

Proposition 3. *If* $(\overline{x},\overline{y}),(\overline{P},\overline{L})$ *is a pseudo FE with production then for every* $\beta\in\mathbb{R}^{S}_{++}$, *there exist financial asset prices* $\overline{q}\in\mathbb{R}^{n}_{++}$ *and investment portfolios* $\overline{z}=(z(1),\cdots,z(n))\in\mathbb{R}^{n}_{++}$ such that $(\overline{x},\overline{y},\overline{z}),(\overline{p},\overline{q})$ is a $(\overline{x},\overline{y})$ allocational equivalent FE with production.

Proof. Using (Definition 3), let $\overline{q}=\sum_{s=1}^{S}\left(\Gamma\left(\overline{P}_1,\overline{\phi}\right)\right)$, let $\overline{p}_1=proj\left(\left(\frac{1}{\beta(s)}\right)^{T}\Box\overline{P}_1(s)\right)$, and let $z_1=\sum_{i=2}^{m}z_i$. The equivalence of a pseudo equilibrium with production and a financial markets with production then follows from similar arguments as in [16]. □

Long run financial payoffs depend on the technology of the firm, its production capacity installed via financial markets, and on a set of regular prices. Equilibrium does not exist for critical prices. The next step is therefore to introduce rank dependant payoff maps, and to exhibit a class of transverse price, technology, and capacity dependent maps. We will show that equilibria exists for this smooth rank dependent real asset structure, denoted π^{ρ}.

Definition 5. *Define the rank dependent long run payoff maps* $\pi^{\rho}:\mathbb{R}^{l}_{++}\times\mathbb{R}^{l}\times\mathbb{R}_{+}\to\mathcal{A}^{\rho}$ *for* $0\le\rho\le n$. *The set of reduced rank matrices* A^{ρ} *of order* $(S\times n)$ *with* $rank\left(A^{\rho}\right)=(n-\rho)$ *is denoted* $\mathcal{A}^{\rho}$ *and is of order* $(S\times n)$.

Lemma 3. a) *For* $1\le\rho<n$, A^{ρ} *is a submanifold of A of codimension* $(S-n+\rho)\rho$. b) *for* $\rho=n$ *the set* $\mathcal{A}^{\rho}=\{\varnothing\}$ *is empty, and* c) *for* $\rho=0$, $\mathcal{A}^{\rho}=\mathcal{A}$ *the set of reduced rank matrices is equivalent to the set of full rank matrices.*

Proof. Consider the open set *U* of $(S\times n)$ matrices $\tilde{a}=\left[\begin{array}{c|c}A_{(n-\rho)\times(n-\rho)} & B_{(n-\rho)\times\rho}\\ \hline C_{(S-n+\rho)\times(n-\rho)} & D_{(S-n+\rho)\times\rho}\end{array}\right]$ of $rank(\tilde{a})=(n-\rho)$ since $\det\overline{A}\neq0$. There exists a matrix $b_{(n-\rho)\times\rho}$ such that $\begin{bmatrix}B\\D\end{bmatrix}=\begin{bmatrix}A\\C\end{bmatrix}b\Leftrightarrow b=A^{-1}B$, and $D=CA^{-1}B$[4]. □

The lemma states that, for $1\le\rho<n$, the incomplete income transfer space is rank reduced. The rank dependent endogenized long run asset structure has following properties.

Proposition 4. a) $\pi^{\rho}\pitchfork\mathcal{A}^{\rho}$ *for integers* $1\le\rho\le n$. b) $\Gamma^{\rho}\pitchfork\mathcal{A}^{\rho}$ *for any* $\beta\in\mathbb{R}^{S}_{++}$ *and integers* $1\le\rho\le n$. c) $\Gamma^{\rho}\cap\mathcal{A}$ *is generic, since it is dense and open.*

Proof. a) The linear map $D_y\pi^{\rho}$ is surjective everywhere in *Y*. b) This property does not change for any $\beta\in\mathbb{R}^{S}_{++}$. c) Immediate consequence of the transversality theorem for maps. Since each set $\pitchfork\left(\Gamma^{\rho},\mathcal{A};\mathcal{A}^{\rho}\right)$ is residual, their intersection is residual. □

Definition 6. *Denote* Ψ^{ρ} *the vector bundle defined by* a) *a basis* $P^{\rho}=\left\{P\in\mathbb{R}^{l(S+1)}_{++}:\text{rank}\left(\Gamma^{\rho}\left(P_1,\phi\right)\right)=(n-\rho)\right\}$, and b) orthogonal income transfer space $L^{\perp}\subset\left\langle\Gamma^{\rho}\left(P_1,\phi\right)\right\rangle^{\perp}$,

$$\Psi^{\rho}=\left\{\left(P,\left\langle\Gamma^{\rho}\left(P_1,\phi\right)\right\rangle^{\perp},L^{\perp}\right)\in P^{\rho}\times G^{S-n+\rho}\left(\mathbb{R}^{S}\right)\times G^{S-n}\left(\mathbb{R}^{S}\right):L^{\perp}\subset\left\langle\Gamma^{\rho}\left(P_1,\phi\right)\right\rangle^{\perp}\right\}. \quad (11)$$

We thus have defined a fiber bundle Ψ^{ρ} of codimension $l(S+1)-1-\rho^{2}$ containing the spot price system and income transfer space consisting of a base vector P^{ρ} and fiber $G^{S-n}\left(\mathbb{R}^{S-n+\rho}\right)$. We can now state the main result.

[4]It is known that $\mathcal{A}^{\rho}$ constitutes a submanifold complex of $\mathcal{A}$. See Hirsch [13].

Theorem 5. *There exists a pseudo FE with production* $(\bar{x},\bar{y}),(\bar{P},\bar{L}) \in \mathbb{R}_{++}^{l(S+1)m} \times \mathbb{R}_{+}^{l(S+1)n} \times \mathfrak{S}' \times G^n(\mathbb{R}^S)$ *for generic endowments. Moreover, by the relational propositions, a FE with production* $(\bar{x},\bar{y},\bar{z}),(\bar{p},\bar{q}) \in \mathbb{R}_{++}^{l(S+1)m} \times \mathbb{R}_{+}^{l(S+1)n} \times \mathbb{R}^{nm} \times \mathfrak{S} \times \mathbb{R}_{++}^{n}$ *exists for generic endowments.*

Proof. By (Proposition 4) and using (Definition 6) define an evaluation map Z^{ρ} on $\Psi^{\rho} \times \mathbb{R}_{++}^{l(S+1)m}$, where denote $\Omega = \mathbb{R}_{++}^{l(S+1)m}$ the set of the economy's total initial endowments, such that the excess demand map $Z^{\rho}: \Psi^{\rho} \times \Omega \to N$.

For the Arrow-Debreu agent have

$$Z_1^{\rho}: \Psi^{\rho} \times \Omega \to N. \tag{12}$$

The evaluation map is a submersion, since $D_{\omega_1} Z_1^{\rho}$ $\forall \omega_1 \in \Omega$ is surjective everywhere. $\exists$ for each $\omega_1 \in \Omega$

$$Z_{1,\omega_1 \in \Omega}^{\rho}: \Psi^{\rho} \to N \pitchfork_{\omega \in \Omega_{\rho}} \{0\}, \tag{13}$$

where $\{0\} \subset N$, and $\rho = 0$. The dimension of the preimage $Z_{1,\omega_1 \in \Omega}^{-1}(\{0\})$ is $l(S+1)-1$. By Thom's parametric transversality theorem[5], it follows that the subset $\Omega_{\rho} \cap \Omega$ is generic since it is open and dense. Equilibria exist. By the equivalence propositions 2 and 3 know that full rank financial markets equilibria with production exist.

For all $1 \le \rho \le n$ the preimage of the rank reduced evaluation map has dimension $l(S+1)-1-\rho^2$. This implies that for generic endowments $\omega \in \cap_{\rho}(\Omega_{\rho})$, for $\rho = 1,\cdots,n$, there is no reduced rank equilibrium, since for $Z_1^{\rho}(.,\omega)$ the set of $\{0\} = \varnothing$. □

4. Conclusion

This paper links the real and the financial sector in a general equilibrium model with incomplete financial markets. Production capacity available to a firm is endogenized and depends on the financial decisions of the firm in period one. At varianve to utility maximizing objective functions of firms, the model developed here considers a long run profit maximization objective function. This rehabilitates the decentralization property of the standard Arrow-Debreu model. It is shown by a parametric transversality theorem that equilibria exists.

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[5]See *i.e.* Hirsch for an exposition of Thom's parametric transversality theorem [13]. For more on transversality see R. Abraham and J. Robbin (1967), Transversal Mappings and Flows (W. A. Benjamin).

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