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Semimartingale Property and Its Connections to Arbitrage*

Sallieu Kabay Samura, Junjun Mao, Dengbao Yao
School of Mathematical Science, Anhui University, Hefei, China
Email: ssallieu@yahoo.com, maojunjun@ahu.edu.cn, yaodengbao@126.com

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ABSTRACT

In this paper, we prove the celebrated Bichteler-Dellaccherie Theorem which states that the class of stochastic processes X allowing for a useful integration theory consists precisely of those processes which can be written in the form $X = X_0 + M + A$, where $M_0 = A_0 = 0$, M is a local martingale, and A is of finite variation process. We obtain this decomposition rather direct form an elementary discrete-time Doob-Meyer decomposition. By moving to convex combination we obtain a direct continuous time decomposition, which then yield the desired decomposition. We also obtain a characterization of semi-martingales in terms of a variant no free lunch with vanishing risk.

Keywords: Bichteler-Dellaccherie Theorem; Doob-Meyer Decomposition; Semi-Martingales; Arbitrage; Komlos Lemma

1. Introduction

In this paper, $\left(\Omega, \mathbb{F}, \left(\mathbb{F}_t\right)_{t\in\mathbb{R}_+}, P\right)$ is assumed to be a filtered probability space where $\left(\mathbb{F}_t\right)_{t\in\mathbb{R}_+}$ is a filtration satisfying $\mathbb{F}_t \subseteq \mathbb{F}$ for all $t \in \mathbb{R}_+$, the usual condition of right continuity and completeness. The random movement of $d \in \mathbb{N}$ risky assets in the market is modeled via cadlag, nonnegative stochastic processes X_i, where $i \in \{1, \cdots, d\}$. We assume that all wealth processes are discounted by another special asset which is considered a baseline. In the market described above, economic agents can trade in order to reallocate their wealth.

Consider a simple predictable process

$$\phi = \sum_{j=0}^{n-1} \eta_j \mathbb{I}_{|\tau_{j-1}, \tau_j|}.$$

where $\tau_0 = 0$, and for all $j \in \{1, \cdots, n\}$, τ_j is a finite stopping time and $\eta_j = \left(\phi_j^i\right)_{i=1,\cdots,d}$ is $\mathbb{F}_{\tau_{j-1}}$-measurable. Each τ_{j-1}, $j \in \{1, \cdots, n\}$, is an instance when some give economic agent may trade in the market, then, η_j^i is the number of unit from the ith risky assets that the agent will hold in the trading interval $\left[\tau_{j-1}, \tau_j\right]$. This form of trading is called simple, as it comprises of finite number of buy-and-hold strategies, in contrast to continuous trading where one is able to change the position of the assets in a continuous fashion. The last form of trading is only theoretical value, since it cannot be implemented in reality, even if one ignores market frictions.

Starting from initial capital $x \in \mathbb{R}_+$ and following the strategy described by the simple predictable process $\phi = \sum_{j=0}^{n-1} \eta_j \mathbb{I}_{|\tau_{j-1}, \tau_j|}$, the agent's discounted process is given by

$$X^{x,\phi} = x + \sum_{j=0}^{n-1} \eta_{j+1} \left(X_{t\wedge\tau_{j+1}} - X_{t\wedge\tau_j}\right).$$

where $n \in \mathbb{N}$, $0 = \tau_0 \le \tau \le \cdots \le \tau_n$ are a.s. finite stoping times with respect to $\mathbb{F}_t$ and the ϕ_j are $\mathbb{F}_{\tau_j}$-measurable real random variables. Note that the trader is allowed to trade on an infinite time horizon, because we do not restrict to bounded stoping times for the re-allocation of the capital. Of course trading on a finite time horizon [0, T] is covered by switching to the process $\left(X_{t\wedge T}, \mathbb{F}_{t\wedge T}\right)$.

Theorem 1.1. [1,2] A real valued, cadlag, adapted process $X = \left(X_t\right)_{0\le t\le T}$ the following are equivalent:

1) X is a good integrator.

2) X may be decomposed as $X = M + A$, where $M = \left(M_t\right)_{0\le t\le T}$ is a local martingale and $A = \left(A_t\right)_{0\le t\le T}$ is an adapted process of finite variation.

Defination 1.1. [1,3] A real valued, cadlag, adapted

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process $X=\left(X_t\right)_{0\le t\le T}$ allows for A Free Lunch With Vanishing Risk for simple integrands if there is a sequence $\left(\phi^n\right)_{n=1}^{\infty}$ of simple integrands such that for $n\to\infty$,

$$\left(\phi^n\cdot X\right)_T^{+}\nrightarrow 0 \text{ in probability}.$$

and

$$\sup_{0\le t\le T}\left\|\left(\phi^n\cdot X\right)_t^{-}\right\|_{\infty}=\left\|\left(\phi^n\cdot X\right)^{-}\right\|\to 0$$

In contrast, X therefore admits No Free Lunch With Vanishing Risk (NFLVR) for simple integrands if for every sequence $\left(\phi^n\right)_{n=1}^{\infty}\in SI$ satisfying (VR) we have

(NFL) $\left(\phi^n\cdot X\right)_T\to 0$ in probability.

A free lunch with vanishing risk (FLVR) for simple integrands indicates that S allows for a sequence of trading schemes $\left(\phi^n\cdot X\right)_{n=1}^{\infty}$, each ϕ^n involving only finitely many rebalancing of the portfolio, such that the losses tend to Zero in the sense that of (VR) ,while the terminal gains (FL) remain substantial as n goes to infinity. It is important to note that the condition (VR) of vanishing risk pertains the maximal losses of the trading strategy ϕ^n during the entire interval [0,*T*]: if the left hand side of (VR) equals ε_n this implies that, with probability one, the strategy ϕ^n never, *i.e.* for not $t\in\left[0,T\right]$, cause an accumulated loss of more than ε_n.

Resently, it has been argued that existence of an Equivalent Martingale Measure(EMM) is not necessary for viability of the market; to see this effect, see [4-6]. In [7], the concept of strictly positive supermartingale deflator which is weaker than the existence of an EMM, that allows for consistent theory to be developed.In this paper, we investigate the relation between the no free lunch with vanishing risk property for simple integands and the semimartingale property.

Theorem 1.2. [1,8] Let $\left(X_t\right)_{0\le t\le T}$ be a real-valued, cadlag, locally bounded process based on and adepted to a filtered probability space $\left(\Omega,\mathbb{F},\left(\mathbb{F}\right)_{0\le t\le T},\mathbb{P}\right)$. If S satisfies the condition of no free lunch with vanishing risk (NFLVR) for simple integrands then S is a semimartingale.

Theorem 1.3. For a locally bounded, adopted, cadlag process *X* the following are equivalent

1) *X* satisfies NFLVR + LI(little Investment)

2) *X* is a classical semimartingale.

Theorem 1.4. For an adapted cadlag process *X* the following are equivalent.

1) For all sequences $\left(\phi^n\right)_{n\ge 1}$ of simple predictable processes,

a) $\lim\left\|\phi^n\right\|_{\infty}=0$

b) $\lim_{n\to\infty}\sup_{0\le t\le T}\left(\phi^n\cdot X\right)_t^{-}=0$

together imply $\left(\phi^n\cdot X\right)_t^{-}\to 0$ in probability.

2) *X* is a classical semimartingale.

Proposition 1.5. Let $X=\left(X_t\right)_{0\le t\le 1}$ be cadlag and adapted, with X_0 and such that $\left\|X\right\|\le 1$ and *X* satisfies NFLVR + LI For all $\epsilon>0$ there is $C>0$ and a sequence of stopping times $\left(\tau_n\right)_{n\ge 1}$ such that, for all *n*

1) τ_n takes values in $D_n\bigcup\{\infty\}$.

2) $P\left(\tau_n<\infty\right)<\varepsilon$.

3) The stopped processes A^{n,τ_n} and M^{n,τ_n} satisfy, for all *n*, $\left\|M_1^{n,\tau_n}\right\|_{L^2}^2\le C$ and

$$TV\left(A^{n,\tau_n}\right)=\sum_{j=1}^{2^n}\left|A_{j2^{-n}}^{n,\tau_n}-A_{(j-1)2^{-n}}^{n,\tau_n}\right|\le C$$

Lemma 1.6. Under the assumptions as in the proposition above with

$$Q^n=\sum_{j=1}^{2^n}\left(X_{j2^{-n}}-X_{(j-1)2^{-n}}\right)^2,$$

the sequence $\left(Q^n\right)_{n\ge 1}$ is bounded in probability.

Proof. For all *n*, let

$$\phi^n=-\sum\nolimits_{j=1}^{2^n}X_{(j-1)2^{-n}}I\left[\left(j-1\right)2^{-n},j2^n\right]$$

a simple predictable process, then $\left\|\phi^n\right\|_{\infty}\le 1$ since $\left\|X\right\|_{\infty}\le 1$

$$\left(\phi^n\cdot X\right)_t=-\sum\nolimits_{j=1}^{2^n}X_{t\wedge(j-1)2^n}\left(X_{t\wedge j2^{-n}}-X_{t\wedge(j-1)2^{-n}}\right)$$

$$=\frac{1}{2}\sum_{j=1}^{2^n}\left(X_{t\wedge j2^{-n}}-X_{t\wedge(j-1)2^{-n}}\right)^2+\frac{1}{2}\left(X_0^2-X_t^2\right)\ge-\frac{1}{2}.$$

$$t=1$$

$$\left(\phi^n\cdot X\right)_1=\frac{1}{2}Q^n+\frac{1}{2}\left(X_0^1-X_t^2\right)$$

since X satisfies NFLVR + LI, $\left(\left(\phi^n\cdot X\right)_1,n\ge 1\right)$ is bounded in $L^0\left(P\right)$. □

For $c>0$ define a sequence of stopping times

$$\sigma_n\left(c\right)=\inf\left\{\frac{k}{2^n}:\sum_{j=1}^{k}\left(X_{j2^{-n}}-X_{(j-1)2^{-n}}\right)^2\ge c-4\right\}.$$

Given $\varepsilon\ge 0$ there is c_1 such that

$$P\left[\sigma_n\left(c_1\right)<\infty\right]<\frac{\varepsilon}{2}$$

Lemma 1.7. Under the same assumptions as in Proposition 1.5 the stopped martingales $M^{n,\sigma_n(c_1)}$ satisfy $\left\|M_1^{n,\sigma_n(c_1)}\right\|_{L^2}^2\le C_1$.

Proof. For $n\ge 1$ and $k=1,\cdots,2^n$, since the $A^n s$ are predictable and the $M^n s$ are martingales,

$$E\left[\left(X_{k2^{-n}}^{\sigma_n(c_1)}-X_{(k-1)2^{-n}}^{\sigma_n(c_1)}\right)^2\right]$$
$$=E\left[\left(M_{k2^{-n}}^{\sigma_n(c_1)}-M_{(k-1)2^{-n}}^{\sigma_n(c_1)}\right)^2\right]+E\left[\left(A_{k2^{-n}}^{\sigma_n(c_1)}-A_{(k-1)2^{-n}}^{\sigma_n(c_1)}\right)^2\right]$$
$$\geq E\left[\left(M_{k2^{-n}}^{\sigma_n(c_1)}\right)^2-\left(M_{(k-1)2^{-n}}^{\sigma_n(c_1)}\right)^2\right]$$

we write $E\left[\left(M_1^{\sigma_n(c_1)}\right)^2\right]$ as a telescoping series and simplifying to get

$$E\left[\left(M_1^{\sigma_n(c_1)}\right)^2\right]$$
$$=\sum_{k2^{-n}\leq\sigma_n(c_1)}E\left[\left(X_{k2^{-n}}^{\sigma_n(c_1)}-X_{(k-1)2^{-n}}^{\sigma_n(c_1)}\right)^2\right]$$
$$+E\left[\left(X_{\sigma_n(c_1)}-X_{\sigma_n(c_1)2^{-n}}\right)^2\right]$$
$$\leq(c_1-4)+2^2=c_1$$

□

Lemma 1.8. Let

$$V^n=TV\left(A^{n,\sigma_n(c_1)}\right)=\sum_{i=1}^{2^n(\sigma_n(c_1)\wedge 1)}\left|A_{j2^{-n}}^n-A_{(j-1)2^{-n}}^n\right|.$$

Under the assumption of Proposition 1.5 the sequence $\left(V^n\right)_{n\geq1}$ is bounded in probability.

Proof. Assume for contradiction that $\left(v^n\right)_{n\geq1}$ is not bounded in probability. Then there is $\alpha>0$ such that for all k there is n_k such that $p\left[V^{n_k}\right]\geq k\geq\alpha$. For $n\geq1$ define

$$b_{j-1}^n=sign\left(A_{j2^{-n}}^{n,\sigma_n(c_1)}-A_{(j-1)2^{-n}}^{n,\sigma_n(c_1)}\right)\in\mathbb{F}_{(j-1)2^{-n}}$$

and

$$\phi^n(t)=\sum_{j=1}^{2^n}b_{j-1}^n1_{\left[(j-1)2^{-n},j2^{-n}\right]}(t).$$

Then $\left\|\phi^n\right\|_u\leq1$ and

$$\left(\phi^{n,\sigma_n(c_1)}\cdot X\right)_t$$
$$=\sum_{j\leq\left[t2^n\right]}b_{j2^{-n}}^n\left(X_{j2^{-n}}^{\sigma_n(c_1)}-X_{(j-1)2^{-n}}^{\sigma_n(c_1)}\right)$$
$$+b_{\left[t2^n\right]}\left(X_t^{\sigma_n(c_1)}-X_{t2^{-n}}^{\sigma_n(c_1)}\right)$$
$$\geq\left(\phi^n,\sigma_{n(c_1)}\cdot A^n\right)_{\left[t2^n\right]2^{-n}}+\left(\phi^n,\sigma_{n(c_1)}\cdot M^n\right)_{\left[t2^n\right]2^{-n}}-2$$

and at time $t=1$ we have

$$\left(\phi^{n,\sigma_n(c_1)}\cdot X\right)_1=V^n+\left(\phi^{n,\sigma_n(c_1)}\cdot M^n\right)_1.$$

But the second summand is bounded in L^2, so we conclude that $\left(\phi^{n,\sigma_n(c_1)}\cdot X\right)_1$ is not bounded in probability.

We defined a sequence of stopping times

$$\eta_n(c)=\inf\left\{\frac{j}{2^n}:\left|\left(\phi^{n,\sigma_n(c_1)}\cdot M^n\right)_{j2^{-n}}\geq c\right.\right\}.$$

Because

$$E\left[(\sup_{1\leq j\leq2^n}\left(\left(h^{n,\sigma_n(c_1)}\cdot M^n\right)_{j2^{-n}}\right)^2\right]\leq4c_1$$

by Doob's sub-martingale in-equality,(see [9,10]) $\left(\phi^{n,\sigma_n(c_1)}\cdot M^n\right)$ is bounded in probability. Therefore there is $c'>0$ such that $P\left[\eta_n(c')<\infty\right]\leq\alpha/2$. Note that $\phi^{n,\sigma_n(c_1)\wedge\eta_n(c')}\cdot X$ is uniformly bounded below by c'. We claim $\left(\phi^{n,\sigma_n(c_1)\wedge\eta_n(c')}\cdot X\right)_1$ is not bounded in probability. Indeed, for any n and any k,

$$\alpha\leq p\left[\left(\phi^{n,\sigma_n(c_1)\wedge\eta_n(c')}\cdot X\right)_1\geq k\right]$$
$$\leq p\left[\left(\phi^{n,\sigma_n(c_1)}\cdot X\right)_1\geq k,\eta_n(c')=\infty\right]+P\left[\eta_n(c')<\infty\right].$$

Since $P\left[\eta_n(c')<\infty\right]\leq\alpha/2$, the probability of the other event is at least $\alpha/2$. This gives the desired contradiction because it is now easy to construct a FLVR + LI.

Proof of Proposition 1.5: Defined a sequence of stopping times

$$\tau_n(c):=\inf\left\{\frac{k}{2^n}:\sum_{j=1}^k\left|A_{j2^{-n}}-A_{(j-1)2^{-n}}\right|\geq c\right\}.$$

By Lemma 1.8 there is c_2 such that $P\left[\tau_n(c_2)<\infty\right]<\varepsilon/2$. Take $C:=c_1\vee c_2$ and $\rho_n=\sigma_n(c_1)\wedge\tau_n(c_2)$.

Lemma 1.9. [11]. Let $f\cdot g:[0,1]\to\Re$ be measurable functions, where f is left continuous and takes finitely many values. Say $f=\sum_{k=1}^k f(X_k)1_{(X_{k-1},s_k]}$. Define

$$(f\cdot g)=\sum_{k=1}^k f(X_{k-1})\left(g(X_k)-g(X_{k-1})\right)$$
$$+f\left(X_{k(t)}\right)\left(g(t)-g\left(X_{k(t)}\right)\right)$$

where $k(t)$ is the biggest of the k such that X_k less than or equal to t. Then for all partition $0\leq t_0\leq\cdots\leq t_M\leq1$,

$$\sum_{i=1}^M\left|(f\cdot g)(t_i)-(f\cdot g)(t_{i-1})\right|$$
$$\leq2TV(f)\|g\|_\infty+\left(\sum_{i=1}^M\left|g(t_i)-g(t_{i-1})\right|\right)\|f\|_\infty.$$

Proposition 2.0. Let $X=(X_t)_{0\leq t\leq1}$ be cadlag and adopted, with $X_0=0$ and such that $\|X\|_u\leq1$ and X satisfies NFLVR + LI. For all $\varepsilon>0$ there is C and a $[0,1]\cup\{\infty\}$ valued stopping time α such that $p[\alpha<\infty]<\varepsilon$ and sequence $\left(M^n\right)_{n\geq1}$ and $\left(A^n\right)_{n\geq1}$ of continuous time cadlag processes such that for all n,

1) $A_0^n=M_0^n=0$
2) $X^\alpha=A^{n,\alpha}+M^{n,\alpha}$

3) $M^{n,\alpha}$ is a martingale with $\left\|M_1^{n,\alpha}\right\|_{L^2}^2 \le C$

4) $\sum_{j=1}^{2^n}\left\|A_{j2^{-n}}^{n,\alpha} - A_{(j-1)2^{-n}}\right\| \le C$

Proof. Let $\varepsilon \ge 0$ be given. Let C, M^n, A^n, and ρ_n be as in proposition 1.5. Extended M^n and A^n to all $t \in [0,1]$ by defining $M_t^n = E\left[M_1^n \middle| F_t\right]$ and $A^n = X_t - M_t$. Not that the extended A^n is no longer predictable, and currently we only have control of the total variation of A^{n,ρ_n} over D_n, *i.e.*

$$\sum_{j=1}^{2^n(\rho_n \Lambda 1)}\left|A_{j2^{-n}}^n - A_{(j-1)2^{-n}}\right| \le C.$$

Notice that, for $t \in \left[(j-1)2^{-n}, j2^{-n}\right]$,

$$\begin{aligned} A^n = X_t - M_t^n &= X_t - E\left[M_{j2^{-n}}^n \middle| F_t\right] \\ &= X_t - E\left[X_{j2^{-n}}^n - A_{j2^{-n}}^n \middle| F_t\right] \\ &= A_{j2^{-n}}^n - \left(E\left[X_{j2^{-n}}^n \middle| F_t\right] - X_t\right) \end{aligned}$$

From this and $\|X\|_u \le 1$ it follow that $\left\|A_t^n - A_{j2^{-n}}^n\right\|_\infty \le 2$, so $\left\|A^{n,\rho_n}\right\|_u \le C+2$. How do we fine the limit of the sequence of stopping times $(\rho_n)_{n\ge1}$? The trick is to define $R^n = 1_{[0,\rho_n \Lambda 1]}$, a simple predicator process, and note that stopping at ρ_n is like integrating R_n, *i.e.* $A^{n,\rho_n} = R^n \cdot A^n$ and $M^{n,\rho_n} = R^n \cdot M^n$. We have that

$$1 \ge E\left[R_1^n\right] = E\left[1_{\rho n} = \infty\right] = 1 - P\left[\rho_n \le \infty\right] \ge 1 - \varepsilon.$$

Apply Komlos' Lemma to obtain convex weights $\left(\mu_n^n, \cdots, \mu_{N_n}^n\right)$ such that

$$R^n = \sum_{i=n}^{\infty}\mu_i^n R_1^i \to R_1$$

a.s as $n \to \infty$ By the dominated convergence theorem, $E\left[R_1\right] \ge 1 - \varepsilon$. Observe that

$$R^n \cdot X = \sum_{i=n}^{\infty}\mu_i^n\left(R_1^i \cdot M^i\right) + \sum_{i=n}^{\infty}\mu_i^n\left(R_1^i \cdot A^i\right)$$

Define $\alpha_n = \inf\left\{t : R_t^n \le 1/2\right\}$. Each R^n is left continuous, decreasing process. In particular, $R_{\alpha_n} \ge 1/2 > 0$, so we can divide by this quantity. We claim that $P\left[\alpha_n < \infty\right] < \varepsilon$. In deed, on the event $\left[\alpha_n < \infty\right]$, $R_1^n \le R_{\alpha_n+}^n \le 1/2$ so

$$\begin{aligned} R_1^n \le \varepsilon \ge E\left[1 - R_1^n\right] &\ge E\left[\left(1 - R_{\alpha_n+}^n\right)1_{\alpha<\infty}\right] \\ &\ge 1/2\, P\left[\alpha_n < \infty\right]. \end{aligned}$$

Define new processes $T_t^n = 1_{[0,\alpha_n]}(t)\big/R_t^n$. Then $\left\|T^n\right\|_n \le 2$ and $T^n \cdot \left(R^n \cdot X\right) = X^{\alpha_n}$. Thus we define M^n and A^n by

$$\begin{aligned} X^{\alpha_n} &= T^n \cdot \left(\sum_{i=n}^{\infty}\mu_i^n\left(R^i \cdot M^i\right)\right) + T^n \cdot \left(\sum_{i=n}^{\infty}\mu_i^n\left(R^i \cdot A^i\right)\right) \\ &= M^n + A^n. \end{aligned}$$

The total variation of T^n over D_n is bounded by 3. By Lemma 1.9,

$$\begin{aligned} \sum_{j=1}^{2^n}\left|A_{j2^{-n}} - A_{(j-1)}^n\right| &\le 2TV_n\left(T^n\right)\left\|\sum_{i=n}^{\infty}\mu_i^n\left(R^i \cdot A^i\right)\right\|_\infty \\ &\quad + \left\|T^n\right\|_\infty TV_n\left(\sum_{i=n}^{\infty}\mu_i^n\left(R^i \cdot A^i\right)\right) \\ &\le 6(C+2) + 2C \end{aligned}$$

That $\left\|M_1^n\right\|_{L^2}^2 \le C$ follows from the fact that $\left\|M_1^{n,\alpha_n}\right\|_{L^2}^2 \le C$. To finish the proof, we show that there is a subsequence $\left(\alpha_{n_k}\right)_{k\ge1}$ such that $\alpha = \inf_k \alpha_{n_k}$ satisfies $P[\alpha < \infty] \le 4\varepsilon$. We know $P\left[R_1 \le 2/3\right] \le 3\varepsilon$ because $E\left[R_1\right] \ge 1 - \varepsilon$. Since $R_1^n \to R_1$ a.s there is a subsequence such that $P\left[\left|R_1^n - R_1\right| \ge 1/15\right] \le \varepsilon 2 - k$. Finally,

$$\begin{aligned} P[\alpha < \infty] &\le P\left[\inf_k R_1^{n_k} \le 2/3\right] \\ &\le 3\varepsilon + P\left[\inf_k R_1^{n_k} \le 3/5, R_1 > 2/3\right] \\ &\le 3\varepsilon + \sum_{k=1}^{\infty}P\left[R_1^{n_k} \le 3/5, R_1 > 2/3\right] \\ &\le 3\varepsilon + \sum_{k=1}^{\infty}P\left[\left\|R_1^{n_k} - R_1\right\| \le 1/15\right] \\ &\le 4\varepsilon \end{aligned}$$

Therefore $\left(M^n\right)_{n\ge1}$, $\left(A^n\right)_{n\ge1}$ and α have the desired properties.

Proof of the Main Theorems

Proof of Theorem 1.3. We may assume the hypothesis of proposition. Let $\varepsilon > 0$ and take C, α, $\left(M^n\right)_{n\ge1}$, $\left(A^n\right)_{n\ge1}$ as in proposition. Apply komlos lemma to find convex weights $\left(\lambda_n^n, \cdots, \lambda_{N_n}^n\right)$ such that

$$\lambda_n^n M_1^{n,\alpha} + \cdots + \lambda_{N_n}^n M_1^{N,\alpha} \to M_1$$

$$\lambda_n^n A_t^{n,\alpha} + \cdots + \lambda_{N_n}^n A_t^{N,\alpha} \to A_t$$

for all t, where the convergence is a.s. For all *n*, $\sum_{j=1}^{2^n}\left|A_{j2^{-n}}^{n,\alpha} - A_{(j-1)2^{-n}}\right| \le C$ so the total variation of *A* over *D* is bounded by *C*. Further, we have $X^\alpha = M_t + A_t$. *A* is a cadlag on *D*, so define it on all of [0,1] to make it cadlag. *M* is L^2 martingale so it has a cadlag modification. Since $P[\alpha < \infty] < \varepsilon$ and $\varepsilon > 0$ was arbitrary, and the class of classical semimartingales is local, *X* must be a classical semimartingale. □

Proof of Theorem 1.4. We no longer assume that *X* is locally bounded. The trick is to leverage the result for locally bounded processes by subtracting the big jump from *X*. Assume without loss of generality that X_0 and defined $J_t = \sum_{s\le t}\Delta X_s 1_{|\Delta S_s|\ge1}$. Then *X* = *X* − *J* is an adopted, cadlag locally bounded process. We will show that theorem 1.4 for X implies NFLVR + LI for X, so that we may apply theorem 1.3 to X. Then since J is finite variation, this will then imply X is a classical semimartingale .

Suppose $\phi^n \in X$ are such that $\left\|\phi^n\right\|_u \to 0$ and $\left\|\left(\phi^n \cdot X\right)^-\right\|_u \to 0$. We need to prove that $\left(\phi^n \cdot X\right)_t \to 0$ in probability . First we will show that $\left\|\left(\phi^n \cdot X\right)^-\right\|_u \to 0$.

$$\sup_{0\le t\le T}\left(\phi^n \cdot X\right)_t^-$$
$$\le \sup_{0\le t\le T}\left(\phi^n \cdot X\right)_t^- + \sup_{0\le t\le T}\left|\left(\phi^n \cdot J\right)_t\right|$$
$$\le \sup_{0\le t\le T}\left(\phi^n \cdot X\right)_t^- + \left(\left\|\phi^n\right\|_\infty \cdot TV(J)\right)_T \to 0$$

by the assumptions on ϕ^n

By (1), $\left(\phi^n \cdot X\right)_T \to 0$ in probability. Since $\left(\phi^n \cdot J\right)_T \to 0$ in probability, we conclude that

$$\left(\phi^n \cdot X\right)_T = \left(\phi^n \cdot J\right)_T - \left(\phi^n \cdot J\right)_T \to 0$$

in probability. Therefore X satisfies NFLVR + LI. □

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