

On Historical Value at Risk under Distribution Uncertainty

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Abstract

We investigate the asymptotics of the historical value-at-risk under capacities defined by sublinear expectations. By generalizing Glivenko-Cantelli lemma, we show that the historical value-at-risk eventually lies between the upper and lower value-at-risks quasi surely.

Keywords

Value-at-Risk, Sublinear Expectation, Capacities, Glivenko-Cantelli Lemma

1. Introduction

In financial industry, the value-at-risk has been one of main tools for risk management (see, e.g., McNeil *et al.* [1], and Föllmer and Schied [2]). In this framework, the random variables for assets or asset returns are assumed to have distributions without uncertainty. In other words, it is implicitly assumed that there are true asset distributions and the estimation difficulty comes from our limited capability. However, it should be remarked that there is a possibility that the assets have the distribution uncertainty, *i.e.*, the assets may have Knightian uncertainty (see Knight [3]).

To capture the distribution uncertainty, the theory of sublinear expectation is introduced and developed (see Peng [4] [5] and the references therein). In this theory, the term probability is replaced by the ones of the upper and lower capacities induced by the upper and lower expectations, respectively, and the distribution uncertainty is described by the gap between the upper and lower expectations.

In this paper, we consider the value-at-risk type risk measure under the sublinear expectation, where the reference probability measure in the classical framework is replaced by the upper and lower capacities. We call these the upper and lower value-at-risk, respectively. Our aim is to study the asymptotic behavior of the historical value-at-risk under uncertainty. In doing so, we prove a generalization of Glivenko-Cantelli lemma under

uncertainty, and then show that the historical value-at-risk eventually lies in between the upper and lower value-at-risks quasi surely.

This paper is organized as follows: In Section 2, we recall the theory of sublinear expectation. Section 3 is devoted to the statement of the main results and its proofs.

2. Sublinear Expectation and Capacities

In this section, we recall the basis of the sublinear expectation, introduced by Peng [4]. Let Ω be a given set and \mathcal{H} a linear space of \mathbb{R} -valued functions on Ω . We assume that $\varphi(X_1, \dots, X_n) \in \mathcal{H}$ whenever $X_1, \dots, X_n \in \mathcal{H}$ and φ is a bounded function on \mathbb{R}^n or $\varphi \in C_{1,Lip}(\mathbb{R}^n)$ where $C_{1,Lip}(\mathbb{R}^n)$ denotes the linear space of functions φ on \mathbb{R}^n satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad x, y \in \mathbb{R}^n,$$

for some $C > 0$ and $m \in \mathbb{N}$ depending on φ . We call an element in \mathcal{H} a random variable.

We consider a sublinear expectation $\mathbb{E}: \mathcal{H} \rightarrow \mathbb{R}$, in the sense of [4]. Namely, \mathbb{E} is assumed to satisfy the following conditions: for any $X, Y \in \mathcal{H}$,

- 1) Monotonicity: if $X \leq Y$ then $\mathbb{E}[X] \leq \mathbb{E}[Y]$.
- 2) Constant preserving: $\mathbb{E}[x] = x$ for $x \in \mathbb{R}$.
- 3) Subadditivity: $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]$.
- 4) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for $\lambda \geq 0$.

Moreover, we assume that $\mathbb{E}[X_i] \rightarrow 0$ for $\{X_i\} \subset \mathcal{H}$ with $X_i(\omega) \searrow 0$ for each $\omega \in \Omega$. Then, by Theorem 2.1 and Remark 2.2 in [5], there exists a set \mathcal{P} of probability measures on $(\Omega, \sigma(\mathcal{H}))$ such that

$$\mathbb{E}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad X \in \mathcal{H},$$

where E_P denotes the linear expectation with respect to $P \in \mathcal{P}$. Then, the $[0, 1]$ -valued set functions

$$C^*(A) = \mathbb{E}[1_A], \quad C_*(A) = -\mathbb{E}[-1_A], \quad A \in \sigma(\mathcal{H}),$$

define capacities, where 1_A denotes the indicator function of a set A . That is, each $C \in \{C^*, C_*\}$ satisfies the following:

- 1) $C(\emptyset) = 0$, $C(\Omega) = 1$.
- 2) If $A, B \in \sigma(\mathcal{H})$ satisfy $A \subset B$ then $C(A) \leq C(B)$.

We refer to Denenberg [6] for the theory of capacities. Throughout this paper, we assume that each $C \in \{C^*, C_*\}$ satisfies the following:

- 3) If $\{A_n\}_{n=1}^\infty \subset \sigma(\mathcal{H})$ satisfies $A_1 \subset A_2 \subset \dots$, then $\lim_{n \rightarrow \infty} C(A_n) = C\left(\bigcup_{k=1}^\infty A_k\right)$.
- 4) If $\{A_n\}_{n=1}^\infty \subset \sigma(\mathcal{H})$ satisfies $A_1 \supset A_2 \supset \dots$, then $\lim_{n \rightarrow \infty} C(A_n) = C\left(\bigcap_{k=1}^\infty A_k\right)$.

Let us recall several concepts in the sublinear expectation theory. The random variable Y_n is said to be independent of $X := (Y_1, Y_2, \dots, Y_{n-1})$, where $Y_i \in \mathcal{H}$, $i = 1, \dots, n$, if

$$\mathbb{E}[\varphi(X, Y_n)] = \mathbb{E}\left[\mathbb{E}[\varphi(x, Y_n)]\Big|_{x=X}\right], \quad \varphi \in C_{1,Lip}(\mathbb{R}^{1+n}).$$

We say that $X \in \mathcal{H}$ and $Y \in \mathcal{H}$ have the same distribution if

$$\mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(Y)], \quad \varphi \in C_{1,Lip}(\mathbb{R}).$$

A sequence $\{X_i\}_{i=1}^\infty$ is called the one of independent, identically distributed random variables if X_i and X_1 have the same distribution, and if X_{i+1} is independent of $Y := (X_1, \dots, X_i)$ for any $i \geq 1$. As in the linear case, we call a sequence of independent, identically distributed random variables an IID sequence. We say that the distribution of X has an uncertainty if $\mathbb{E}[\varphi(X)]$ is nonlinear in φ . In particular, set

$$\bar{\mu} = \mathbb{E}[X], \quad \underline{\mu} = -\mathbb{E}[-X], \quad \bar{\sigma}^2 = \mathbb{E}[X^2], \quad \underline{\sigma}^2 = -\mathbb{E}[-X^2].$$

Then if $\underline{\mu} \neq \bar{\mu}$, then we say that X has the mean uncertainty. Similarly, X is said to have the volatility or variance uncertainty if $\underline{\sigma} \neq \bar{\sigma}$.

3. Main Results

For any $X \in \mathcal{H}$, we define the functions $F^*, F_* : \mathbb{R} \rightarrow [0, 1]$ by

$$F^*(x) = C^*(X \leq x) = \sup_{P \in \mathcal{P}} P[X \leq x], \quad F_*(x) = C_*(X \leq x) = \inf_{P \in \mathcal{P}} P[X \leq x], \quad x \in \mathbb{R}. \quad (1)$$

Proposition 3.1 *Let $X \in \mathcal{H}$ and let F^* and F_* be as in (1). Then each $F \in \{F^*, F_*\}$ satisfies the following:*

- 1) $F(x_1) \leq F(x_2)$ for $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$.
- 2) $\lim_{x \rightarrow -\infty} F(x) = 0$, $\lim_{x \rightarrow \infty} F(x) = 1$.
- 3) $\lim_{y \searrow x} F(y) = F(x)$ for $x \in \mathbb{R}$.

Proof. The assertion (1) follows from the monotonicity of C^* and C_* .

To prove (2), take $C \in \{C^*, C_*\}$ and set $F(x) = C(X \leq x)$. We will see $\lim_{n \rightarrow \infty} F(x_n) = 0$ for any sequence $\{x_n\}$ with $x_1 > x_2 > \dots \rightarrow -\infty$. By setting $A_n = (-\infty, x_n]$ we have $F(x_n) = C(A_n)$ and $A_1 \supset A_2 \supset \dots, \bigcap_{n=1}^{\infty} A_n = \emptyset$. It follows from the definition of the capacity that

$$\lim_{n \rightarrow \infty} F(x_n) = C\left(\lim_{n \rightarrow \infty} A_n\right) = 0.$$

Similarly, $\lim_{x \rightarrow \infty} F(x) = 1$ follows.

Finally, by an argument similar to the proof of (2) with $A_n = (-\infty, x + \varepsilon_n]$, we can show $\lim_{n \rightarrow \infty} F(x + \varepsilon_n) = F(x)$ for any sequence $\{\varepsilon_n\}$ with $\varepsilon_1 > \varepsilon_2 > \dots \rightarrow 0$, implying (3). \square

The proposition above justifies the following definition:

Definition 3.2 *For a random variable X , we call the function F^* and F_* as in (1) the upper and lower cumulative distribution functions of X , respectively.*

For an IID sequence $\{X_i\}_{i=1}^{\infty}$, the empirical distribution function $\hat{F}_n : \mathbb{R} \times \Omega \rightarrow [0, 1]$ of $\{X_i\}$ is defined by

$$\hat{F}_n(x, \omega) = \frac{1}{n} \sum_{i=1}^n I_{[X_i(\omega), \infty)}(x), \quad x \in \mathbb{R}, \quad \omega \in \Omega, \quad n \in \mathbb{N}.$$

If $\{X_i\}_{i=1}^{\infty}$ satisfies $\mathbb{E}[|X_1|^p] < \infty$ for some $p > 1$, then by Strong law of large number under sublinear expectation (see Theorem 1 in Chen [7]),

$$C_*\left(F_*(x) \leq \liminf_{n \rightarrow \infty} \hat{F}_n(x) \leq \limsup_{n \rightarrow \infty} \hat{F}_n(x) \leq F^*(x)\right) = 1, \quad x \in \mathbb{R}. \quad (2)$$

Indeed,

$$\begin{aligned} \mathbb{E}\left[1_{[X_i, \infty)}(x)\right] &= C^*\left(1_{[X_i, \infty)}(x) = 1\right) = C^*(X_i \leq x) = F^*(x), \\ -\mathbb{E}\left[-1_{[X_i, \infty)}(x)\right] &= C_*\left(1_{[X_i, \infty)}(x) = 1\right) = C_*(X_i \leq x) = F_*(x). \end{aligned}$$

We show a stronger result, which is a generalization of Glivenko-Cantelli lemma.

Theorem 3.3 *Let $\{X_i\}_{i=1}^{\infty}$ be an IID sequence with $\mathbb{E}[|X_1|^p] < \infty$ for some $p > 1$. Denote by F^* and F_* the upper and lower cumulative distribution functions of X respectively, and denote by $\hat{F}_n(x)$ the empirical distribution function of $\{X_i\}_{i=1}^{\infty}$. Then,*

$$C_*\left(F_*(x) \leq \liminf_{n \rightarrow \infty} \hat{F}_n(x) \leq \limsup_{n \rightarrow \infty} \hat{F}_n(x) \leq F^*(x), \quad x \in \mathbb{R}\right) = 1.$$

We need the following lemma for the proof of the theorem.

Lemma 3.4 *Under the assumptions imposed in Theorem 3.3, for $\varepsilon \in (0, 1)$, there exist $k \geq 2$ and*

$\{t_j\}_{j=0}^k \subset \mathbb{R} \cup \{\pm\infty\}$ such that $-\infty = t_0 < t_1 < \dots < t_k = \infty$ and

$$F^*(t_{j+1}-) - F^*(t_j) \leq \varepsilon, \quad F_*(t_{j+1}-) - F_*(t_j) \leq \varepsilon, \quad 0 \leq j \leq k-1,$$

where $F(s-) = \lim_{u \searrow s} F(u)$, $s \in \mathbb{R}$, for each $F \in \{F^*, F_*\}$.

Proof. Let $\varepsilon \in (0, 1)$. Then there exists $t_1 > -\infty$ such that $F^*(t_1), F_*(t_1) \leq \varepsilon$. Starting with $t_0 = -\infty$, we recursively define $\{t_j\}$ by

$$t_{j+1} = \sup\{z \geq t_j : F^*(z) \leq F^*(t_j) + \varepsilon, F_*(z) \leq F_*(t_j) + \varepsilon\}, \quad j = 1, 2, \dots$$

By this recursion, we can find $k \geq 2$ such that $F^*(t_{k-1}) \geq 1 - \varepsilon$ or $F_*(t_{k-1}) \geq 1 - \varepsilon$ hold, and set $t_k = +\infty$. With this sequence, the lemma follows. \square

With the help of Lemma 3.4, we can show Theorem 3.3.

Proof of Theorem 3.3. Let $\varepsilon > 0$ be fixed. By Lemma 3.4, there exists a partition of \mathbb{R} such that $-\infty = t_0 < t_1 < \dots < t_k = \infty$ and

$$F^*(t_{j+1}-) - F^*(t_j) \leq \varepsilon, \quad F_*(t_{j+1}-) - F_*(t_j) \leq \varepsilon, \quad 0 \leq j \leq k-1. \quad (3)$$

By (2), we have, $C_*(A_q) = 1$ for any $q \in \mathbb{Q}$, where

$$A_s = \left\{ F_*(s) \leq \liminf_{n \rightarrow \infty} \hat{F}_n(s) \leq \limsup_{n \rightarrow \infty} \hat{F}_n(s) \leq F^*(s) \right\}, \quad s \in \mathbb{R}.$$

Thus, $P(A_q) = 1$ for any $P \in \mathcal{P}$ and so $P\left(\bigcap_{q \in \mathbb{Q}} A_q\right) = 1$ for any $P \in \mathcal{P}$. Hence

$$P(A_j) = 1, \quad P(A_{j-}) = 1, \quad P \in \mathcal{P}, \quad j = 1, \dots, k-1,$$

where for $s \in \mathbb{R}$

$$A_{s-} = \left\{ F_*(s-) \leq \liminf_{n \rightarrow \infty} \hat{F}_n(s-) \leq \limsup_{n \rightarrow \infty} \hat{F}_n(s-) \leq F^*(s-) \right\}.$$

So we have

$$C_*\left(\bigcap_{j=0}^k (A_{t_j} \cap A_{t_{j-}})\right) = 1.$$

Now, for any $x \in \mathbb{R}$ there exists j such that $x \in [t_{j-1}, t_j)$. Thus, by (3),

$$\hat{F}_n(x) - F^*(x) \leq \hat{F}_n(t_j-) - F^*(t_{j-1}) \leq \hat{F}_n(t_j-) - F^*(t_j-) + \varepsilon,$$

$$\hat{F}_n(x) - F_*(x) \geq F_n(t_{j-1}) - F_*(t_j-) \geq \hat{F}_n(t_{j-1}) - F_*(t_{j-1}) - \varepsilon,$$

so

$$F_*(x) + \hat{F}_n(t_{j-1}) - F^*(t_{j-1}) - \varepsilon \leq \hat{F}_n(x) \leq F^*(x) + \hat{F}_n(t_j^-) - F_*(t_j^-) + \varepsilon.$$

Therefore, on $\bigcap_{j=0}^k (A_{t_j} \cap A_{t_{j-}})$, letting $n \rightarrow \infty$ we get $F_*(x) - \varepsilon \leq \liminf_{n \rightarrow \infty} \hat{F}_n(x)$ and

$F^*(x) + \varepsilon \geq \limsup_{n \rightarrow \infty} \hat{F}_n(x)$ for all $x \in \mathbb{R}$. Thus

$$C_*\left(F_*(x) - \varepsilon \leq \liminf_{n \rightarrow \infty} \hat{F}_n(x) \leq \limsup_{n \rightarrow \infty} \hat{F}_n(x) \leq F^*(x) + \varepsilon, \quad x \in \mathbb{R}\right) = 1, \quad \varepsilon > 0.$$

If we write B_ε for the event inside the brace above and denote $B = \bigcap_{m=1}^{\infty} B_{1/m}$, then by the continuity of the capacity

$$C_*(B) = C_*\left(\lim_{m \rightarrow \infty} B_{1/m}\right) = \lim_{m \rightarrow \infty} C_*(B_{1/m}) = 1,$$

meaning the assertion of the theorem. \square

Recall that for a function $F: \mathbb{R} \rightarrow [0,1]$ satisfying (1)-(3) in Proposition 3.1 and for $\alpha \in (0,1)$, the α -quantile $q_\alpha(F)$ of F is defined by

$$q_\alpha(F) = \inf \{x \in \mathbb{R} : F(x) > \alpha\}.$$

Then we have the following:

Theorem 3.5 Let $\{X_i\}_{i=1}^\infty$ be an IID sequence with $\mathbb{E}[|X_1|^p] < \infty$ for some $p > 1$. Denote by F^* and F_* the upper and lower cumulative distribution functions of X respectively, and denote by $\hat{F}_n(x)$ the empirical distribution function of $\{X_i\}_{i=1}^\infty$. Consider the upper, lower, and historical value-at-risk defined respectively by

$$\text{VaR}^*(\alpha) := -q_\alpha(F^*), \quad \text{VaR}_*(\alpha) := -q_\alpha(F_*), \quad \widehat{\text{VaR}}_n(\alpha) := -q_\alpha(\hat{F}_n).$$

Suppose that for $\alpha \in (0,1)$

$$F^*(t) < \alpha, \quad t < -\text{VaR}^*(\alpha). \quad (4)$$

Then, the historical value-at-risk eventually lies in between the upper and lower value-at-risk, i.e.,

$$C_* \left(\text{VaR}_*(\alpha) \leq \liminf_{n \rightarrow \infty} \widehat{\text{VaR}}_n(\alpha) \leq \limsup_{n \rightarrow \infty} \widehat{\text{VaR}}_n(\alpha) \leq \text{VaR}^*(\alpha) \right) = 1, \quad \alpha \in (0,1).$$

Proof. Consider the event A defined by

$$A = \left\{ F_*(x) \leq \liminf_{n \rightarrow \infty} \hat{F}_n(x) \leq \limsup_{n \rightarrow \infty} \hat{F}_n(x) \leq F^*(x), \quad x \in \mathbb{R} \right\}.$$

In view of Theorem 3.3, it suffices to show that for a given $\varepsilon > 0$ and $\omega \in A$ there exists $N = N(\omega)$ such that

$$q_\alpha(F^*) - \varepsilon \leq q_\alpha(\hat{F}_n(\cdot, \omega)) \leq q_\alpha(F_*) + \varepsilon, \quad n \geq N. \quad (5)$$

To this end, fix $\omega \in A$ and set $q^* = q_\alpha(F^*)$, $q_* = q_\alpha(F_*)$. By the definition of the infimum and the condition (4),

$$F^*(q^* - \varepsilon/2) < \alpha < F_*(q_* + \varepsilon/2).$$

Thus we can take $\delta > 0$ such that

$$F^*(q^* - \varepsilon/2) < \alpha - \delta < \alpha + \delta < F_*(q_* + \varepsilon/2).$$

Next, take $N = N(\omega)$ such that

$$F_*(x) - \delta \leq \hat{F}_n(x, \omega) \leq F^*(x) + \delta, \quad x \in \mathbb{R}, \quad n \geq N,$$

and set $q_n(\omega) = q_\alpha(\hat{F}_n(\omega))$. Then,

$$F_*(q_n(\omega) - \varepsilon/2) \leq \hat{F}_n(q_n(\omega) - \varepsilon/2, \omega) + \delta \leq \alpha - \delta < F_*(q_* + \varepsilon/2).$$

So we have $q_n(\omega) < q_* - \varepsilon$. Similarly, we see

$$F^*(q_n(\omega) + \varepsilon/2) \geq \hat{F}_n(q_n(\omega) + \varepsilon/2, \omega) - \delta > \alpha - \delta > F^*(q^* - \varepsilon/2),$$

leading to $q_n(\omega) > q^* - \varepsilon$. Thus (5) follows. \square

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