

Risk-Sensitive Asset Management under a Wishart Autoregressive Factor Model

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ABSTRACT

The risk-sensitive asset management problem with a finite horizon is studied under a financial market model having a Wishart autoregressive stochastic factor, which is positive-definite symmetric matrix-valued. This financial market model has the following interesting features: 1) it describes the stochasticity of the market covariance structure, interest rates, and the risk premium of the risky assets; and 2) it admits the explicit representations of the solution to the risk-sensitive asset management problem.

Keywords: Risk-Sensitive Asset Management; Wishart Autoregressive Stochastic Factor; Stochastic Covariance; Stochastic Interest Rate; Stochastic Risk Premium; Riccati Differential Equation

1. Introduction

1.1. Risk-Sensitive Asset Management

Consider a continuous-time financial market that consists of one riskless asset and n risky assets. The price process of the riskless asset $S^0 := (S_t^0)_{t \geq 0}$ and that of the n risky assets $S := (S^1, \dots, S^n)^T$, $S^i := (S_t^i)_{t \geq 0}$, where $(\cdot)^T$ denotes the transpose of a vector or matrix, are semimartingales defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$. Define the wealth process $X^\pi := (X_t^\pi)_{t \geq 0}$ of a self-financing investor governed by the following stochastic differential equation (SDE):

$$\frac{dX_t^\pi}{X_t^\pi} = \left\{ \sum_{i=1}^n \pi_t^i \frac{dS_t^i}{S_t^i} + \left(1 - \sum_{i=1}^n \pi_t^i \right) \frac{dS_t^0}{S_t^0} \right\}, \quad (1.1)$$

$$X_0^\pi = x,$$

where $x \in \mathbb{R}_{++}$ is the initial wealth of the investor, and $\pi := (\pi^1, \dots, \pi^n)^T$, $\pi^i := (\pi_t^i)_{t \geq 0}$ is the dynamic investment strategy of the investor. Let

$$G_t^\pi := \frac{1}{t} \log X_t^\pi, \quad (1.2)$$

be the growth rate of the wealth at time t . For given constants $T \in (0, \infty)$ and $\gamma \in (0, \infty)$, define the risk-sensitized expected value of G_T^π by

$$C_T^\pi := \frac{1}{(-\gamma T)} \log \mathbb{E} \exp(-\gamma T G_T^\pi),$$

which is rewritten as

$$\exp(-\gamma T G_T^\pi) = \mathbb{E} \exp(-\gamma T G_T^\pi)$$

and interpreted as the certainty equivalent value of G_T^π with respect to the exponential criterion function $\exp(-\gamma T (\cdot))$. We are interested in maximizing G_T^π , that is,

$$\Gamma_T(\gamma) := \sup_{\pi \in \mathcal{A}_T} \frac{1}{(-\gamma T)} \log \mathbb{E} \exp(-\gamma T G_T^\pi) \quad (1.3)$$

$$= \sup_{\pi \in \mathcal{A}_T} \frac{1}{(-\gamma T)} \log \mathbb{E} (X_T^\pi)^{-\gamma},$$

which we call the risk-sensitive asset management problem. Here, \mathcal{A}_T is a space of admissible investment strategies and is a subset of $\mathcal{S}_{2,T}^n$, the totality of n -dimensional \mathcal{F}_t -progressively measurable processes $(p_t)_{t \in [0, T]}$ on the time interval $[0, T]$ such that

$$\int_0^T |p_t|^2 dt < \infty \quad \text{almost surely.}$$

Remark 1.1 The risk-sensitive asset management problem (1.3) has been well-studied under a linear-Gaussian market model, for example, by [1-7]. In those works, the price processes (S^0, S) are given by the solutions to the following system of SDEs:

$$\begin{aligned} dS_t^0 &= S_t^0 r(Y_t) dt, \quad S_0^0 = 1, \\ dS_t &= \text{diag}(S_t) \{ \mu(Y_t) dt + \Sigma dW_t \}, \quad S_0 \in \mathbb{R}_{++}^n, \\ dY_t &= b(Y_t) dt + \Lambda dW_t, \quad Y_0 \in \mathbb{R}^m \end{aligned} \quad (1.4)$$

on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ endowed with the d -dimensional \mathcal{F}_t -Brownian motion $W := (W^1, \dots, W^d)^\top$, $W^i := (W_t^i)_{t \geq 0}$. Here, $\text{diag}(x)$ denotes the diagonal matrix whose (i, i) th element is equal to the i th element x^i of $x := (x^1, \dots, x^n)^\top \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n \times d}$, $\Lambda \in \mathbb{R}^{m \times d}$, and

$$\begin{aligned} r(y) &:= r_0 + r_1^\top y, \\ \mu(y) &:= r(y)\mathbf{1} + \theta(y), \\ \theta(y) &:= \theta_0 + \Theta_1 y, \\ b(y) &:= b_0 + B_1 y \end{aligned} \tag{1.5}$$

with $r_0 \in \mathbb{R}$, $r_1 \in \mathbb{R}^m$, $\mathbf{1} := (1, \dots, 1)^\top \in \mathbb{R}^n$, $\theta_0 \in \mathbb{R}^n$, $\Theta_1 \in \mathbb{R}^{n \times m}$, $b_0 \in \mathbb{R}^m$, and $B_1 \in \mathbb{R}^{m \times m}$. We reformulate (1.3) with (1.1), (1.2), (1.4), and (1.5) as a linear exponential quadratic Gaussian stochastic control problem, and the optimal investment strategy (portfolio)

$\hat{\pi}^{(T)} := (\hat{\pi}_t^{(T)})_{t \in [0, T]}$ for (1.3) is represented explicitly:

$$\hat{\pi}_t^{(T)} := \frac{1}{1 + \gamma} (\Sigma \Sigma^\top)^{-1} \theta(Y_t) - \frac{\gamma}{1 + \gamma} (\Sigma \Sigma^\top)^{-1} \Sigma \Lambda^\top (P_t + P_t Y_t).$$

Here, $(P_t)_{t \in [0, T]}$ is the solution to a matrix differential Riccati equation, and $(P_t)_{t \in [0, T]}$ is the solution to a linear differential equation, including P .

Remark 1.2 Intuitively, recalling the cumulant expansion,

$$C_T^\pi = \mathbb{E} G_T^\pi - \frac{\gamma}{2} \mathbb{V} \sqrt{T} G_T^\pi + O(\gamma^2) \text{ as } \gamma \rightarrow 0,$$

where $\mathbb{V}[\cdot]$ denotes variance, we interpret (1.3) as a risk-sensitized optimization of the expected growth rate maximization,

$$\sup_{\pi} \mathbb{E} G_T^\pi.$$

1.2. Wishart Factor Model

The main aim of the present paper is to introduce a simple and tractable market model that satisfies the following requirements:

- The model describes the stochasticity of the covariance structure of $\left(\frac{dS^i}{S^i}, dY\right)$ ($i = 1, \dots, n$), interest rates, and mean-return rates of S .
- The model admits an explicit representation of the optimal investment strategy for (1.3).

For the purpose, we employ a Wishart autoregressive process as a stochastic factor, which is positive-definite symmetric matrix-valued. Such matrix-valued processes have been introduced and studied by [8], and recently, generalizations have been intensively studied, for example, see [9,10], and the references therein. Moreover,

these processes are now extensively utilized for financial modeling. We can refer to the examples given below.

- Modeling of multivariate stochastic volatility (covariance) under the risk-neutral probability: see [11-16].
- Modeling of multivariate asset price process under physical probability with stochastic covariance and mean-return rates: see [14,17,18].
- Modeling of (term structure of) interest rates and stochastic intensity for credit risk: see [14,17,19,20].

Our market model defined by (2.1)-(2.4) in Section 2 is an extension of the model employed by [18], (see Example 2.1), who studied the expected CRRA-utility maximization of terminal wealth, which is essentially equivalent to (1.3). A main contribution of the present paper is a rigorous mathematical analysis of portfolio optimization problem (1.3) under a flexible Wishart autoregressive stochastic factor model: We strengthen the mathematical results in [18] by formulating an appropriate space of admissible trading strategies (see (3.5)) and showing a verification theorem for the candidate of the optimal strategy (see Theorem 3.1), both of which are omitted in [18].

In the next section, we introduce our market model with a Wishart autoregressive factor and present preliminary calculations of the associated Hamilton-Jacobi-Bellman (HJB) equation for solving risk-sensitive asset management problem (1.3). In Section 3, we introduce our main results. In Section 4, we show the proof of the main theorem after preparing lemmas.

2. Setup

2.1. Market Model with Wishart Autoregressive Factor

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ be a filtered probability space endowed with the $(d \times d)$ -dimensional \mathcal{F}_t -Brownian motion $B := (B_t)_{t \in [0, T]}$, $B_t := (B_t^{ij})_{1 \leq i, j \leq d}$, and the

d -dimensional \mathcal{F}_t -Brownian motion $z := (z_t)_{t \in [0, T]}$,

$z_t := (z_t^1, \dots, z_t^d)^\top$, which is independent of B . Using a constant vector $\rho := (\rho^1, \dots, \rho^d)^\top \in \mathbb{R}^d$ so that $|\rho| \leq 1$, we define another d -dimensional Brownian motion $w := (w^1, \dots, w^d)^\top$, $w^i := (w_t^i)_{t \geq 0}$ by

$$w_t := B_t \rho + \sqrt{1 - |\rho|^2} z_t,$$

which is correlated with B as

$$d \langle w^i, B^{jk} \rangle_t = \delta_{ij} \rho^k dt,$$

where $\langle \cdot, \cdot \rangle$ denotes the quadratic covariation, and δ_{ij} denotes Kronecker's delta. We consider the price processes (S^0, S) , described by the following system of

SDEs:

$$\begin{aligned} dS_t^0 &= r(Y_t)S_t^0 dt, \\ dS_t &= \text{diag}(S_t) \left\{ \mu(Y_t) dt + \Sigma \sqrt{Y_t} dw_t \right\}, \\ dY_t &= (LL^T + KY_t + Y_t K^T) dt + \sqrt{Y_t} dB_t \Lambda^T + \Lambda dB_t^T \sqrt{Y_t}, \end{aligned} \tag{2.1}$$

with the initial values $S_0^0 = 1$, $S_0 \in \mathbb{R}_{++}^n$ and $Y_0 \in \mathbb{S}_{++}^d$. Here, we denote by \mathbb{S}^d the totality of d -dimensional, real, symmetric matrices, and $\mathbb{S}_{++}^d := \{M \in \mathbb{S}^d; M > 0\}$. Furthermore, for $y \in \mathbb{S}^d$, we define

$$\begin{aligned} r(y) &:= r_0 + \text{tr}(R_1 y), \\ \mu(y) &:= r(y)\mathbf{1} + (\Sigma y \Sigma^T) \lambda, \end{aligned} \tag{2.2}$$

where $r_0 \in \mathbb{R}$, $R_1 \in \mathbb{S}^d$, and $\lambda \in \mathbb{R}^n$. Also, we assume that $\Sigma \in \mathbb{R}^{n \times d}$ is full rank, that is,

$$\Sigma^T \Sigma > 0 \text{ if } d \leq n, \text{ and } \Sigma \Sigma^T > 0 \text{ if } d \geq n. \tag{2.3}$$

We also assume that $K \in \mathbb{R}^{d \times d}$, and that $L, \Lambda \in \mathbb{R}^{d \times d}$ satisfies

$$LL^T \geq (n+1)\Lambda\Lambda^T. \tag{2.4}$$

Condition (2.4) ensures $Y > 0$ almost everywhere on (t, ω) , which was established by Mayerhofer *et al.* (2011) using a generalized form of SDE, including a jump martingale part. The \mathbb{S}_{++}^d -valued process Y is a stochastic factor process, which linearly depends on the covariance structure of (S, Y) as

$$\begin{aligned} \frac{d\langle S^i, S^j \rangle_t}{S_t^i S_t^j} &= (\Sigma Y_t \Sigma^T)_{ij} dt \quad 1 \leq i, j \leq n, \\ \frac{d\langle S^i, Y^{jk} \rangle_t}{S_t^i} &= \left\{ (\Sigma Y_t)_{ij} (\Lambda \rho)_k + (\Sigma Y_t)_{ik} (\Lambda \rho)_j \right\} dt \\ &\quad 1 \leq i \leq n, 1 \leq j, k \leq d, \\ d\langle Y^{ij}, Y^{kl} \rangle_t &= \left\{ Y_t^{ik} (\Lambda \Lambda^T)_{jl} + Y_t^{il} (\Lambda \Lambda^T)_{jk} + Y_t^{jk} (\Lambda \Lambda^T)_{il} \right. \\ &\quad \left. + Y_t^{jl} (\Lambda \Lambda^T)_{ik} \right\} dt \quad 1 \leq i, j, k, l \leq d, \end{aligned} \tag{2.5}$$

as well as on the interest rate

$$r(Y_t) := r_0 + \text{tr}(R_1 Y_t), \tag{2.6}$$

and on the so-called risk premium of S ,

$$\mu(Y_t) - r(Y_t)\mathbf{1} = (\Sigma Y_t \Sigma^T) \lambda. \tag{2.7}$$

Remark 2.1 From (2.6), we see that the interest rate process $(r(Y_t))_{t \geq 0}$ is included in the so-called affine class: $r(Y_t)$ is an affine function of Y_t and the process Y , whose infinitesimal generator is given by

$$\mathbb{L}f := 2\text{tr}(yD\Lambda\Lambda^T D)f + \text{tr}\left((LL^T + Ky + yK^T)Df\right), \tag{2.8}$$

where $f \in C^2(\mathbb{S}^d)$ and

$$D := \left(\partial_{y_{ij}} \right)_{1 \leq i, j \leq d} \tag{2.9}$$

is indeed an affine diffusion. To review affine processes and their financial applications, see, for example, [21], [22], and the references therein.

Remark 2.2 The condition (2.7) on the structure of the risk-premium vector is rewritten as

$$\lambda = (\Sigma Y_t \Sigma^T)^{-1} \left\{ \mu(Y_t) - r(Y_t)\mathbf{1} \right\}.$$

So we interpret that the so-called mean-variance term in portfolio optimization theory is assumed to be constant.

The following are concrete examples of setting up (2.1) and (2.2).

Example 2.1 (Stochastic Covariance) Let

$$d = n, R_1 = 0, \text{ and } \Sigma = I_n.$$

Concretely, we have

$$\begin{aligned} dS_t^0 &= r_0 S_t^0 dt, \quad S_0^0 = 1, \\ dS_t &= \text{diag}(S_t) \left\{ (r_0 \mathbf{1} + Y_t \lambda) dt + \sqrt{Y_t} dw_t \right\}, S_0 \in \mathbb{R}_{++}^n \end{aligned}$$

with the third equation in (2.1). Y describes the infinitesimal covariance and the risk premium of S as

$$Y_t^{ij} dt = \frac{d\langle S^i, S^j \rangle_t}{S_t^i S_t^j} \quad \text{for } 1 \leq i, j \leq n$$

and

$$\mu(Y_t) - r_0 \mathbf{1} = Y_t \lambda.$$

This is exactly the model employed in Section 1 of [18] to study expected CRRA-utility maximization of terminal wealth.

Example 2.2 (Stochastic Covariance and Interest Rate) We present a slight generalization of Example 2.1 to include stochasticity of interest rates. Let

$$d = n + 1, R_1 = (r_{ij})_{1 \leq i, j \leq d}, \text{ and } \Sigma = (I_n, 0),$$

where we set $r_{ij} = 0$ if $1 \leq i, j \leq n$. Then, letting

$$Y_1(t) := (Y_t^{ij})_{1 \leq i, j \leq n}, \quad Y_2(t) := (Y_t^{d1}, \dots, Y_t^{dd})^T,$$

and $\tilde{r}_d := (r_{d1}, 2r_{d2}, \dots, 2r_{dd})^T \in \mathbb{R}^d$, we see that

$$r(Y_2(t)) = r_0 + \tilde{r}_d^T Y_2(t)$$

is the risk-free interest rate with the latent factor $Y_2(t)$ and that

$$dS_t = \text{diag}(S_t) \left[\left\{ r(Y_2(t))\mathbf{1} + Y_1(t)\lambda \right\} dt + \sqrt{Y_1(t)} dw_t \right],$$

$$S_0 \in \mathbb{R}_{++}^n,$$

where $Y_1(t)$ describes the infinitesimal covariance and

the risk premium of S as

$$Y_t^{ij} dt = \frac{d\langle S^i, S^j \rangle_t}{S_t^i S_t^j} \quad \text{for } 1 \leq i, j \leq n$$

and

$$\mu(Y_t) - r(Y_2(t))\mathbf{1} = Y_1(t)\lambda.$$

Example 2.3 (Cox-Ingersoll-Ross Interest Rate Factor) Let

$$d = n = 1, r_0 = 0 \text{ and } R_1 = 1.$$

Then, we see

$$\begin{aligned} dS_t^0 &= Y_t S_t^0 dt, \quad S_0^0 = 1, \\ dS_t &= S_t \left\{ (1 + \Sigma^2 \lambda) Y_t dt + \Sigma \sqrt{Y_t} dw_t \right\}, \quad S_0 > 0, \\ dY_t &= (L^2 + 2KY_t) dt + 2\Lambda \sqrt{Y_t} dB_t, \quad Y_0 > 0. \end{aligned}$$

This financial market model with Cox-Ingersoll-Ross's interest rate Y is treated in [23] to study (1.3).

Under the financial market model comprising (2.1) and (2.2) with the assumptions (2.3) and (2.4), we are interested in treating the risk-sensitive asset management problem (1.3).

2.2. Deriving the HJB Equation

To tackle (1.3), we employ a dynamic programming approach: Recall that wealth process (1.1) of a self-financing investor, combined with (2.1), is rewritten as

$$\begin{aligned} dX_t^\pi &= X_t^\pi \pi_t^\top \left[\{ \mu(Y_t) - r(Y_t) \mathbf{1} \} dt + \Sigma \sqrt{Y_t} dw_t + r(Y_t) dt \right], \\ X_0^\pi &= x. \end{aligned}$$

So, we see

$$\begin{aligned} d(X_t^\pi)^{-\gamma} &= (X_t^\pi)^{-\gamma} \left[-\gamma \ell(Y_t, \pi_t) dt - \gamma \pi_t^\top \Sigma \sqrt{Y_t} dw_t \right], \\ (X_0^\pi)^{-\gamma} &= x^{-\gamma}, \end{aligned}$$

where we set

$$\ell(y, \pi) := r(y) + \pi^\top \Sigma y \Sigma^\top \lambda - \frac{1+\gamma}{2} \pi^\top \Sigma y \Sigma^\top \pi. \quad (2.10)$$

Hence, we have

$$(X_T^\pi)^{-\gamma} = x^{-\gamma} M_T^{(\gamma\pi)} \exp \left\{ -\gamma \int_0^T \ell(Y_t, \pi_t) dt \right\},$$

where we define

$$M_t^{(\gamma\pi)} := \exp \left(-\gamma \int_0^t \pi_u^\top \Sigma \sqrt{Y_u} dw_u - \frac{\gamma^2}{2} \int_0^t \pi_u^\top \Sigma Y_u \Sigma^\top \pi_u du \right). \quad (2.11)$$

Let

$$\mathcal{A}_T^1 := \left\{ \pi \in \mathcal{S}_{2,T}^n; (M_t^{(\gamma\pi)})_{t \in [0, T]} \text{ is a martingale} \right\}.$$

For $\pi \in \mathcal{A}_T^1$, we define the probability measure $\tilde{\mathbb{P}}_T^{(\gamma\pi)}$ on (Ω, \mathcal{F}_T) by the formula

$$\frac{d\tilde{\mathbb{P}}_T^{(\gamma\pi)}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} := M_t^{(\gamma\pi)}, \quad t \in [0, T].$$

By Cameron-Martin-Maruyama-Girsanov's theorem, we see that the $\mathbb{R}^{n \times n}$ -valued process $\tilde{B} := (\tilde{B}_t)_{t \in [0, T]}$, defined by

$$\tilde{B}_t := B_t + \gamma \left(\int_0^t \sqrt{Y_u} \Sigma^\top \pi_u du \right) \rho^\top,$$

is a $(\tilde{\mathbb{P}}_T^{(\gamma\pi)}, \mathcal{F}_t)$ -Brownian motion. Moreover, we see that Y has the $\tilde{\mathbb{P}}_T^{(\gamma\pi)}$ -dynamics

$$\begin{aligned} dY_t &= (LL^\top + KY_t + Y_t K^\top) dt \\ &\quad + \sqrt{Y_t} (d\tilde{B}_t - \gamma \sqrt{Y_t} \Sigma^\top \pi_t \rho^\top dt) \Lambda^\top \\ &\quad + \Lambda (d\tilde{B}_t - \gamma \sqrt{Y_t} \Sigma^\top \pi_t \rho^\top dt) \sqrt{Y_t} \\ &= \left\{ LL^\top + (K - \gamma \Lambda \rho \pi_t^\top \Sigma) Y_t + Y_t (K - \gamma \Lambda \rho \pi_t^\top \Sigma)^\top \right\} dt \\ &\quad + \sqrt{Y_t} d\tilde{B}_t \Lambda^\top + \Lambda d\tilde{B}_t \sqrt{Y_t}. \end{aligned}$$

Recall that, for $\pi \in \mathcal{A}_T^1$, we have

$$\begin{aligned} &\frac{1}{(-\gamma)} \log \mathbb{E} (X_T^\pi)^{-\gamma} \\ &= \log x + \frac{1}{(-\gamma)} \log \tilde{\mathbb{E}}_T^{(\gamma\pi)} \exp \left\{ -\gamma \int_0^T \ell(Y_t, \pi_t) dt \right\}, \end{aligned} \quad (2.12)$$

where $\tilde{\mathbb{P}}_T^{(\gamma\pi)}(\cdot)$ denotes the expectation with respect to $\tilde{\mathbb{P}}_T^{(\gamma\pi)}$. We now consider, for $0 \leq t \leq T < \infty$,

$$\bar{V}_t^{(T)} := \text{ess sup}_{\pi \in \mathcal{A}_{t,T}^1} \frac{1}{(-\gamma)} \log \tilde{\mathbb{E}}_t^{(\gamma\pi)} \left[e^{-\gamma \int_t^T \ell(Y_u, \pi_u) du} \Big| \mathcal{F}_t \right]$$

where

$$\mathcal{A}_{t,T}^1 := \left\{ \pi \mathbf{1}_{[t, T]}; \pi \in \mathcal{A}_T^1 \right\}.$$

The associated HJB equation is written as

$$\begin{aligned} -\partial_t V &= 2\text{tr} \left(yD(\Lambda \Lambda^\top D) \right) V - 2\gamma \text{tr} \left(yDV \Lambda \Lambda^\top DV \right) \\ &\quad + \sup_{\pi \in \mathbb{R}^n} \left[\text{tr} \left(\left\{ LL^\top + (K - \gamma \Lambda \rho \pi^\top \Sigma) y \right. \right. \right. \\ &\quad \left. \left. \left. + y(K - \gamma \Lambda \rho \pi^\top \Sigma)^\top \right\} DV \right) + \ell(y, \pi) \right], \end{aligned} \quad (2.13)$$

$$V(T, y) = 0.$$

By direct calculation, we can see the following.

Lemma 2.1 1) If $d \leq n$ and $\Sigma^\top \Sigma > 0$, then HJB Equation (2.13) is rewritten as

$$\begin{aligned} -\partial_t V &= 2\text{tr} \left(yD(\Lambda \Lambda^\top D) \right) V - \text{tr} \left(yDV \Lambda N^{-1} \Lambda^\top DV \right) \\ &\quad + \text{tr} \left((K_1 y + y K_1^\top + LL^\top) DV \right) + \text{tr} (yA) + r_0, \end{aligned} \quad (2.14)$$

$$V(T, y) = 0,$$

where we define

$$\begin{aligned} N^{-1} &:= 2\gamma \left(I - \frac{\gamma}{1+\gamma} \rho \rho^T \right), \\ K_1 &:= K - \frac{\gamma}{1+\gamma} \Lambda \rho \lambda^T \Sigma, \\ A &:= \frac{1}{2(1+\gamma)} \Sigma^T \lambda \lambda^T \Sigma + R_1. \end{aligned} \tag{2.15}$$

The maximizer for (2.13) is given by

$$\bar{\pi}^{(T)}(t, y) := \frac{1}{1+\gamma} \left\{ \lambda - 2\gamma \Sigma (\Sigma^T \Sigma)^{-1} DV(t, y) \Lambda \rho \right\}.$$

2) If $d \geq n$ and $\Sigma \Sigma^T > 0$, then HJB equation (2.13) is rewritten as

$$\begin{aligned} -\partial_t V &= 2\text{tr}(yD(\Lambda \Lambda^T D))V - 2\gamma \text{tr}(yDV \Lambda \Lambda^T DV) \\ &\quad + \frac{2\gamma^2}{1+\gamma} \text{tr} \left(y \Sigma^T (\Sigma y \Sigma^T)^{-1} \Sigma y DV \Lambda \rho \rho^T \Lambda^T DV \right) \\ &\quad + \text{tr} \left((K_1 y + y K_1^T + LL^T) DV \right) + \text{tr}(yA) + r_0, \\ V(T, y) &= 0, \end{aligned} \tag{2.16}$$

where $K_1 \in \mathbb{R}^{n \times d}$ and $A \in \mathbb{R}^{d \times d}$ are given by (2.15). The maximizer for (2.13) is given by

$$\bar{\pi}^{(T)}(t, y) := \frac{1}{1+\gamma} \left\{ \lambda - 2\gamma (\Sigma y \Sigma^T)^{-1} \Sigma y DV(t, y) \Lambda \rho \right\}.$$

3. Results

With the help of Lemma 2.1, it is straightforward to see the following.

Proposition 3.1 (Solution to the HJB equation) 1) If $d \leq n$, then

$$\hat{V}^{(T)}(t, y) := \text{tr} \left(P^{(T)}(t) y \right) + p^{(T)}(t) \tag{3.1}$$

solves (2.13), or equivalently (2.14). Here, $P := P^{(T)} : [0, T] \rightarrow \mathbb{S}^d$ and $p := p^{(T)} : [0, T] \rightarrow \mathbb{R}$ solve the following system of ordinary differential equations:

$$\frac{d}{dt} P - P \Lambda N^{-1} \Lambda^T P + P K_1 + K_1^T P + A = 0, P(T) = 0, \tag{3.2}$$

$$\frac{d}{dt} p + \text{tr}(PLL^T) + r_0 = 0, p(T) = 0.$$

2) If $d \geq n$, then

$$\hat{V}^{(T)}(t, y) := \text{tr} \left(\Sigma^T P^{(T)}(t) \Sigma y \right) + p^{(T)}(t) \tag{3.3}$$

solves (2.13), or equivalently (2.16). Here, $P := P^{(T)} : [0, T] \rightarrow \mathbb{S}^n$ and $p := p^{(T)} : [0, T] \rightarrow \mathbb{R}$ solve the following system of ordinary differential equations:

$$\begin{aligned} \frac{d}{dt} P - P \Sigma \Lambda N^{-1} \Lambda^T \Sigma^T P + P \Sigma K_1 \Sigma^T (\Sigma \Sigma^T)^{-1} \\ + (\Sigma \Sigma^T)^{-1} \Sigma K_1^T \Sigma^T P + (\Sigma \Sigma^T)^{-1} \Sigma A \Sigma^T (\Sigma \Sigma^T)^{-1} = 0, P(T) = 0, \\ \frac{d}{dt} p + \text{tr}(P \Sigma LL^T \Sigma^T) + r_0 = 0, p(T) = 0. \end{aligned} \tag{3.4}$$

Using this proposition, we obtain the following.

Theorem 3.1 (Verification and optimal strategy)

Define the filtration $(\mathcal{F}_t^B)_{t \in [0, T]}$ by

$$\mathcal{F}_t^B := \sigma(B_u; u \in [0, t]).$$

$$\mathcal{A}_T^{Bb} := \left\{ (\pi_t)_{t \in [0, T]} \left| \begin{array}{l} \text{bounded,} \\ \mathcal{F}_t^B \text{-progressively measurable} \end{array} \right. \right\} \tag{3.5}$$

and consider (1.3) with $\mathcal{A}_T := \mathcal{A}_T^{Bb}$. Then, the following assertions hold.

1) If $d \leq n$, then $\hat{\pi}^{(T)} := (\hat{\pi}_t^{(T)})_{t \in [0, T]}$, defined by

$$\hat{\pi}_t^{(T)} := \frac{1}{1+\gamma} \lambda - \frac{2\gamma}{1+\gamma} \Sigma (\Sigma^T \Sigma)^{-1} P^{(T)}(t) \Lambda \rho, \tag{3.6}$$

is optimal for (1.3). It holds that

$$\Gamma_T(\gamma) = \frac{1}{T} \left\{ \log x + \hat{V}^{(T)}(0, Y_0) \right\}. \tag{3.7}$$

2) If $d \geq n$, then $\hat{\pi}^{(T)} := (\hat{\pi}_t^{(T)})_{t \in [0, T]}$, defined by

$$\hat{\pi}_t^{(T)} := \frac{1}{1+\gamma} \lambda - \frac{2\gamma}{1+\gamma} P^{(T)}(t) \Sigma \Lambda \rho, \tag{3.8}$$

is optimal for (1.3). The relation (3.7) holds.

The proof of the above theorem is given in Subsection 4.2 after preparing lemmas in Subsection 4.1.

4. Proofs

4.1. Lemmas for Exponential Martingale

We prepare the following two lemmas.

Lemma 4.1 Let $(F, f) : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^{d \times d} \times \mathbb{R}^d$ be \mathcal{F}_t^B -progressively measurable so that $\int_0^T (|F_t|^2 + |f_t|^2) dt < \infty$ almost surely for all $T > 0$.

Define

$$\mathcal{E}_1(t) = \mathcal{E}_1(t; B) := \exp \left\{ \int_0^t \text{tr}(F_u dB_u) - \frac{1}{2} \int_0^t \text{tr}(F_u F_u^T) du \right\},$$

$$\mathcal{E}_2(t) = \mathcal{E}_2(t; B, z) := \exp \left\{ \int_0^t f_u^T dz_u - \frac{1}{2} \int_0^t |f_u|^2 du \right\}.$$

Then, $\mathcal{E}_1 \mathcal{E}_2$ is an \mathcal{F}_t -martingale if and only if \mathcal{E}_1 is an \mathcal{F}_t -martingale.

Proof. Denote $\mathcal{F}_t^z := \sigma(z_u; u \in [0, t])$. For

$0 \leq s \leq t \leq T$, we have

$$\begin{aligned} & \mathbb{E}[\mathcal{E}_1(t)\mathcal{E}_2(t)|\mathcal{F}_s] \\ &= \mathbb{E}\left[\mathbb{E}\left[\mathcal{E}_1(t)\mathcal{E}_2(t)\middle|\mathcal{F}_t^B \vee \mathcal{F}_s^z\right]\middle|\mathcal{F}_s\right] \\ &= \mathcal{E}_2(s)\mathbb{E}\left[\mathcal{E}_1(t)\mathbb{E}\left[\mathcal{E}_2(t)\mathcal{E}_2^{-1}(s)\middle|\mathcal{F}_t^B \vee \mathcal{F}_s^z\right]\middle|\mathcal{F}_s\right] \\ &= \mathcal{E}_2(s)\mathbb{E}\left[\mathcal{E}_1(t)\mathbb{E}\left[\mathcal{E}_2(t)\mathcal{E}_2^{-1}(s)\middle|\mathcal{F}_t^B\right]\middle|\mathcal{F}_s\right] \\ &= \mathcal{E}_2(s)\mathbb{E}\left[\mathcal{E}_1(t)\mathbb{E}\left[\mathcal{E}_2(t,l)\mathcal{E}_2^{-1}(s,l)\right]\middle|_{l=B}\middle|\mathcal{F}_s\right] \\ &= \mathcal{E}_2(s)\mathbb{E}\left[\mathcal{E}_1(t)\middle|\mathcal{F}_s\right]. \end{aligned}$$

Lemma 4.2 *Let $F : \mathbb{R}_+ \times \mathbb{S}^d \times \Omega \rightarrow \mathbb{R}^{d \times d}$ satisfy the following: for each $t \geq 0$, $(s, y, \omega) \mapsto F(s, y, \omega)$ is $\mathcal{B}([0, t]) \otimes \mathcal{B}(\mathbb{B}^d) \otimes \mathcal{F}_t / \mathcal{B}(\mathbb{R}^{d \times d})$ -measurable, and*

$$|FF^T(t, y, \omega)| \leq C_0(t, \omega)|y| \text{ for all } (t, y, \omega) \in \mathbb{R}_+ \times \mathbb{S}^d \times \Omega$$

with some bounded C_0 , where we write $|y| := \sqrt{\text{tr}(y^2)}$ for $y \in \mathbb{S}^d$. Then, the process $\mathcal{E} := (\mathcal{E}_t)_{t \geq 0}$, defined by

$$\mathcal{E}_t := \exp\left\{\int_0^t \text{tr}(F(u, Y_u)dB_u) - \frac{1}{2}\int_0^t \text{tr}(FF^T)(u, Y_u)du\right\},$$

is a martingale.

Proof. The lemma follows from Lemma 4.1.5 of [24], an extension of Lemma 4.1.1 of [25]. Below, we reproduce the proof for self-containedness. Note that it suffices to show

$$\mathbb{E}\mathcal{E}_t = 1 \text{ for each } t > 0. \tag{4.1}$$

$$d\left(\frac{\mathcal{E}_t\phi(Y_t)}{1+\epsilon\mathcal{E}_t\phi(Y_t)}\right) = \frac{\phi(Y_t)d\mathcal{E}_t + \mathcal{E}_td\phi(Y_t)}{(1+\epsilon\mathcal{E}_t\phi(Y_t))^2} - \epsilon\frac{\phi(Y_t)^2 d\langle\mathcal{E}\rangle_t + 2\phi(Y_t)\mathcal{E}_td\langle\mathcal{E}, \phi(Y)\rangle_t + \mathcal{E}_t^2 d\langle\phi(Y)\rangle_t}{(1+\epsilon\mathcal{E}_t\phi(Y_t))^3} =: dM_t + dA_t,$$

where

$$M_t := \int_0^t \frac{\mathcal{E}_u}{(1+\epsilon\mathcal{E}_u\phi(Y_u))^2} \times \left\{ \phi(Y_u)\text{tr}(F(u, Y_u)dB_u) + \frac{1}{\phi(Y_u)}\text{tr}(Y_u^{3/2}dB_u\Lambda^T + \Lambda dB_u^T Y_u^{3/2}) \right\}$$

is the local-martingale part and $(A_t)_{t \geq 0}$ is the bounded-variation part, which satisfies

$$\begin{aligned} A_t &\leq \frac{\phi(Y_0)}{1+\epsilon\phi(Y_0)} + \int_0^t \frac{\mathcal{E}_u\mathbb{L}\phi(Y_u)}{(1+\epsilon\mathcal{E}_u\phi(Y_u))^2} du \\ &+ C_4 \int_0^t \frac{\epsilon\mathcal{E}_u^2}{(1+\epsilon\mathcal{E}_u\phi(Y_u))^3} \left\{ \phi(Y_u)^2 \text{tr}(FF^T(u, Y_u)) + 2\phi(Y_u)\left|\text{tr}(F(u, Y_u)Y_u^{1/2})\right| + \frac{\text{tr}(Y_u^3)}{\phi(Y_u)^2} \right\} du \end{aligned}$$

with some constant C_4 , independent of ϵ . We can check that $\mathbb{E}\langle M \rangle_t < \infty$; hence, M is a square-integrable martingale. Further, using (4.2) and recalling that $|AB| \leq |A||B|$ for conformable matrices A and B , we can check that

$$A_t \leq A_0 + C_5 \int_0^t \frac{\mathcal{E}_s\phi(Y_s)}{1+\epsilon\mathcal{E}_s\phi(Y_s)} ds$$

Recall that $(t, \omega) \mapsto F(t, Y_t(\omega), \omega)$ is progressively measurable. The proof of (4.1) consists of several steps.

First, writing $\phi(y) := |y|$, we recall that

$$\mathbb{L}\phi \leq C_1\phi \text{ with some } C_1 \in \mathbb{R}_{++}, \tag{4.2}$$

where \mathbb{L} is defined by (2.8). From this, we can check that

$$\mathbb{E}\phi(Y_t) \leq \phi(Y_0)e^{C_2 t}$$

for each $t \geq 0$ with some constant C_2 . Also, we can check that

$$\mathcal{E}_t\phi(Y_t) \in L^1(\mathbb{P}) \text{ for each } t \geq 0. \tag{4.3}$$

This follows from the relation

$$\mathbb{E}\left[\frac{\mathcal{E}_t\phi(Y_t)}{1+\epsilon\mathcal{E}_t\phi(Y_t)}\right] \leq |Y_0|e^{C_3 t}, \tag{4.4}$$

where $\epsilon > 0$ is arbitrary and the constant C_3 is independent of ϵ . Indeed, in (4.4), letting $\epsilon \downarrow 0$ and using Fatou's lemma, (4.3) is deduced. To see (4.4), use (2.1) and Itô's formula to deduce

$$d\mathcal{E}_t = \mathcal{E}_t\text{tr}(F(t, Y_t)dB_t),$$

$$d\phi(Y_t) = \text{tr}\left(D\phi(Y_t)\left(\sqrt{Y_t}dB_t\Lambda^T + \Lambda dB_t^T\sqrt{Y_t}\right)\right) + \mathbb{L}\phi(Y_t)dt,$$

where we use notation (2.8) and (2.9). From these, we see, from Itô's formula,

with some positive constant C_5 , independent of $\epsilon > 0$. So, taking the expectation, we deduce that

$$\mathbb{E}\left[\frac{\mathcal{E}_t\phi(Y_t)}{1+\epsilon\mathcal{E}_t\phi(Y_t)}\right] \leq \frac{\phi(Y_0)}{1+\epsilon\phi(Y_0)} + C_5 \int_0^t \mathbb{E}\left[\frac{\mathcal{E}_s\phi(Y_s)}{1+\epsilon\mathcal{E}_s\phi(Y_s)}\right] ds$$

and that (4.4) follows from Gronwall's inequality.

Next, use Itô's formula for the following computation:

$$d\left(\frac{\mathcal{E}_t}{1+\epsilon\mathcal{E}_t}\right) = \frac{d\mathcal{E}_t}{(1+\epsilon\mathcal{E}_t)^2} - \epsilon \frac{d\langle\mathcal{E}\rangle_t}{(1+\epsilon\mathcal{E}_t)^3}. \tag{4.5}$$

Here, we see that

$$\mathbb{E}\int_0^t \frac{d\langle\mathcal{E}\rangle_s}{(1+\epsilon\mathcal{E}_s)^4} \leq C_6 \mathbb{E}\int_0^t \text{tr}(FF^T(s, Y_s)) ds < \infty$$

with some constant C_6 ; hence, the first term of the right-hand side of (4.5) is a square-intergrable martingale. Also, we can deduce that

$$\epsilon \int_0^t \frac{d\langle\mathcal{E}\rangle_s}{(1+\epsilon\mathcal{E}_s)^3} \leq C_7 \int_0^t \mathcal{E}_s |Y_s| ds$$

where C_7 is a positive constant, independent of ϵ . Taking the expectation, we see

$$\mathbb{E} \frac{\mathcal{E}_t}{1+\epsilon\mathcal{E}_t} = \frac{1}{1+\epsilon} - \epsilon \mathbb{E} \int_0^t \frac{d\langle\mathcal{E}\rangle_s}{(1+\epsilon\mathcal{E}_s)^3}.$$

Letting $\epsilon \downarrow 0$ and using the dominated convergence theorem, we obtain (4.1).

4.2. Proof of Theorem 3.1

Let $\hat{V} := \hat{V}^{(T)} : [0, T] \times \mathbb{S}_{++}^d \rightarrow \mathbb{R}$ be given by (3.1). Fix $(t, y) \in [0, T] \times \mathbb{S}_{++}^d$ and take $\pi \in \mathcal{A}_{T-t}^{Bb}$. Using these, define

$$\Phi_s(\pi) := M_s^{(\gamma\pi)} \exp\left[-\gamma\left\{\hat{V}(t+s, Y_s) + \int_0^s \ell(Y_u, \pi_u) du\right\}\right], \tag{4.6}$$

where we use (2.10), (2.11), and the process Y given by (2.1), and we set $Y_0 = y \in \mathbb{S}_{++}^d$. Using Itô's formula, we see that

$$\frac{d\Phi_s(\pi)}{\Phi_s(\pi)} = \frac{d\mathcal{E}_s(\pi)}{\mathcal{E}_s(\pi)} - \gamma \left(\mathcal{L}_s^{(\pi)}\hat{V}\right)(t+s, Y_s) ds, \tag{4.7}$$

where we define the process $\mathcal{E}(\pi) := (\mathcal{E}_s(\pi))_{s \in [0, T-t]}$ by (see below)

Combining (4.6)-(4.8), we have, for $s \in [0, T-t]$,

$$\begin{aligned} \Phi_s(\pi) &= \mathcal{E}_s(\pi) \times \exp\left[-\gamma\left\{\hat{V}(t, y) + \int_0^s \left(\mathcal{L}_u^{(\pi)}\hat{V}\right)(t+u, Y_u) du\right\}\right]. \end{aligned}$$

Here, note that $(\mathcal{E}_s(\pi))_{s \in [0, T-t]}$ is a martingale for any $\pi \in \mathcal{A}_{T-t}^{Bb}$ by using Lemma 4.1 and 4.2 and that $(\mathcal{L}_s^{(\pi)}\hat{V})(t+s, Y_s) \leq 0$ almost everywhere on $(s, \omega) \in [0, T-t] \times \Omega$ since \hat{V} solves HJB-equation (2.13). So we deduce that $(\Phi_s(\pi))_{s \in [0, T-t]}$ is a submartingale for each $\pi \in \mathcal{A}_{T-t}^{Bb}$. Taking the expectation, we see that

$$\begin{aligned} \exp\{-\gamma\hat{V}(t, y)\} &\leq \mathbb{E}\Phi_{T-t}(\pi) \\ &= \mathbb{E}_T^{(\gamma\pi)} \exp\left\{-\gamma\int_0^{T-t} \ell(Y_u, \pi_u) du\right\}. \end{aligned}$$

Thus, we see that

$$\frac{1}{(-\gamma)} \log \mathbb{E}_T^{(\gamma\pi)} \exp\left\{-\gamma\int_0^{T-t} \ell(Y_u, \pi_u) du\right\} \leq \hat{V}(t, y) \tag{4.9}$$

for any $\pi \in \mathcal{A}_{T-t}^{Bb}$. Furthermore, if we define $\tilde{\pi} := (\tilde{\pi}_s)_{s \in [0, T-t]} \in \mathcal{A}_{T-t}^{Bb}$ by

$$\tilde{\pi}_s := \hat{\pi}_{t+s}^{(T)},$$

then, we deduce that $(\mathcal{L}_s^{(\tilde{\pi})}\hat{V})(t+s, Y_s) = 0$ almost everywhere on $(s, \omega) \in [0, T-t] \times \Omega$, from which we see that $(\Phi_s(\tilde{\pi}))_{s \in [0, T-t]}$ is a martingale. Therefore, taking the expectation, we see that

$$\begin{aligned} \exp\{-\gamma\hat{V}(t, y)\} &= \mathbb{E}\Phi_{T-t}(\tilde{\pi}) \\ &= \mathbb{E}_T^{(\gamma\tilde{\pi})} \exp\left\{-\gamma\int_0^{T-t} \ell(Y_u, \tilde{\pi}_u) du\right\}, \end{aligned}$$

that is,

$$\frac{1}{(-\gamma)} \log \mathbb{E}_T^{(\gamma\tilde{\pi})} \exp\left\{-\gamma\int_0^{T-t} \ell(Y_u, \tilde{\pi}_u) du\right\} = \hat{V}(t, y). \tag{4.10}$$

Combining (4.9) and (4.10), we deduce that

$$\hat{V}(t, y) = \sup_{\pi \in \mathcal{A}_{T-t}^{Bb}} \frac{1}{(-\gamma)} \log \mathbb{E}_T^{(\gamma\pi)} \exp\left\{-\gamma\int_0^{T-t} \ell(Y_u, \pi_u) du\right\}. \tag{4.11}$$

Thus, letting $t = 0$ in (4.11), we have that

$$\frac{d\mathcal{E}_s(\pi)}{\mathcal{E}_s(\pi)} = -\gamma \text{tr}\left(\sqrt{Y_s} dB_s \left\{2\Lambda^T D\hat{V}(t+s, Y_s) + \rho\pi_s^T \Sigma\right\}\right) - \gamma \sqrt{1-|\rho|^2} \pi_s^T \Sigma \sqrt{Y_s} dz_s, \quad \mathcal{E}_0(\pi) = 1 \tag{4.8}$$

and write

$$\begin{aligned} \left(\mathcal{L}_s^{(\pi)}\hat{V}\right)(t+s, y) &:= \partial_s \hat{V}(t+s, y) + 2\text{tr}(yD\Lambda\Lambda^T D)\hat{V}(t+s, y) - 2\gamma \text{tr}(yD\hat{V}\Lambda\Lambda^T D\hat{V})(t+s, y) \\ &\quad + \text{tr}\left(\left\{LL^T + (K - \gamma\Lambda\rho\pi_s^T \Sigma)y + y(K - \gamma\Lambda\rho\pi_s^T \Sigma)^T\right\} D\hat{V}\right)(t+s, y) + \ell(y, \pi_s). \end{aligned}$$

$$\begin{aligned}\hat{V}^{(T)}(0, Y_0) &= \sup_{\pi \in \cdot \rightarrow T^{Bb}} \frac{1}{(-\gamma)} \log \mathbb{E}_T^{(\gamma\pi)} \exp \left\{ -\gamma \int_0^T \ell(Y_u, \pi_u) du \right\} \\ &= \frac{1}{(-\gamma)} \log \mathbb{E}_T^{(\gamma\hat{\pi}^{(T)})} \exp \left\{ -\gamma \int_0^T \ell(Y_u, \hat{\pi}_u^{(T)}) du \right\}.\end{aligned}$$

(3.7) follows from relation (2.12).

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