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Risk Measures and Nonlinear Expectations*

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ABSTRACT

Coherent and convex risk measures, Choquet expectation and Peng's *g*-expectation are all generalizations of mathematical expectation. All have been widely used to assess financial riskiness under uncertainty. In this paper, we investigate differences amongst these risk measures and expectations. For this purpose, we constrain our attention of coherent and convex risk measures, and Choquet expectation to the domain of *g*-expectation. Some differences among coherent and convex risk measures and Choquet expectations are accounted for in the framework of *g*-expectations. We show that in the family of convex risk measures, only coherent risk measures satisfy Jensen's inequality. In mathematical finance, risk measures and Choquet expectations are typically used in the pricing of contingent claims over families of measures. The different risk measures will typically yield different pricing. In this paper, we show that the coherent pricing is always less than the corresponding Choquet pricing. This property and inequality fails in general when one uses pricing by convex risk measures. We also discuss the relation between static risk measure and dynamic risk measure in the framework of *g*-expectations. We show that if *g*-expectations yield coherent (convex) risk measures then the corresponding conditional *g*-expectations or equivalently the dynamic risk measure is also coherent (convex). To prove these results, we establish a new converse of the comparison theorem of *g*-expectations.

Keywords: Risk Measure; Coherent Risk; Convex Risk; Choquet Expectation; *g*-Expectation; Backward Stochastic Differential Equation; Converse Comparison Theorem; BSDE; Jensen's Inequality

1. Introduction

The choice of financial risk measures is very important in the assessment of the riskiness of financial positions. For this reason, several classes of financial risk measures have been proposed in the literature. Among these are coherent and convex risk measures, Choquet expectations and Peng's *g*-expectations. Coherent risk measures were first introduced by Artzner, Delbaen, Eber and Heath [1] and Delbaen [2]. As an extension of coherent risk measures, convex risk measures in general probability spaces were introduced by Föllmer & Schied [3] and Frittelli & Rosazza Gianin [4]. *g*-expectations were introduced by Peng [5] via a class of nonlinear backward stochastic differential equations (BSDEs), this class of nonlinear BSDEs being introduced earlier by Pardoux and Peng [6]. Choquet [7] extended probability measures to nonadditive probability measures (capacity), and introduced the so called Choquet expectation.

Our interest in this paper is to explore the relations among risk measures and expectations. To do so, we restrict our attention of coherent and convex risk measures and Choquet expectations to the domain of *g*-expectations. The distinctions between coherent risk measure and convex risk measure are accounted for intuitively in the framework of *g*-expectations. We show that 1) in the family of convex risk measures, only coherent risk measures satisfy Jensen's inequality; 2) coherent risk measures are always bounded by the corresponding Choquet expectation, but such an inequality in general fails for convex risk measures. In finance, coherent and convex risk measures and Choquet expectations are often used in the pricing of a contingent claim. Result 2) implies coherent pricing is always less than Choquet pricing, but the pricing by a convex risk measure no longer has

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this property. We also study the relation between static and dynamic risk measures. We establish that if *g*-expectations are coherent (convex) risk measures, then the same is true for the corresponding conditional *g*-expectations or dynamic risk. In order to prove these results, we establish in Section 3, Theorem 1, a new converse comparison theorem of *g*-expectations. Jiang [8] studies *g*-expectation and shows that some cases give rise to risk measures. Here we are able to show, in the case of *g*-expectations, that coherent risk measures are bounded by Choquet expectation but this relation fails for convex risk measures; see Theorem 4. Also we show that convex risk measures obey Jensen's inequality; see Theorem 3.

The paper is organized as follows. Section 2 reviews and gives the various definitions needed here. Section 3 gives the main results and proofs. Section 4 gives a summary of the results, putting them into a Table form for convenience of the various relations.

2. Expectations and Risk Measures

In this section, we briefly recall the definitions of *g*-expectation, Choquet expectation, coherent and convex risk measures.

2.1. *g*-Expectation

Peng [5] introduced *g*-expectation via a class of backward stochastic differential equations (BSDE). Some of the relevant definition and notation are given here.

Fix $T\in[0,\infty)$ and let $(W_t)_{0\le t\le T}$ be a d-dimensional standard Brownian motion defined on a completed probability space $(\Omega,\mathcal{F},P)$. Suppose $\{\mathcal{F}_t\}_{0\le t\le T}$ is the natural filtration generated by $(W_t)_{0\le t\le T}$, that is $\mathcal{F}_t=\sigma\{W_s;s\le t\}$. We also assume $\mathcal{F}_T=\mathcal{F}$. Denote

$$L^2(\Omega,\mathcal{F}_t,P)=\left\{\xi:\xi \text{ is } \mathcal{F}_t\text{—measurable random variables with } E|\xi|^2<\infty\right\},\ t\in[0,T];$$

$$L^2\left(0,T,\mathbb{R}^d\right)=\left\{X: X \text{ is } \mathbb{R}^d\text{—valued}, \mathcal{F}_t\text{—adapted processes with } E\int_0^T|X_s|^2\,\mathrm{d}s<\infty\right\}$$

Let $g:\ \Omega\times\mathbb{R}\times\mathbb{R}^d\times[0,T]\to\mathbb{R}$ satisfy

(H1) For any $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, $\{g(y,z,t)\}_{t\ge0}$ is a continuous progressively measurable process with

$$E\left[\int_0^T|g(y,z,s)|^2\,\mathrm{d}s\right]<\infty .$$

(H2) There exists a constant $K\ge0$ such that for any

$$(y_1,z_1),(y_2,z_2)\in\mathbb{R}\times\mathbb{R}^d$$
$$|g(y_1,z_1,t)-g(y_2,z_2,t)|$$
$$\le K(|y_1-y_2|+|z_1-z_2|),\ t\in[0,T].$$

(H3) $g(y,0,t)=0,\ \forall(y,t)\in\mathbb{R}\times[0,T].$

In Section 3, Corollary 3 we will consider a special case of $\mathbb{R}^d$ with $d=1$.

Under the assumptions of (H1) and (H2), Pardoux and Peng [6] showed that for any $\xi\in L^2(\Omega,\mathcal{F},P)$, the BSDE

$$y_t=\xi+\int_t^T g(y_s,z_s,s)\,\mathrm{d}s-\int_t^T z_s\,\mathrm{d}W_s,\ 0\le t\le T \quad (1)$$

has a unique pair solution

$$(y_t,z_t)_{t\ge0}\in L^2(0,T,\mathbb{R})\times L^2\left(0,T,\mathbb{R}^d\right).$$

Using the solution y_t of BSDE (1), which depends on ξ, Peng [5] introduced the notion of *g*-expectations.

Definition 1 *Assume that* (*H*1), (*H*2) *and* (*H*3) *hold on g and* $\xi\in L^2(\Omega,\mathcal{F},P)$. *Let* (y_s,z_s) *be the solution of BSDE* (1).

$\mathcal{E}_g[\xi]$ defined by $\mathcal{E}_g[\xi]:=y_0$ is called the *g*-expectation of the random variable ξ.

$\mathcal{E}_g[\xi|\mathcal{F}_t]$ defined by $\mathcal{E}_g[\xi|\mathcal{F}_t]:=y_t$ is called the conditional *g*-expectation of the random variable ξ.

Peng [5] also showed that *g*-expectation $\mathcal{E}_g[\cdot]$ and conditional g-expectation $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ preserve most of basic properties of mathematical expectation, except for linearity. The basic properties are summarized in the next Lemma.

Lemma 1 (Peng) *Suppose that*

$$\xi,\xi_1,\xi_2\in L^2(\Omega,\mathcal{F},P).$$

1) Preservation of constants: For any constant c, $\mathcal{E}_g[c]=c$.

2) Monotonicity: If $\xi_1\ge\xi_2$, then $\mathcal{E}_g[\xi_1]\ge\mathcal{E}_g[\xi_2]$.

3) Strict monotonicity: If $\xi_1\ge\xi_2$, and $P(\xi_1>\xi_2)>0$, then $\mathcal{E}_g[\xi_1]>\mathcal{E}_g[\xi_2]$.

4) Consistency: For any $t\in[0,T]$,

$$\mathcal{E}_g\left[\mathcal{E}_g[\xi|\mathcal{F}_t]\right]=\mathcal{E}_g[\xi].$$

5) If g does not depend on y, and η is $\mathcal{F}_t$-measurable, then

$$\mathcal{E}_g[\xi+\eta|\mathcal{F}_t]=\mathcal{E}_g[\xi|\mathcal{F}_t]+\eta.$$

In particular, $\mathcal{E}_g\left[\xi-\mathcal{E}_g[\xi|\mathcal{F}_t]\,\middle|\,\mathcal{F}_t\right]=0$.

6) Continuity: If $\xi_n\to\xi$ as $n\to\infty$ in $L^2(\Omega,\mathcal{F},P)$, then $\lim_{n\to\infty}\mathcal{E}_g[\xi_n]=\mathcal{E}_g[\xi]$.

The following lemma is from Briand *et al.* [9, Theorem 2.1]. We can rewrite it as follows.

Lemma 2 (Briand *et al.*) *Suppose that* $\{X_t\}$ *is of the form* $X_t = x + \int_0^t \sigma_s \mathrm{d}W_s,\ \ 0 \le t \le T$, *where* $\{\sigma_t\}$ *is a continuous bounded process. Then*

$$\lim_{s\downarrow t} \frac{\mathcal{E}_g\left[X_s|\mathcal{F}_t\right] - E\left[X_s|\mathcal{F}_t\right]}{s-t} = g\left(X_t, \sigma_t, t\right),\ t \ge 0,$$

where the limit is in the sense of $L^2\left(\Omega, \mathcal{F}, P\right)$.

2.2. Choquet Expectation

Choquet [7] extended the notion of a probability measure to nonadditive probability (called capacity) and defined a kind of nonlinear expectation, which is now called Choquet expectation.

Definition 2

1) A real valued set function $V : \mathcal{F} \to [0,1]$ is called a capacity if

a) $V(\varnothing) = 0, V(\Omega) = 1$;

b) $V(A) \le V(B)$, whenever $A, B \in \mathcal{F}$ and $A \subset B$.

2) Let V be a capacity. For any $\xi \in L^2(\Omega, \mathcal{F}, P)$, the Choquet expectation $\mathcal{C}_V(\xi)$ is defined by

$$\mathcal{C}_V(\xi) := \int_{-\infty}^0 \left[V(\xi \ge t) - 1\right]\mathrm{d}t + \int_0^\infty V(\xi \ge t)\mathrm{d}t$$

Remark 1 *A property of Choquet expectation is positive homogeneity*, *i.e. for any constant* $a \ge 0$,

$$\mathcal{C}_V(a\xi) = a\mathcal{C}_V(\xi).$$

2.3. Risk Measures

A risk measure is a map $\rho : \mathcal{G} \to \mathbb{R}$, where $\mathcal{G}$ is interpreted as the "habitat" of the financial positions whose riskiness has to be quantified. In this paper, we shall consider $\mathcal{G} = L^2(\Omega, \mathcal{F}, P)$.

The following modifications of coherent risk measures (Artzner *et al*.[1]) is from Roorda *et. al.* [10].

Definition 3 *A risk measure* ρ *is said to be coherent if it satisfies*

1) Subadditivity: $\rho(X_1 + X_2) \le \rho(X_1) + \rho(X_2)$, $X_1, X_2 \in \mathcal{G}$;

2) Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$, for all real number $\lambda \ge 0$;

3) Monotonicity: $\rho(X) \le \rho(Y)$, whenever $X \le Y$;

4) Translation invariance: $\rho(X + \alpha) = \rho(X) + \alpha$ for all real number α.

As an extension of coherent risk measures, Föllmer and Schied [3] introduced the axiomatic setting for convex risk measures. The following modifications of convex risk measures of Föllmer and Schied [3] is from Frittelli and Rosazza Gianin [4].

Definition 4 *A risk measure is said to be convex if it satisfies*

1) Convexity:

$$\rho(\lambda X_1 + (1-\lambda) X_2) \le \lambda\rho(X_1) + (1-\lambda)\rho(X_2),$$

$\forall \lambda \in [0,1],\ \ X_1, X_2 \in \mathcal{G}$;

2) Normality: $\rho(0) = 0$;

3) Properties (3) and (4) in Definition 3.

A functional $\rho(\cdot)$ in Definitions 3 and 4 is usually called a static risk measure. Obviously, a coherent risk measure is a convex risk measure.

As an extension of such a functional $\rho(\cdot)$, Artzner *et al*. [11,12], Frittelli and Rosazza Gianin [13] introduced the notion of dynamic risk measure $\rho_t(\cdot)$, which is random and depends on a time parameter t.

Definition 5 *A dynamic risk measure*

$$\rho_t(\cdot) : L^2(\Omega, \mathcal{F}, P) \to L^2(\Omega, \mathcal{F}_t, P)$$

is a random functional which depends on t, *such that for each* t *it is a risk measure. If* $\rho_t(\cdot)$ *satisfies for each* $t \in [0,T]$ *the conditions in Definition* 3, *we say* $\rho_t(\cdot)$ *is a dynamic coherent risk measure. Similarly if* $\rho_t(\cdot)$ *satisfies for each* $t \in [0,T]$ *the conditions in Definition* 4, *we say* $\rho_t(\cdot)$ *is a dynamic convex risk measure.*

3. Main Results

In order to prove our main results, we establish a general converse comparison theorem of *g*-expectation. This theorem plays an important role in this paper.

Theorem 1 *Suppose that* g, g_1 *and* g_2 *satisfy* (*H*1), (*H*2) *and* (*H*3). *Then the following conclusions are equivalent.*

1) For any $\xi, \eta \in L^2(\Omega, \mathcal{F}, P)$,

$$\mathcal{E}_g[\xi + \eta] \le \mathcal{E}_{g_1}[\xi] + \mathcal{E}_{g_2}[\eta].$$

2) For any $(y_1, z_1, t), (y_2, z_2, t) \in \mathbb{R} \times \mathbb{R}^d \times [0,T]$,

$$g(y_1 + y_2, z_1 + z_2, t) \le g_1(y_1, z_1, t) + g_2(y_2, z_2, t). \tag{2}$$

Proof: We first show that inequality (2) implies inequality 3).

Let (y_t^1, z_t^1), (y_t^2, z_t^2) and (Y_t, Z_t) be the solutions of the following BSDE corresponding to the terminal value $X = \xi, \eta$ and $\xi + \eta$, and the generator $\overline{g} = g_1, g_2$ and g, respectively

$$y_t = X + \int_t^T \overline{g}(y_s, z_s, s)\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s. \tag{3}$$

Then $\mathcal{E}_{g_1}[\xi] = y_0^1$, $\mathcal{E}_{g_2}[\eta] = y_0^2$, $\mathcal{E}_g[\xi + \eta] = Y_0$.

For fixed (y_t^1, z_t^1), consider the BSDE

$$y_t = \xi + \eta + \int_t^T \left[g_2(y_s - y_s^1, z_s - z_s^1, s) + g_1(y_s^1, z_s^1, s)\right]\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s. \tag{4}$$

It is easy to check that $\left(y_t^1+y_t^2, z_t^1+z_t^2\right)$ is the solution of the BSDE (4).

Comparing BSDEs (4) and (3) with $X=\xi+\eta$ and $\bar{g}=g$, assumption (2), (2) then yields

$$g\left(y_t^1+y_t^2, z_t^1+z_t^2, t\right) \le g_1\left(y_t^1, z_t^1, t\right) + g_2\left(y_t^2, z_t^2, t\right),\ t\ge 0.$$

Applying the comparison theorem of BSDE in Peng [5], we have $Y_t \le y_t^1+y_t^2,\ t\ge 0.$ Taking $t=0$, thus by the definition of g-expectation, the proof of this part is complete.

We now prove that inequality (1) implies (2). We distinguish two cases: the former where g does not depend on y, the latter where g may depend on y.

Case 1, g does not depend on y. The proof of this case 1 is done in two steps.

Case 1, Step 1: We now show that for any $t\in[0,T]$, we have

$$\mathcal{E}_g\left[\xi+\eta|\mathcal{F}_t\right] \le \mathcal{E}_{g_1}\left[\xi|\mathcal{F}_t\right] + \mathcal{E}_{g_2}\left[\eta|\mathcal{F}_t\right],\quad \forall\, \xi,\eta\in L^2\left(\Omega,\mathcal{F},P\right).$$

Indeed, for $\forall t\in[0,T]$, set

$$A_t=\left\{\omega: \mathcal{E}_g\left[\xi+\eta|\mathcal{F}_t\right] > \mathcal{E}_{g_1}\left[\xi|\mathcal{F}_t\right] + \mathcal{E}_{g_2}\left[\eta|\mathcal{F}_t\right]\right\}.$$

If for all $t\in[0,T]$, we have $P\left(A_t\right)=0,$ then we obtain our result.

If not, then there exists $t\in[0,T]$ such that $P\left(A_t\right)>0$. We will now obtain a contradiction.

For this t,

$$I_{A_t}\mathcal{E}_g\left[\xi+\eta|\mathcal{F}_t\right] > I_{A_t}\left(\mathcal{E}_{g_1}\left[\xi|\mathcal{F}_t\right] + \mathcal{E}_{g_2}\left[\eta|\mathcal{F}_t\right]\right).$$

That is

$$I_{A_t}\left(\mathcal{E}_g\left[\xi+\eta|\mathcal{F}_t\right] - \mathcal{E}_{g_1}\left[\xi\,|\,\mathcal{F}_t\right] - \mathcal{E}_{g_2}\left[\eta|\mathcal{F}_t\right]\right)>0.$$

Taking g-expectation on both sides of the above inequality, and apply the strict monotonicity of g-expectation in Lemma 1 (3), it follows

$$\mathcal{E}_g\left[I_{A_t}\left(\mathcal{E}_g\left[\xi+\eta|\mathcal{F}_t\right] - \mathcal{E}_{g_1}\left[\xi|\mathcal{F}_t\right] - \mathcal{E}_{g_2}\left[\eta|\mathcal{F}_t\right]\right)\right]>0.$$

But by Lemma 1 (4) and (5),

$$\begin{aligned}&\mathcal{E}_g\left[I_{A_t}\left(\mathcal{E}_g\left[\xi+\eta|\mathcal{F}_t\right] - \mathcal{E}_{g_1}\left[\xi|\mathcal{F}_t\right] - \mathcal{E}_{g_2}\left[\eta|\mathcal{F}_t\right]\right)\right]\\ &= \mathcal{E}_g\left[I_{A_t}\left(\left(\xi+\eta\right) - \mathcal{E}_{g_1}\left[I_{A_t}\xi|\mathcal{F}_t\right] - \mathcal{E}_{g_2}\left[I_{A_t}\eta|\mathcal{F}_t\right]\right)\right].\end{aligned}$$

Note that by Lemma 1(v)

$$\mathcal{E}_{g_i}\left[I_{A_t}\xi - \mathcal{E}_{g_i}\left[I_{A_t}\xi|\mathcal{F}_t\right]\right]=0,\ i=1,2.$$

Thus

$$\begin{aligned}0 &< \mathcal{E}_g\left[I_{A_t}\left(\xi+\eta\right) - \mathcal{E}_{g_1}\left[I_{A_t}\xi|\mathcal{F}_t\right] - \mathcal{E}_{g_2}\left[I_{A_t}\eta|\mathcal{F}_t\right]\right]\\ &= \mathcal{E}_g\left[I_{A_t}\xi - \mathcal{E}_{g_1}\left[I_{A_t}\xi|\mathcal{F}_t\right] + I_{A_t}\eta - \mathcal{E}_{g_2}\left[I_{A_t}\eta|\mathcal{F}_t\right]\right]\\ &\le \mathcal{E}_{g_1}\left[I_{A_t}\xi - \mathcal{E}_{g_1}\left[I_{A_t}\xi|\mathcal{F}_t\right]\right]\\ &\quad + \mathcal{E}_{g_2}\left[I_{A_t}\eta - \mathcal{E}_{g_2}\left[I_{A_t}\eta|\mathcal{F}_t\right]\right]\\ &=0\end{aligned}$$

This induces a contradiction, thus concluding the proof of this Step 1.

Case 1, Step 2: For any $\tau,t\in[0,T]$ with $\tau\ge t$ and $z_i\in\mathbb{R}^d$, let us choose $X_\tau^i=z_i\left(W_\tau-W_t\right),\quad i=1,2$. Obviously, $X_\tau^i\in L^2\left(\Omega,\mathcal{F},P\right)$.

By Step 1,

$$\mathcal{E}_g\left[X_\tau^1+X_\tau^2|\mathcal{F}_t\right] \le \mathcal{E}_{g_1}\left[X_\tau^1|\mathcal{F}_t\right] + \mathcal{E}_{g_2}\left[X_\tau^2|\mathcal{F}_t\right],\ t\in[0,T].$$

Thus

$$\begin{aligned}&\frac{\mathcal{E}_g\left[X_\tau^1+X_\tau^2|\mathcal{F}_t\right] - E\left[X_\tau^1+X_\tau^2|\mathcal{F}_t\right]}{\tau-t}\\ &\le \frac{\mathcal{E}_{g_1}\left[X_\tau^1|\mathcal{F}_t\right] - E\left[X_\tau^1|\mathcal{F}_t\right]}{\tau-t}\\ &\quad + \frac{\mathcal{E}_{g_2}\left[X_\tau^2|\mathcal{F}_t\right] - E\left[X_\tau^2|\mathcal{F}_t\right]}{\tau-t}.\end{aligned}$$

Let $\tau\to t,$ applying Lemma 2, since g does not depend on $y,$ we rewrite $g\left(y,z,t\right)$ simply as $g\left(z,t\right),$ thus $g\left(z_1+z_2,t\right)\le g_1\left(z_1,t\right)+g_2\left(z_2,t\right),\ t\ge 0.$ The proof of Case 1 is complete.

Case 2, g depends on y. The proof is similar to the proof of Theorem 2.1 in Coquet *et al.* [14]. For each $\varepsilon>0$ and $\left(y_1,z_1\right),\left(y_2,z_2\right)\in\mathbb{R}\times\mathbb{R}^d,$ define the stopping time

$$\begin{aligned}\tau_\varepsilon &= \tau_\varepsilon\left(y_1,z_1;y_2,z_2\right)\\ &= \inf\left\{t\ge 0; g_1\left(y_1,z_1,t\right)+g_2\left(y_2,z_2,t\right)\right.\\ &\quad \left.\le g\left(y_1+y_2,z_1+z_2,t\right)-\varepsilon\right\}\wedge T.\end{aligned}$$

Obviously, if for each $\left(y_1,z_1\right),\left(y_2,z_2\right)\in\mathbb{R}\times\mathbb{R}^d$, $P\left(\tau_\varepsilon\left(y_1,z_1;y_2,z_2\right)<T\right)=0$, for all $\varepsilon,$ then the proof is done. If it is not the case, then there exist $\varepsilon>0$ and

$$\left(y_1,z_1\right),\left(y_2,z_2\right)\in\mathbb{R}\times\mathbb{R}^d,$$

such that

$$P\left(\tau_\varepsilon\left(y_1,z_1;y_2,z_2\right)<T\right)>0.$$

Fix $\varepsilon, y_i, z_i, (i=1,2),$ and consider the following (forward) SDEs defined on the interval $\left[\tau_\varepsilon,T\right]$

$$\begin{cases}\mathrm{d}Y^i\left(t\right)=-g_i\left(Y^i\left(t\right),z_i,t\right)\mathrm{d}t+z_i\mathrm{d}W_t,\\ Y^i\left(\tau_\varepsilon\right)=y_i,\ t\ge\tau_\varepsilon,\ i=1,2\end{cases}$$

and

$$\begin{cases} \mathrm{d}Y^3(t) = -g\left(Y^3(t), z_1+z_2, t\right)\mathrm{d}t + \left(z_1+z_2\right)\mathrm{d}W_t, \\ Y^3(\tau_\varepsilon) = y_1 + y_2, \quad t \ge \tau_\varepsilon. \end{cases}$$

Obviously, the above equations admit a unique solution Y^i which is progressively measurable with

$$E\left[\sup_{0\le t\le T}\left|Y^i(t)\right|^2\right] < \infty.$$

Define the following stopping time

$$\tau_\delta := \inf\left\{t \ge \tau_\varepsilon; g_1\left(Y^1(t), z_1, t\right) + g_2\left(Y^2(t), z_2, t\right) \ge g\left(Y^3(t), z_1+z_2, t\right) - \frac{\varepsilon}{2}\right\} \wedge T.$$

It is clear that $\tau_\varepsilon \le \tau_\delta \le T$ and note that $\tau_\delta = T$ whenever $\tau_\varepsilon = T$, thus, $\{\tau_\varepsilon < \tau_\delta\} = \{\tau_\varepsilon < T\}$. Hence $P(\tau_\varepsilon < \tau_\delta) > 0$.

Moreover, we can prove

$$Y^1(\tau_\delta) + Y^2(\tau_\delta) > Y^3(\tau_\delta), \text{ on } \{\tau_\varepsilon < \tau_\delta\}.$$

In fact, setting

$$\hat{Y}(t) = Y^3(t) - Y^1(t) - Y^2(t),$$

then

$$\mathrm{d}\hat{Y}(t) = \left[-g\left(Y^3(t), z_1+z_2, t\right) + g_1\left(Y^1(t), z_1, t\right) + g_2\left(Y^2(t), z_2, t\right)\right]\mathrm{d}t.$$

Thus $\dfrac{\mathrm{d}\hat{Y}(t)}{\mathrm{d}t} \le -\dfrac{\varepsilon}{2}, \ t \in [\tau_\varepsilon, \tau_\delta), \ \hat{Y}(\tau_\varepsilon) = 0.$

It follows that on $[\tau_\varepsilon, \tau_\delta)$, $\hat{Y}(\tau_\delta) \le -\dfrac{\varepsilon}{2}(\tau_\delta - \tau_\varepsilon) < 0.$

This implies

$$P\left(Y^3(\tau_\delta) < Y^1(\tau_\delta) + Y^2(\tau_\delta)\right) \ge P(\tau_\varepsilon < \tau_\delta) > 0. \quad (5)$$

By the definition of Y^1, Y^2 and Y^3, the pair processes $\left(Y^1(t), z_1\right), \left(Y^2(t), z_2\right)$ and $\left(Y^3(t), z_1+z_2\right)$ are the solutions of the following BSDEs with terminal values $Y^1(T)$, $Y^2(T)$ and $Y^3(T)$,

$$y_t = Y^i(T) + \int_t^T g_i(y_s, z_i, s)\mathrm{d}s - \int_t^T z_i \mathrm{d}W_s, \quad i = 1, 2$$

and

$$y_t = Y^3(T) + \int_t^T g(y_s, z_1+z_2, s)\mathrm{d}s - \int_t^T (z_1+z_2)\mathrm{d}W_s.$$

Hence,

$$\mathcal{E}_{g_1}\left[Y^1(\tau_\delta)\middle|\mathcal{F}_{\tau_\varepsilon}\right] = \mathcal{E}_{g_1}\left[Y^1(T)\middle|\mathcal{F}_{\tau_\varepsilon}\right] = y_1,$$

$$\mathcal{E}_{g_2}\left[Y^2(\tau_\delta)\middle|\mathcal{F}_{\tau_\varepsilon}\right] = \mathcal{E}_{g_2}\left[Y^2(T)\middle|\mathcal{F}_{\tau_\varepsilon}\right] = y_2$$

and

$$\mathcal{E}_{g}\left[Y^3(\tau_\delta)\middle|\mathcal{F}_{\tau_\varepsilon}\right] = \mathcal{E}_{g}\left[Y^3(T)\middle|\mathcal{F}_{\tau_\varepsilon}\right] = y_1 + y_2.$$

Applying the strict comparison theorem of BSDE and inequality (5), by the assumptions of this Theorem, we have

$$\begin{aligned} y_1 + y_2 &= \mathcal{E}_g\left[Y^3(\tau_\delta)\right] < \mathcal{E}_g\left[Y^1(\tau_\delta) + Y^2(\tau_\delta)\right] \\ &\le \mathcal{E}_{g_1}\left[Y^1(\tau_\delta)\right] + \mathcal{E}_{g_2}\left[Y^2(\tau_\delta)\right] = y_1 + y_2. \end{aligned}$$

This induces a contradiction. The proof is complete.

Lemma 3 *Suppose that g satisfies* (*H*1), (*H*2) *and* (*H*3). *For any constant* $c \ne 0$, *let*

$$\bar{g}(y, z, t) = cg\left(\frac{1}{c}y, \frac{1}{c}z, t\right).$$

Then for any $\xi \in L^2(\Omega, \mathcal{F}, P)$, $\mathcal{E}_{\bar{g}}[c\xi] = c\mathcal{E}_g[\xi]$.

Proof: Letting $\bar{y}_t = \mathcal{E}_{\bar{g}}\left[c\xi\middle|\mathcal{F}_t\right]$, then $\bar{y}_t$ is the solution of BSDE

$$\bar{y}_t = c\xi + \int_t^T \bar{g}(\bar{y}_s, \bar{z}_s, s)\mathrm{d}s - \int_t^T \bar{z}_s \mathrm{d}W_s.$$

Since

$$\bar{g}(y, z, t) = cg\left(\frac{1}{c}y, \frac{1}{c}z, t\right),$$

the above BSDE can be rewritten as

$$\bar{y}_t = c\xi + \int_t^T cg\left(\frac{1}{c}\bar{y}_s, \frac{1}{c}\bar{z}_s, s\right)\mathrm{d}s - \int_t^T \bar{z}_s \mathrm{d}W_s. \quad (6)$$

Let $y_t = \mathcal{E}_g\left[\xi\middle|\mathcal{F}_t\right]$, then cy_t satisfies

$$cy_t = c\xi + \int_t^T cg(y_s, z_s, s)\mathrm{d}s - \int_t^T cz_s \mathrm{d}W_s. \quad (7)$$

Comparing with BSDE (6) and BSDE (7), by the uniqueness of the solution of BSDE, we have

$$(cy_t, cz_t) = (\bar{y}_t, \bar{z}_t).$$

Let $t = 0$, then $cy_0 = \bar{y}_0$. The conclusion of the Lemma now follows by the definition of *g*-expectation. This concludes the proof.

Applying Theorem 1 and Lemma 3, immediately, we obtain several relations between *g*-expectation $\mathcal{E}_g[\cdot]$ and g. These are given in the following Corollaries.

Corollary 1 *The g-expectation* $\mathcal{E}_g[\cdot]$ *is the classical mathematical expectation if and only if g does not depend on* y *and is linear in* z.

Proof: Applying Theorem 1, $\mathcal{E}_g[\cdot]$ is linear if and only if $g(y, z, t)$ is linear in (y, z). By assumption (H3), that is $g(y, 0, t) = 0$ for all (y, t). Thus g does not depend on y. The proof is complete.

Corollary 2 *The* g *-expectation* $\mathcal{E}_g[\cdot]$ *is a convex risk measure if and only if* g *does not depend on* y *and is convex in* z.

Proof: Obviously, g -expectation $\mathcal{E}_g[\cdot]$ is convex risk measure if and only if for any $\lambda \in (0,1)$

$$\mathcal{E}_g\left[\lambda\xi+(1-\lambda)\eta\right]\le\lambda\mathcal{E}_g[\xi]+(1-\lambda)\mathcal{E}_g[\eta],\quad \forall\,\xi,\eta\in L^2(\Omega,\mathcal{F},P). \tag{8}$$

For a fixed $\lambda\in(0,1)$, let

$$g_1(y,z,t)=\lambda g\left(\frac{1}{\lambda}y,\frac{1}{\lambda}z,t\right),$$

$$g_2(y,z,t)=(1-\lambda)g\left(\frac{1}{1-\lambda}y,\frac{1}{1-\lambda}z,t\right).$$

Applying Lemma 3,

$$\mathcal{E}_{g_1}[\lambda\xi]=\lambda\mathcal{E}_g[\xi],\ \mathcal{E}_{g_2}\left[(1-\lambda)\xi\right]=(1-\lambda)\mathcal{E}_g[\xi].$$

Inequality (8) becomes

$$\mathcal{E}_g\left[\lambda\xi+(1-\lambda)\eta\right]\le\mathcal{E}_{g_1}[\lambda\xi]+\mathcal{E}_{g_2}\left[(1-\lambda)\eta\right],\quad \forall\,\xi,\eta\in L^2(\Omega,\mathcal{F},P).$$

Applying Theorem 1, for any

$$(y_i,z_i,t)\in\mathbb{R}\times\mathbb{R}^d\times[0,T],\ i=1,2,$$

$$\begin{aligned}&g\left(\lambda y_1+(1-\lambda)y_2,\lambda z_1+(1-\lambda)z_2,t\right)\\&\le g_1(\lambda y_1,\lambda z_1,t)+g_2\left((1-\lambda)y_2+(1-\lambda)z_2,t\right)\\&=\lambda g(y_1,z_1,t)+(1-\lambda)g(y_2,z_2,t)\end{aligned}$$

which then implies that g is convex. By the explanation of Remark for Lemma 4.5 in Briand *et al.* [9], the convexity of g and the assumption (H3) imply that g does not depend on y. The proof is complete.

The function g is positively homogeneous in z if for any $a\ge 0$, $g(\cdot,az,\cdot)=ag(\cdot,z,\cdot)$.

Corollary 3 *The g-expectation $\mathcal{E}_g[\cdot]$ is a coherent risk measure if and only if g does not depend on y and it is convex and positively homogenous in z. In particular, if $d=1$, g is of the form*

$$g(z,t)=a_t|z|+b_tz\quad \textit{with}\quad a\ge 0.$$

Proof: By Corollary 2, the g-expectation $\mathcal{E}_g[\cdot]$ is a convex risk measure if and only if g does not depend on y and is convex in z. Applying Theorem 1 and Lemma 3 again, it is easy to check that g-expectation $\mathcal{E}_g[\cdot]$ is positively homogeneous if and only if g is positively homogeneous (that is for all $a>0$ and ξ, $\mathcal{E}_g[a\xi]=a\mathcal{E}_g[\xi]$ if and only if for any $a\ge 0$, $g(\cdot,az,\cdot)=ag(\cdot,z,\cdot)$.

In particular, if $d=1$, notice the fact that g is convex and positively homogeneous on $\mathbb{R}$, and that g does not depend on y. We write it as $g(z,t)$ then

$$\begin{aligned}g(z,t)&=g(z,t)I_{[z\ge0]}+g(z,t)I_{[z\le0]}\\&=g(1,t)zI_{[z\ge0]}+g(-1,t)(-z)I_{[z\le0]}.\end{aligned}\tag{9}$$

Note that $zI_{[z\ge0]}=z^+$, $(-z)I_{[z\le0]}=z^-$, but

$$z^+=\frac{|z|+z}{2},\ z^-=\frac{|z|-z}{2}.$$

Thus from (9)

$$g(z,t)=\frac{g(1,t)+g(-1,t)}{2}|z|+\frac{g(1,t)-g(-1,t)}{2}z.$$

Defining

$$a_t:=\frac{g(1,t)+g(-1,t)}{2},\ b_t:=\frac{g(1,t)-g(-1,t)}{2}.$$

Obviously $a\ge 0$, since the convexity of g yields

$$\frac{g(1,t)+g(-1,t)}{2}\ge g(0,t)=0.$$

The proof is complete.

Remark 2 *Corollaries 2 and 3 give us an intuitive explanation for the distinction between coherent and convex risk measures. In the framework of g-expectations, convex risk measures are generated by convex functions, while coherent measures are generated only by convex and positively homogenous functions. In particular, if d = 1, it is generated only by the family $g(z,t)=a_t|z|+b_tz$ with $a\ge 0$. Thus the family of coherent risk measures is much smaller than the family of convex risk measures.*

Jensen's inequality for mathematical inequality is important in probability theory. Chen *et al.* [15] studied Jensen's inequality for g-expectation.

We say that g-expectation satisfies Jensen's inequality if for any convex function $\varphi:\mathbb{R}\to\mathbb{R}$, then

$$\varphi\left(\mathcal{E}_g[\xi]\right)\le\mathcal{E}_g\left[\varphi(\xi)\right],\quad \text{whenever }\ \xi,\varphi(\xi)\in L^2(\Omega,\mathcal{F},P).\tag{10}$$

Lemma 4 [*Chen et al.* [15] *Theorem* 3.1] *Let g be a convex function and satisfy $(H1)$, $(H2)$ and $(H3)$. Then*

1) Jensen's inequality (10) holds for g-expectations if and only if g does not depend on y and is positively homogeneous in z;

2) If $d=1$, the necessary and sufficient condition for Jensen's inequality (10) to hold is that there exist two adapted processes $a\ge 0$ and b such that

$$g(z,t)=a_t|z|+b_tz.$$

Now we can easily obtain our main results. Theorem 2 below shows the relation between static risk measures and dynamic risk measures.

Theorem 2 *If g-expectation $\mathcal{E}_g[\cdot]$ is a static convex (coherent) risk measure, then the corresponding conditional g-expectation $\mathcal{E}_g\left[\cdot\middle|\mathcal{F}_t\right]$ is dynamic convex (coherent) risk measure for each $t\in(0,T)$.*

Proof: This follows directly direct from Theorem 1.

Theorem 3 below shows that in the family of convex risk measure, only coherent risk measure satisfies Jensen's inequality.

Theorem 3 *Suppose that* $\mathcal{E}_g[\cdot]$ *is a convex risk measure. Then* $\mathcal{E}_g[\cdot]$ *is a coherent risk measure if and only if* $\mathcal{E}_g[\cdot]$ *satisfies Jensen's inequality.*

Proof: If $\mathcal{E}_g[\cdot]$ is a convex risk measure, then by Corollary 2, g is convex. Applying Lemma 4, $\mathcal{E}_g[\cdot]$ satisfies Jensen's inequality if and only if g is positively homogenous. By Corollary 2, the corresponding $\mathcal{E}_g[\cdot]$ is coherent risk measure. The proof is complete.

Theorem 4 and Counterexample 1 below give the relation between risk measures and Choquet expectation.

Theorem 4 *If* $\mathcal{E}_g[\cdot]$ *is a coherent risk measure, then* $\mathcal{E}_g[\cdot]$ *is bounded by the corresponding Choquet expectation, that is* $\mathcal{E}_g[\xi]\le \mathcal{C}_V(\xi),\ \xi\in L^2(\Omega,\mathcal{F},P)$ *where* $V(A)=\mathcal{E}_g[I_A]$. *If* $\mathcal{E}_g[\cdot]$ *is a convex risk measure then inequality above fails in general. By construction there exists a convex risk measure and random variables* ξ_1 *and* ξ_2 *such that*

$$\mathcal{E}_g[\xi_1]\le \mathcal{C}_V(\xi_1) \text{ and } \mathcal{E}_g[\xi_2]> \mathcal{C}_V(\xi_2)$$

The prove this theorem uses the following lemma.

Lemma 5 *Suppose that* g *does not depend on* y. *Suppose that the* g*-expectation* $\mathcal{E}_g[\cdot]$ *satisfies* (1) $\mathcal{E}_g[I_A+I_B]\le \mathcal{E}_g[I_A]+\mathcal{E}_g[I_B],\ \forall A,B\in\mathcal{F}$ (2) *For any positive constant* $a<1$,

$$\mathcal{E}_g[a\xi]\le a\mathcal{E}_g[\xi],\ \xi\in L^2(\Omega,\mathcal{F},P).$$

Then for any $\xi\in L^2(\Omega,\mathcal{F},P)$ the g-expectation $\mathcal{E}_g[\cdot]$ is bounded by the corresponding Choquet expectation, that is

$$\mathcal{E}_g[\xi]\le \int_{-\infty}^{0}\left[\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]-1\right]\mathrm{d}x+\int_0^{\infty}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x. \quad (11)$$

Proof: The proof is done in three steps.

Step 1. We show that if $\xi\ge 0$ is bounded by $N>0$, then inequality (11) holds.

In fact, for the fixed N, denote $\xi^{(n)}$ by

$$\xi^{(n)}:=\sum_{i=0}^{2^n-1}\frac{iN}{2^n}I_{\left\{\frac{iN}{2^n}\le\xi<\frac{(i+1)N}{2^n}\right\}}.$$

Then $\xi^{(n)}\to\xi, n\to\infty$ in $L^2(\Omega,\mathcal{F},P)$.
Moreover, $\xi^{(n)}$ can be rewritten as

$$\xi^{(n)}=\sum_{i=1}^{2^n}\frac{N}{2^n}I_{\left\{\xi\ge\frac{iN}{2^n}\right\}}.$$

But by the assumptions (1) and (2) in this lemma, we have

$$\begin{aligned}\mathcal{E}_g\left[\xi^{(n)}\right]&=\mathcal{E}_g\left[\sum_{i=1}^{2^n}\frac{N}{2^n}I_{\left\{\xi\ge\frac{iN}{2^n}\right\}}\right]\\&\le\sum_{i=1}^{2^n}\frac{N}{2^n}\mathcal{E}_g\left[I_{\left\{\xi\ge\frac{iN}{2^n}\right\}}\right].\end{aligned}\quad (12)$$

Note that

$$\sum_{i=1}^{2^n}\frac{N}{2^n}\mathcal{E}_g\left[I_{\left\{\xi\ge\frac{iN}{2^n}\right\}}\right]\to\int_0^{\infty}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x,\ n\to\infty.$$

and $\mathcal{E}_g\left[\xi^{(n)}\right]\to\mathcal{E}_g[\xi],\ n\to\infty.$

Thus, taking limits on both sides of inequality (12), it follows that $\mathcal{E}_g[\xi]\le\int_0^{\infty}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x.$ The proof of Step 1 is complete.

Step 2. We show that if ξ is bounded by $N>0$, that is $|\xi|\le N$, then inequality (11) holds.

Let $\xi^N=\xi+N,$ then $0\le\xi^N\le 2N.$ Applying Step 1,

$$\mathcal{E}_g[\xi+N]\le\int_0^{\infty}\mathcal{E}_g\left[I_{\{\xi+N\ge x\}}\right]\mathrm{d}x. \quad (13)$$

But by Lemma 1(v), $\mathcal{E}_g[\xi+N]=\mathcal{E}_g[\xi]+N.$ On the other hand,

$$\begin{aligned}\int_0^{\infty}\mathcal{E}_g\left[I_{\{\xi+N\ge x\}}\right]\mathrm{d}x&=\int_0^{2N}\mathcal{E}_g\left[I_{\{\xi\ge x-N\}}\right]\mathrm{d}x\\&=\int_{-N}^{N}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x\\&=\int_{-N}^{0}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x\\&\quad+\int_0^{N}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x.\end{aligned}$$

Thus by (13)

$$\mathcal{E}_g[\xi]+N\le\int_{-N}^{0}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x+\int_0^{N}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x.$$

Therefore

$$\mathcal{E}_g[\xi]\le\int_{-N}^{0}\left[\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]-1\right]\mathrm{d}x+\int_0^{N}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x.$$

Step 3. For any $\xi\in L^2(\Omega,\mathcal{F},P).$ let $\xi^N=\xi I_{[|\xi|\le N]}$, then $\left|\xi^N\right|\le N$. By Step 2,

$$\mathcal{E}_g\left[\xi^N\right]\le\int_{-N}^{0}\left[\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]-1\right]\mathrm{d}x+\int_0^{N}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x.$$

Letting $N\to\infty$, it follows that

$$\mathcal{E}_g[\xi]\le\int_{-\infty}^{0}\left[\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]-1\right]\mathrm{d}x+\int_0^{\infty}\mathcal{E}_g\left[I_{\{\xi\ge x\}}\right]\mathrm{d}x.$$

The proof is complete.

Proof of Theorem 4: If the g-expectation $\mathcal{E}_g[\cdot]$ is a coherent risk measure, then it is easy to check that the g-expectation $\mathcal{E}_g[\cdot]$ satisfies the conditions of Lemma 5.

Let $V(A)=\mathcal{E}_g[I_A]\ \forall A\in\mathcal{F}$. By Lemma 5 and the definition of Choquet expectation, we have $\mathcal{E}_g[\xi]\le\mathcal{C}_V[\xi]$. The first part of this theorem is complete.

Counterexample 1 shows that this property of coherent risk measures fails in general for more general convex risk measures. This completes the proof of Theorem 4.

Counterexample 1 *Suppose that* $\{W_t\}$ *is* 1*-dimensional Brownian motion* (*i.e. d* = 1). *Let* $g(z)=(z-1)^+$ *where* $x^+=\max\{x,0\}$. *Then* $\mathcal{E}_g[\cdot]$ *is a convex risk* *measure. Let* $\xi_1 = \frac{1}{2} I_{\{W_T \geq 1\}}$ *and* $\xi_2 = 2I_{\{W_T \geq 1\}}$. *Then* $\mathcal{E}_g[\xi_1] \leq \mathcal{C}_V(\xi_1)$ *However* $\mathcal{E}_g[\xi_2] > \mathcal{C}_V(\xi_2)$. *Here the capacity* V *in the Choquet expectation* $\mathcal{C}_V(\cdot)$ *is given by* $V(A) = \mathcal{E}_g[I_A]$.

Proof of the Inequality in Counterexample 1: The convex function $g(z) = (z-1)^+$ satisfies (H1), (H2) and(H3). Thus, by Corollary 2, g-expectation $\mathcal{E}_g[\cdot]$ is a convex risk measure. This together with the property of Choquet expectation in Remark 1 implies

$$\mathcal{E}_g[\xi_1] = \mathcal{E}_g\left[\frac{1}{2} I_{\{W_T \geq 1\}}\right] \leq \frac{1}{2}\mathcal{E}_g\left[I_{\{W_T \geq 1\}}\right] = \frac{1}{2}\mathcal{C}_V\left(I_{\{W_T \geq 1\}}\right) = \mathcal{C}_V\left(\frac{1}{2} I_{\{W_T \geq 1\}}\right) = \mathcal{C}_V(\xi_1).$$

Moreover, since $d = 1$ by Corollary 3, $\mathcal{E}_g[\cdot]$ is a convex risk measure rather than a coherent risk measure. We now prove that $\mathcal{E}_g[\xi_2] > \mathcal{C}_V(\xi_2)$ In fact, since

$$\mathcal{C}_V\left(2I_{\{W_T \geq 1\}}\right) = 2\mathcal{C}_V\left(I_{\{W_T \geq 1\}}\right) = 2\mathcal{E}_g\left[I_{\{W_T \geq 1\}}\right],$$

we only need to show

$$\mathcal{E}_g\left[2I_{\{W_T \geq 1\}}\right] > 2\mathcal{E}_g\left[I_{\{W_T \geq 1\}}\right].$$

Let (y, z) be the solution of the BSDE

$$y_t = 2I_{\{W_T \geq 1\}} + \int_t^T (z_s - 1)^+ \,\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s. \quad (14)$$

First we prove that

$$(L \times P)\left((t, \omega) \in [0, T) \times \Omega : z_t(\omega) > 1\right) > 0, \quad (15)$$

where L is Lebesgue measure on $[0, T]$.

If it is not true, then $z_t \leq 1$ a.e. $t \in [0, T]$ and BSDE (14) becomes

$$y_t = 2I_{\{W_T \geq 1\}} - \int_t^T z_s \mathrm{d}W_s.$$

Thus

$$y_t = 2E\left[I_{\{W_T \geq 1\}} \middle| \mathcal{F}_t\right] = 2E\left[I_{\{W_T - W_t \geq 1 - W_t\}} \middle| \mathcal{F}_t\right].$$

By the Markov property,

$$y_t = 2P\left(W_T - W_t \geq 1 - W_t \middle| \sigma(W_t)\right).$$

Recall that $W_T - W_t$ and W_t are independent and $W_T - W_t \sim N(0, T-t)$. Thus

$$y_t = 2\int_{1-x}^{\infty} \varphi(y) \mathrm{d}y \Big|_{x = W_t},$$

where $\varphi(x)$ is the density function of the normal distribution $N(0, T-t)$. Thus $z_t = D_t y_t = 2\varphi(1 - W_t)$, where D_t is the Malliavin derivative. Thus z_t can be greater than 1 whenever t is near 0 and W_t is near 0. Thus (15) holds, which contradicts the assumption $z_t \leq 1$ a.e. $t \in [0, T]$.

Secondly we prove that

$$\mathcal{E}_g\left[2I_{\{W_T \geq 1\}}\right] > 2\mathcal{E}_g\left[I_{\{W_T \geq 1\}}\right].$$

Let (Y_t, Z_t) be the solution of the BSDE

$$Y_t = 2I_{\{W_T \geq 1\}} + \int_t^T 2\left(\frac{Z_s}{2} - 1\right)^+ \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s. \quad (16)$$

Obviously,

$$\frac{Y_t}{2} = I_{\{W_T \geq 1\}} + \int_t^T \left(\frac{Z_s}{2} - 1\right)^+ \mathrm{d}s - \int_t^T \frac{Z_s}{2} \mathrm{d}W_s,$$

which means $\left(\frac{Y_t}{2}, \frac{Z_s}{2}\right)$ is the solution of BSDE

$$\overline{y}_t = I_{\{W_T \geq 1\}} + \int_t^T (\overline{z}_s - 1)^+ \mathrm{d}s - \int_t^T \overline{z}_s \mathrm{d}W_s.$$

But $\overline{y}_t = \mathcal{E}_g\left[I_{\{W_T \geq 1\}} \middle| \mathcal{F}_t\right]$. Thus by the uniqueness of the solution of BSDE, $\frac{Y_t}{2} = \mathcal{E}_g\left[I_{\{W_T \geq 1\}} \middle| \mathcal{F}_t\right]$. On the other hand, let (y_t, z_t) be the solution of the BSDE

$$y_t = 2I_{\{W_T \geq 1\}} + \int_t^T (z_s - 1)^+ \mathrm{d}s - \int_t^T z_s \mathrm{d}W_s. \quad (17)$$

Comparing BSDE(17) with BSDE (16), notice (15) and the fact

$$(z-1)^+ \geq 2\left(\frac{z}{2} - 1\right)^+ \text{ and } \frac{\mathcal{E}_g[\xi]}{\mathcal{E}_g[\xi] < \mathcal{C}_V(\xi)}$$

whenever z >1. By the strict comparison theorem of BSDE, we have $y_t > Y_t$, $t \in [0, T)$.

Setting $t = 0$, thus

$$\mathcal{E}_g\left[2I_{\{W_T \geq 1\}}\right] > 2\mathcal{E}_g\left[I_{\{W_T \geq 1\}}\right] = \mathcal{C}_V\left(2I_{\{W_T \geq 1\}}\right).$$

The proof is complete.

Remark 3 *In mathematical finance, coherent and convex risk measures and Choquet expectation are used in the pricing of contingent claim. Theorem* 4 *shows that coherent pricing is always less than Choquet pricing, while Counterexample* 1 *demonstrates that pricing by a convex risk measure no longer has this property. In fact the convex risk price may be greater than or less than the Choquet expectation.*

4. Summary

Coherent risk measures are a generalization of mathematical expectations, while convex risk measures are a generalization of coherent risk measures. In the framework of g-expectation, the summary of our results is given in **Table 1**. In that Table, the Choquet expectation is $V(A) := \mathcal{E}_g[I_A]$.

Counterexample 1 shows that convex risk may be $\geq$ or $\leq$ Choquet expectation. Only in the case of coherent

Table 1. Relations among coherent and convex risk measures $\mathcal{E}_g[\xi]$, choquet expectation and Jensen's inequality.

Risk Measures	Relation to Choquet Expectation	Jensen inequality
	g is linear	
math. expectation	$\mathcal{E}_g[\xi]=\mathcal{C}_V(\xi)$	true
	g is convex and positively homogeneous	
coherent	$\mathcal{E}_g[\xi]<\mathcal{C}_V(\xi)$	true (*)
	g is convex	
convex	Neither $\leq$ nor $\geq$	not true except (*)

risk there is an inequality relation with Choquet expectation.

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