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Some Properties for the American Option-Pricing Model

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ABSTRACT

In this paper we study global properties of the optimal excising boundary for the American option-pricing model. It is shown that a global comparison principle with respect to time-dependent volatility holds. Moreover, we proved a global regularity for the free boundary.

Keywords: American Option Model; Regularity of Free Boundary; Comparison Principle

1. Introduction

It is well-known that, for the American option-pricing model, there is an optimal holding region for contracts holders (see [1-5]). The part of the boundary for the region is unknown (free boundary), which is often referred as the optimal excising boundary for option traders. This free boundary has to be calculated along with the option price of the security. The mathematical model for the problem is highly nonlinear and there is no explicit solution representation even when volatility and interest rate are assumed to be constants (see [4]). On the other hand, for the financial world as well as for the intrinsic interest itself, it is extremely important to find the location of the free boundary along with the option price of the security. Particularly, people would like to know how the price of a security changes near the option expiry time since it may change dramatically [6,7].

During the past few decades, there are many research papers concerning for various option-pricing models. There are several Monographs devoted to this topic (see, for examples, [1,3,4,8]). For the American option model as well as its generalization, the existence and uniqueness are studied by many researchers (here just a few examples, [2,5,9-12]). A basic fact is that the American option-pricing model can be reformulated as a variational inequality of parabolic type. Hence, many known results about existence and uniqueness can be applied to the model. However, the disadvantage of the method is that there is no information about the free boundary. To overcome the shortcoming, several authors employed other methods to establish the existence and uniqueness for the problem (see [7,13-17]). Because of the practical importance, many researchers paid a special attention to the asymptotic behavior for the free boundary near the expiration time(see [6,18-25]). Moreover, various numerical computations for the location of free boundary are also carried out by many people (see, for examples, [14,25-28] and the references therein). More recently, some global property of the free boundary attracts some interest. The authors of [29,30] proved that the free boundary is convex if the volatility in the model is assumed to be a constant. However, this global property is not valid in the real financial market since the volatility depends on time and other economical factors. When the volatility depends on time and the security, the problem becomes much more challenging. In this paper we would like to study some global property of the free boundary. We want to find how the optimal exercising boundary changes when the volatility changes during the life-time of the option contract. This question is very important for structured products in the financial world.

We first recall the classical model for the American option-pricing model with one security or one type of asset. Let $V(s,t)$ be the option price for a security such as a stock with price s at time t. Then it is well-known that $V(s,t)$ satisfies the Black-Scholes equation with no dividend [31,32]:

$$L_0 V := V_t + \frac{\sigma^2}{2} s^2 V_{ss} + rsV_s - rV = 0, (s,t) \in Q_T^1, \quad (1.1)$$

where r is the interest rate and σ represents the market volatility of the stock, Q_T^1 is the region defined below.

For the American put-option model (call-option is similar), in order to avoid loss for option holders, it is desirable to hold the option only when s lies in the region (called optimal holding region):

$$Q_T^1 := \{(s,t) : S(t) < s < \infty, 0 < t < T\},$$

where $s = S(t)$ is the free boundary, which ensures

$V(s,t) \geq 0$, called the optimal exercising boundary.

On the free boundary $s = S(t)$, we know from the continuity of the option price that $V(s,t)$ satisfies:

$$V(S(t),t) = K - S(t), \quad 0 < t < T \tag{1.2}$$

$$V_s(S(t),t) = -1, \quad 0 < t < T, \tag{1.3}$$

where K is the striking price.

We also know the payoff value at the terminal time T once the striking price is given:

$$V(s,T) = (K-s)^+, S(T) \leq s < \infty, \tag{1.4}$$

$$V(s,t) \to 0, s \to \infty. \tag{1.5}$$

For later use, we introduce :

$$Q_T^2 = \{(s,t): 0 < s < S(t), 0 < t < T\},$$

$$Q_T = Q_T^1 \cup Q_T^2 \cup \Gamma_T,$$

where

$$\Gamma_T = \{s = S(t): 0 \leq t < T\}.$$

In financial markets, the volatility σ plays a major role for the option pricing model. Option price often changes dramatically when the stock market is in a chaotic movement. This was the case when the flash-crash happened on May 6, 2010 as well as the case on Oct. 19, 1987. On the other hand, for a relatively stable market, the volatility mainly depends on time. This is particularly true for an index fund such as S&P500 index in the U.S. market. Hence, we assume that $\sigma = \sigma(t)$ throughout this paper. Our question is how the free boundary $S(t)$ changes when the volatility $\sigma(t)$ changes during the life-span of the option contract. We show that there is a global comparison principle for the free boundary with respect to the change of volatility $\sigma(t)$. Moreover, a global existence result is also established as a by-product. Our proof is based on the line method (see [15]), which is different from existing literature (see [21,13] and the references therein). Although the existence of a solution for the problem is already known, our method does have several advantages. One of them is that the free boundary is determined along with the option price at each discrete time simultaneously. Moreover, a global regularity for the free boundary is also obtained. To author's knowledge, this regularity result is new and optimal (see [19, 21,12]).

The paper is organized as follows. In Section 2, we construct a sequence of approximation solutions by using the line method. After deriving some uniform estimates, a global existence is established. Moreover, an optimal global regularity for the free boundary is also obtained. In Section 3, we first derive some comparison properties for the approximation solution and then show that the limit solution preserves the same property. Some concluding remarks are given in Section 4.

Remark 1.1: After this paper is completed, the author learned that E. Ekströn proved a result in [33] (2004) about the monotonicity of option price with respect to volatility. However, there is no result about the comparison result for the free boundary. Moreover, the method in [33] is totally different from ours here. In addition, we also present a regularity result for the free boundary.

2. Existence and Uniqueness

Since our argument in Section 3 is based on the discrete problem, we give the complete details about the construction of the approximation solution sequence. We also show that the approximation sequence is convergent to the solution of the original problem (1.1)-(1.5). As a byproduct, an optimal regularity of the free boundary is obtained.

The following conditions are always assumed throughout this paper.

H(1): Let $\sigma(t), r(t) \in C^\alpha[0,T]$ for some $\alpha \in (0,1)$. There exist positive constants a, b and R such that

$$0 < a \leq \sigma(t) \leq b, 0 \leq r(t) \leq R.$$

Now we construct an approximate solution sequence by using the line method.

Let N be a positive integer. Divide $[0,T]$ into N subintervals with equal length $h = \frac{T}{N}$:

$$0 = t_0 < t_1 < \cdots < t_N = T, t_i = ih, i = 0,1,\cdots,N$$

Define

$$\sigma_k = \frac{1}{h}\int_{t_{k-1}}^{t_k} \sigma(\tau)\,\mathrm{d}\tau,.$$

$$r_k = \frac{1}{h}\int_{t_{k-1}}^{t_k} r(\tau)\,\mathrm{d}\tau, k = 1,2,\cdots N.$$

If we use difference quotient to approximate V_t and replace $\sigma(t)$ and $r(t)$ by σ_n and r_n, we have

$$\frac{V(s,t_{n+1}) - V(s,t_n)}{h} + \frac{1}{2}\sigma_n s^2 V(s,t_n)_{ss}$$
$$+ r_n s V(s,t_n)_s - r_n V(s,t_n) = 0.$$

This leads us to define the approximate solution $V_n(s)$ and S_n as follows:

From the terminal condition, we know

$$V(s,T) = (K-s)^+$$

and $S(T) = K$. So we define

$$V_N(s) = (K-s)^+, 0 \leq s < \infty, S_N = K.$$

Suppose we have obtained $V_{n+1}(s)$ and S_{n+1}, we can define $V_n(s)$ and S_n as follows:

$$\frac{1}{2}\sigma_n s^2 V_n'' + r_n s V_n' - \left(r_n + \frac{1}{h}\right)V_n = -\frac{1}{h}V_{n+1}(s), \quad S_n < s < \infty, \tag{2.1}$$

$$V_n(S_n) = K - S_n, V_n'(S_n) = -1, \tag{2.2}$$

$$V_n(s) \to 0, as\ s \to \infty, \tag{2.3}$$

where we have extended $V_{n+1}(s)$ into the whole interval $[0,\infty)$ by

$$V_{n+1}(s) = K - s, \quad 0 < s < S_{n+1}.$$

It is easy to see that the above free boundary problem (2.1)-(2.3) has a unique solution $(V_n(s), S_n)$ for each n. Actually, since the problem is one-dimensional one can find the solution $V_n(s)$ and S_n explicitly (see [4] for detailed calculation).

Now we use the interpolation to define the free boundary $S^N(t)$ as follows:

$$S^N(t) = \frac{t_n - t}{h}S_{n-1} + \frac{t - t_{n-1}}{h}S_n,$$
$$t_{n-1} \le t \le t_n, n = 1, 2, \cdots, N.$$

Also, we define

$$V^N(s,t) = V_n(s), t_n \le t < t_{n+1}, n = 0, 1, \cdots N-1.$$

We also use the notation

$$V_{n,\bar{h}}(s) = \frac{V_n(s) - V_{n-1}(s)}{h}.$$

$$\Omega_k = Q_T \bigcap \{t = t_k\}, k = 0, 1, \cdots, N.$$

Our goal is to show that the approximate solution sequence $(V^N(s,t), S^N(t))$ is convergent to the solution of the original free boundary problem (1.1)-(1.5). To this end, we need to derive some uniform estimates.

Lemma 2.1: For all $(s,t) \in Q_T$,

$$0 \le V^N(s,t) \le K, 0 \le S^N(t) \le K.$$

Proof: From the definition, we see

$$V^N(s,t) = V_N(s) \ge 0$$

if $t_{N-1} \le t \le t_N = T$. Suppose we have shown that $V_{n+1}(s) \ge 0$, we claim that $V_n(s) \ge 0$. Indeed, if $V_n(s)$ attains a negative minimum at some point $s^* \in (S_n, \infty)$, then at this minimum point, we see

$$\left[\frac{1}{2}\sigma_n s^2 V_n'' + r_n s V_n' - \left(r_n + \frac{1}{h}\right)V_n\right]\Bigg|_{s=s^*} > 0,$$

which contradicts the right-hand side of the Equation (2.1). It follows that $V_n(s) \ge 0$ on $[S_n, \infty)$. By the definition of $V_n(s)$ on $[0, S_n)$, we see $V_n(s) \ge 0$ for $s \in [0, \infty)$. Consequently, $V^N(s,t) \ge 0$ on Q_T.

On the other hand, we claim that $V^N(s,t)$ has an upper bound K. Indeed, it is obviously true for $V_N(s)$, which implies that $V^N(s,t) \le K$ when $t \in [t_{N-1}, t_N]$. We assume that $[t_{n-1}, t_n]$ is the first interval in which $\max_{0\le s<\infty} V_n(s) > K$. Then, suppose that $V_n(s)$ attains a positive maximum at an interior point $s^* \in (S_n, \infty)$, then at $s = s^*$, $V_n''(s^*) \le 0, V_n'(s^*) = 0$. Thus,

$$\left[\frac{1}{2}\sigma_n s^2 V_n'' + r_n s V_n' - \left(r_n + \frac{1}{h}\right)V_n\right]\Bigg|_{s=s^*} \le -\left(r_n + \frac{1}{h}\right)V_n(s^*).$$

It follows from Equation (2.1) that

$$V_n(s^*) \le \frac{1}{1 + r_n h}V_{n+1} \le K,$$

which is a contradiction. On the boundary $s = S_n$,

$$V_n(S_n, t) = K - S_n \le K.$$

Obviously, $V_n(s) \le K$ when $0 \le s \le S_n$. Consequently, $0 \le V^N(s,t) \le K$ in Q_T. Furthermore, from the boundary condition (2.2), we see $0 \le S_n \le K$ for all $n = 0, 1, \cdots, N$.

Q.E.D.

Lemma 2.2: There exists a constant C_1 such that

$$\left\|V^N\right\|_{L^2(\Omega)_m} + h\sum_{k=1}^{m}\left\|\left(V^N\right)_s\right\|_{L^2(\Omega_k)} + h^2\sum_{k=1}^{m}\left\|V_{\bar{h}}^N\right\|_{L^2(\Omega_k)} \le C_1,$$

where C_1 depends only on known data, but not on N.

Proof: This estimate is similar to the energy estimate for a parabolic equation. Indeed, we introduce new variables:

$$x = \ln s, \tau = T - t.$$

Define

$$U(x,\tau) = V(s,t), X(\tau) = S(t).$$

Then the original free boundary problem (1.1)-(1.5) is equivalent to the following one:

$$L_1 U := U_\tau - \frac{1}{2}\hat{\sigma}(\tau)^2 U_{xx} - \left(\hat{r}(\tau) - \frac{1}{2}\hat{\sigma}(\tau)^2\right)U_x + \hat{r}(\tau)U = 0, (x,\tau) \in \hat{Q}_T, \tag{2.4}$$

$$U(X(\tau),\tau) = K - e^{X(\tau)}, \quad 0 < \tau \le T, \tag{2.5}$$

$$U_x(X(\tau),\tau) = -e^{X(\tau)}, \quad 0 < \tau \le T, \tag{2.6}$$

$$U(x,0) = \left(K - e^{S(T)}\right)^+, X(0) < x < \infty, \tag{2.7}$$

where

$$\hat{Q}_T = \{(x,\tau) : X(\tau) < x < \infty, 0 < \tau \le T\},$$

$$\hat{\sigma}(\tau) = \sigma(t), \hat{r}(\tau) = r(t).$$

On the other hand, by the definition we know

$$V(s,t) = K - s, \quad 0 < s < S(t), 0 < t < T.$$

It follows that

$$L_0 V = -\hat{r}(\tau)K, \quad 0 < s < S(t), 0 < t < T.$$

Thus,

$$L_1 U = -r(t)K, \quad -\infty < x < X(\tau), 0 < \tau < T.$$

Now we can extend $U(x,\tau)$ into the region $\hat{Q}_T = R^1 \times (0,T]$, we use the continuity of $U(x,\tau)$ and $U_x(x,\tau)$ in $\hat{Q}_T$ to see that $U(x,\tau)$ is a weak solution of the following problem:

$$L_1 U = f(x,\tau), \quad (x,\tau) \in \hat{Q}_T, \tag{2.8}$$

$$U(x,0) = \left(K - e^x\right)+, \quad -\infty < x < \infty. \tag{2.9}$$

where $f(x,\tau) = 0$ if $-\infty < x < X(\tau), 0 \le \tau \le T$ and $f(x,\tau) = -\hat{r}(\tau)K$ if $X(\tau) < x < \infty, 0 \le \tau < T$.

Now we can use the line method method to define $U_n(x)$ and $U^N(x,\tau)$ which are exactly the same as for a classical parabolic equation (see [34], estimate (5.15) on page 137) and obtain the desired energy estimate. By the definition, we see clearly that $V^N(s,t) = U^N(x,\tau)$ for $(s,t) \in Q_T^1$.

Q.E.D.

Lemma 2.3: There exists a constant C_2 such that

$$\left\|\left(V^N\right)_s\right\|_{L^\infty(Q_T)} \le C_2,$$

where C_2 depends only on known data, but not on N.

Proof: Note that $U(x,0)$ is uniformly Lipschitz continuous on $\left[X(0),\infty\right)$. We may assume that $U(x,0)$ is differentiable with a bounded derivative on $\left[X(0),\infty\right)$.

Define

$$W(x,\tau) = U_x((x,\tau).$$

It follows that W satisfies the following equations:

$$W_\tau - \frac{1}{2}\hat{\sigma}(\tau)^2 W_{xx} - \left(\hat{r}(\tau) - \frac{1}{2}\hat{\sigma}(\tau)^2\right)W_x + \hat{r}(\tau)W = 0, (x,\tau) \in \hat{Q}_T, \tag{2.10}$$

$$W\left(X(\tau),\tau\right) = -e^{X(\tau)}, \quad 0 < \tau \le T, \tag{2.11}$$

$$W(x,0) = U_x(x,0), \quad X(0) < x < \infty, \tag{2.12}$$

The maximum principle yields that $W(x,\tau)$ is uniformly bounded and the bound depends only on known data. By using the same argument, we can easily deduce the uniform bound for $V^N(s,t)_s$.

Q.E.D.

Let $\delta > 0$ be a small number and define

$$Q_T(\delta) = Q_T \cap \left\{(s,t) : 0 \le t \le T - \delta.\right\}.$$

Lemma 2.4: There exists a constant C_3 such that

$$\left\|\left(V^N\right)_{ss}\right\|_{L^\infty(Q_T(\delta))} + \left\|\left(V_{\bar{h}}^N\right)_x\right\|_{L^\infty(Q_T(\delta))} \le C_3,$$

where C_3 depends only on the known data and δ, but not on N.

Proof: From the theory of parabolic equations, we may assume that $U(x,\tau)$ is differentiable up to $X(\tau)$. Set

$$P(x,\tau) = U_{xx}(x,\tau), (x,\tau) \in Q_T.$$

From the boundary condition (2.5), we see

$$U_x\left(X(\tau),\tau\right)X'(\tau) + U_\tau\left(X(\tau),\tau\right) = -e^{X(\tau)}X'(\tau).$$

It follows by (2.6) that

$$U_\tau\left(X(\tau),\tau\right) = 0, \quad X(\tau) < x < \infty.$$

From the Equation (2.4) and the boundary conditions (2.5) and (2.6), we see

$$\frac{1}{2}\hat{\sigma}(\tau)P\left(X(\tau),\tau\right) = \left(\hat{r}(\tau) - \frac{1}{2}\hat{\sigma}(\tau)^2\right)e^{X(\tau)} + (\tau)\left(K - e^{X(\tau)}\right),$$

which is uniformly bounded.

By differentiating Equation (2.4) with respect to x twice, we see $P(x,\tau)$ satisfies

$$P_\tau - \frac{1}{2}\hat{\sigma}(\tau)^2 P_{xx} - \left(\hat{r}(\tau) - \frac{1}{2}\hat{\sigma}(\tau)^2\right)P_x + \hat{r}(\tau)P = 0, (x,\tau) \in Q_T^1.$$

For any $\delta > 0$, the Schauder's theory implies that $P(x,\delta) = U_{xx}(x,\delta)$ is uniformly bounded and the bound depends on known data and δ. Now we can apply the maximum principle again on $Q_T(\delta)$ to conclude that $P(x,\tau)$ is uniformly bounded. One can also use the same argument for W to conclude the estimate for $U_{xt}(x,\tau)$ in $Q_T(\delta)$. Similar estimates hold for the discretized solution $V^N(s,t)s$ and $\left(V_{\bar{h}}(s,t)^N\right)_x$.

Q.E.D.

Lemma 2.5: There exists a constant C_4 such that

$$\int_0^{T-\delta} \left|\frac{\mathrm{d}}{\mathrm{d}t}S^N(t)\right|^2 \mathrm{d}t \le C_4,$$

where C_2 depends only on known data and δ, but not on N.

Proof: First of all, $S^N(t)$ is continuous and is also differentiable on $[0,T]$ except $t = t_n, n = 0,1,\cdots N$. It follows that $S^N(t) \in H^1(0,T)$.

From the definition of $V^N(s,t)$ and the boundary condition (2.2), we know that, for $t_{n-1} \le t < t_n$,

$$\frac{\mathrm{d}}{\mathrm{d}t}S^N(t) = \frac{S_n - S_{n-1}}{h}.$$

Note that $S_{n-1} < S_n$, then

$$S_n - S_{n-1} = \frac{V_n(S_n) - V_n(S_{n-1})}{h}$$
$$= \frac{V^N(S_n,t) - V^N(S_{n-1},t)}{h} = \frac{1}{h}\int_{S_{n-1}}^{S_n} V^N(y,t)_y \,\mathrm{d}y$$

It follows that

$$\int_0^{T-\delta}\left|\frac{\mathrm{d}}{\mathrm{d}t}S^N(t)\right|^2 \mathrm{d}t = \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\left|\frac{\mathrm{d}}{\mathrm{d}t}S^N(t)\right|^2 \mathrm{d}t$$
$$\le C\left\|V_h^N\right\|_{L^\infty(Q_T(\delta))} \le C_5,$$

where C_5 depends only on known data and δ.

Q.E.D.

With the results of Lemmas 2.1-2.5, we are ready to prove the following theorem.

Theorem 2.6: The free boundary problem (1.1)-(1.5) has a unique solution $(V(s,t), S(t))$ with $(V(x,t) \in L0,T;W^{1,2}(\Omega))$ and $S(t) \in C^{1+\alpha}[0,T)$.

Proof: First of all, the existence of a weak solution $U(s,t)$ in Q_T follows the exactly same argument as that in [34] (Theorem 5.1, page 138). The uniqueness follows from the variational inequality. Moreover, regularity theory for parabolic equation implies that

$$U(s,t) \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T) \cap C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T^1).$$

Moreover, since the coefficients of the Equation (2.4) depends only on τ, we use the interior regularity of parabolic equations to conclude that

$$U_s \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T^1(\delta)).$$

To see the regularity of the free boundary, we use Lemma 2.5 to see $S^N(t) \in H^1(0,T-\delta)$ and

$$\left\|S^N\right\|_{H^1(0,T-\delta)} \le C_5.$$

It follows that

$$\sup_{0\le t_1,t_2\le T-\delta}\frac{\left|S^N(t_1) - S^N(t_2)\right|}{\sqrt{|t_1 - t_2|}} \le C.$$

Hence, by Ascoli-Arzela's lemma, we can extract a subsequence, still denoted by $S^N(t)$, such that $S^N(t)$ converges to a function, denoted by $S(t)$. Moreover, $S(t) \in C^{\frac{1}{2}}[0,T-\delta]$. Since $\delta > 0$ is arbitrarily, we have $S(t) \in C^1 2[0,T)$.

Furthermore, since $f(x,\tau) \in L^\infty(\hat{Q}_T)$, we use $W_p^{2,1}$-estimate to obtain that for any $p > 1$,

$$\|U\|_{W_p^{2,1}(\hat{Q}_T(\delta))} \le C,$$

where C depends only on known data, p and δ.

Now we convert back to the original variables to conclude that

$$V(s,t) \in W_p^{2,1}(Q_T(\delta)).$$

By Sobolev's embedding, we know that $V(s,t)$ and $V_s(s,t)$ are continuous over Q_T. On the other hand, since

$$V(s,t) = K - s, \quad 0 \le s \le S(t), 0 \le t \le T,$$

we obtain

$$V(S(t),t) = K - S(t), V_s(S(t),t) = -1,$$
$$0 < t < T.$$

To see more regularity for $S(t)$, we use the boundary condition (2.5)-(2.6). Indeed, from the condition (2.5)-(2.6), we see

$$U_\tau(X(\tau),\tau) = 0, \delta < \tau \le T.$$

We differentiate (2.6) to find

$$U_{xx}(X(\tau),\tau)X'(\tau) + U_{x\tau}(X(\tau),\tau)$$
$$= -e^{X(\tau)}X'(\tau), \delta < \tau \le T.$$

From the Equation (2.4) we obtain

$$U_{xx}(X(\tau),\tau) = -e^{X(\tau)} + \frac{2K\hat{r}(\tau)}{\sigma(\tau)}.$$

It follows that

$$X'(\tau) = -\frac{\hat{\sigma}(\tau)}{2K\hat{r}(\tau)}U_{x\tau}(X(\tau),\tau), \delta < \tau \le T.$$

Now we consider the free boundary problem for $W(x,\tau) = U_x(x,\tau)$ in $Q_T(\delta)$:

$$W_\tau - \frac{1}{2}\hat{\sigma}(\tau)^2 W_{xx} - \left(\hat{r}(\tau) - \frac{1}{2}\hat{\sigma}(\tau)^2\right)W_x + \hat{r}(\tau)W$$
$$= 0, (x,\tau) \in \hat{Q}_T(\delta), loca$$
$$W(X(\tau),\tau) = -e^{X(\tau)}, \quad \delta < \tau \le T, locall$$
$$X'(\tau) = -\frac{\hat{\sigma}(\tau)}{2K\hat{r}(\tau)}W_\tau(X(\tau),\tau), \delta < \tau \le T,$$
$$W(x,0) = U_x(x,0), \quad X(0) < x < \infty, local$$

It is easy to see that a unique solution $(W(x,\tau), X(\tau))$ exists with $X(\tau) \in C^{1+\alpha}[\delta,T]$. It follows that

$$S(t) \in C^{1+\alpha}[0,T-\delta].$$

Q.E.D.

Remark 2.1: For the existence and uniqueness, we only need to assume that $\sigma(t)$ and $r(t)$ are of class $L^\infty(0,T)$ with a positive lower bound for $\sigma(t)$.

3. Properties of Free Boundary

As we mentioned in the introduction, we are interested in how the free boundary changes when $\sigma(t)$ changes. It turns out that a comparison principle holds.

Theorem 3.1: Let $\sigma_1(t)$ and $\sigma_2(t)$ satisfy the assumption H(1). Let $\left(V_1(s,t),S_1(t)\right)$ and $\left(V_2(s,t),S_2(t)\right)$ be the solutions of the problem (1.1)-(1.5) corresponding to $\sigma_1(t)$ and $\sigma_2(t)$.

If $\sigma_1(t)\le\sigma_2(t)$ on $[0,T]$, then

$$V_1(s,t)\le V_2(s,t),\quad S_1(t)\ge S_2(t),$$
$$0<t<T.$$

To prove the theorem, we show that the comparison property holds for the discrete solution under certain condition.

Lemma 3.1: If $h<\dfrac{1}{b^2+R}$, then

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}V_n(s)\ge 0,\quad S_n<S<\infty.$$

Proof: If necessary, we may use an approximation to replace $V_N(s)=(K-s)^+$ by a smooth convex function on $[S_N,\infty)$. Without loss of generality, we may simply assume $V_N(s)''\ge 0$. Then from the regularity theory, we know that $V_n(s)$ is differentiable in (S_n,∞). Let

$$u_n(s)=V_n''(s),\quad S_n<s<\infty.$$

Now for $n=N-1$, we differentiate the Equation (2.1) twice with respect to s to see that $u_n(s)$ satisfies the following equation:

$$\frac{1}{2}\sigma_n^2s^2u_n''+(2\sigma_n+r_n)su_n'+\left(\sigma_n+r_n-\frac{1}{h}\right)u_n$$
$$=-\frac{1}{h}V_N(s)''\le 0,S_n<s<\infty.$$

From the maximum principle, we see that $u_n(s)$ can not attain a negative minimum if $\sigma_n^2+r_n-\dfrac{1}{h}<0$.

On the other hand, from the Equations (2.1) and (2.1) we see

$$u_n(S_n)=\frac{2r_n}{(\sigma_nS_n)^2}.$$

It follows that, if $h<\dfrac{1}{b^2+R}$,

$$u_n(s)\ge 0,\quad S_n<s<\infty.$$

Once we know $u_{N-1}(s)''\ge 0$, we can use the maximum principle to obtain the same conclusion for $n=N-2$. After a finite number of steps, we obtain the desired result of Lemma 3.1.

Q.E.D.

Since we are interested in the relation between $\left(V_n(s),S_n\right)$ and σ_n, for convenience we use $V_n(s,\sigma):=V_n(s)$ and $S_n(\sigma):=S_n$ instead of $V_n(s,\sigma_n)$ and $S_n(\sigma_n)$.

Lemma 3.2: For $n=0,1,\cdots,N-1$,

$$\frac{\partial V_n(s,\sigma)}{\partial\sigma}>0,\quad \frac{\mathrm{d}}{\mathrm{d}\sigma}S_n(\sigma)<0.$$

Proof: Let

$$W(s,\sigma)=\frac{\partial V_n(s,\sigma)}{\partial\sigma}.$$

We differentiate Equation (2.1) for $n=N$ with respect to $\sigma=\sigma_n$ to obtain:

$$LW:=\frac{1}{2}\sigma^2s^2W''+rsW'-\left(r_n+\frac{1}{h}\right)W=-\sigma s^2\frac{\mathrm{d}^2V_n}{\mathrm{d}s^2}.$$

From Lemma 3.1, we see that

$$\frac{1}{2}\sigma^2s^2W''+rsW'-\left(r_n+\frac{1}{h}\right)W\le 0,S_n<s<\infty.$$

The maximum principle implies that $W(s,\sigma)$ can not attain a negative minimum at an interior point in (S_n,∞).

On $S_n=S_n(\sigma)$:

$$V_n\left(S_n(\sigma),\sigma\right)=K-S_n(\sigma),V_{ns}\left(S_n(\sigma),\sigma\right)=-1.$$

We differentiate $V_n\left(S_n(\sigma),\sigma\right)$ with respect to σ to obtain

$$W\left(S_n(\sigma),\sigma\right)=0.$$

It follows that $W(s,\sigma)\ge 0$ for $s\in(S_n,\infty),\sigma\in[a,b]$ when $n=N$. Now we can use the same argument to obtain the same conclusion for $n=N-1,\cdots,0$.

Moreover, from the second boundary condition, we have

$$V_{nss}\left(S_n(\sigma),\sigma\right)\frac{\mathrm{d}S_n(\sigma)}{\mathrm{d}\sigma}+V_{ns\sigma}\left(S_n(\sigma),\sigma\right)=0.$$

Also, from Equation (2.1) we know

$$V_{nss}=\frac{2r_n}{(\sigma_nS_n)^2}.$$

It follows that

$$\frac{\mathrm{d}S_n(\sigma)}{\mathrm{d}\sigma}=-\frac{(\sigma_nS_n)^2}{2r_n}W_s\left(S_n(\sigma),\sigma\right).$$

Since $W(s,\sigma)$ attains its minimum 0 at the boundary $S_n(\sigma)$, by Hopf's lemma, we see $W_s\left(S_n(\sigma),\sigma\right)>0$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}S_n(\sigma)<0.$$

Q.E.D.

Now we are ready to prove the main theorem in this section.

Proof of Theorem 3.1: Let $\sigma_1(t)$ and $\sigma_2(t)$ satisfy the assumption H(1). Let $(V_1(s,t), S_1(t))$ and $(V_2(s,t), S_2(t))$ be the solutions of (1.1)-(1.5) corresponding $\sigma_1(t)$ and $\sigma_2(t)$. If $\sigma_1(t) < \sigma_2(t)$ on $[0,T]$. We define

$$\sigma_{ik} = \frac{1}{h}\int_{t_{k-1}}^{t_k} \sigma_i(\tau)\,\mathrm{d}\tau, i = 1,2; k = 1,2,\cdots,N.$$

Let $(V_{in}(s), S_{in})$ be the solution of the problem (2.1)-(2.3) corresponding to the volatility σ_{in}. It is clear that $\sigma_{1n} < \sigma_{2n}$ for $n = 1,2,\cdots,N$ if $\sigma_1(t) < \sigma_2(t)$ on $[0,T]$. By Lemma 3.1 and Lemma 3.2, if $h < \frac{1}{b^2 + R}$ we have

$$S_{1n}(\sigma_{1n}) > S_{2n}(\sigma_{2n}), \quad n = 0,1,2,\cdots,N.$$

From the definition of $S^N(t)$, we know that

$$S_1^N(t) > S_2^N(t), \quad 0 \le t \le T,$$

provided that $h < \frac{1}{b^2 + R}$.

Since $S_1^N(t)$ and $S_2^N(t)$ are uniformly convergent to $S_1(t)$ and $S_2(t)$, respectively, as $N \to \infty$. It follows that

$$S_1(t) \ge S_2(t), \quad 0 \le t \le T.$$

It is also clear that $V_1(s,t) \le V_2(s,t)$ on Q_T.

Q.E.D.

Remark 3.1: It is clear that the comparison result in Theorem 3.1 still holds if $\sigma(t) \in L^\infty(0,T)$ with a positive lower and upper bounds.

4. Conclusion

When the volatility is a constant, it has been known for a long time that the option price is bigger when the volatility is bigger. However, when the volatility is a function of time, $\sigma = \sigma(t)$, it is not clear how the option price nor the optimal excise boundary change when the volatility changes for the whole time period $[0,T]$. In this paper we answered such a question. We show that a comparison property for option price and the optimal excising boundary hold (Theorem 3.1) when the volatility $\sigma_1(t) \le \sigma_2(t)$. This result is important for option traders. Moreover, we proved a global regularity result for the free boundary by using a very different method from the existing literature.

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