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On Some Class of Distance Functions for Measuring Portfolio Efficiency

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Abstract

Morey and Morey [1] have developed an approach for gauging portfolio efficiencies in the context of the Markowitz model. Following some recent contributions [2,3], this paper analyzes the axiomatic properties of distance functions extending an earlier approach proposed by Morey and Morey. The paper also focuses on the hyperbolic measure and the McFadden gauge function [4]. Among other things, overall, allocative and portfolio improvements possibilities (in term of return expansion or/and risk contraction) based upon the indirect mean-variance utility function are analyzed. Along this line, duality results are established in each case. This enables us to calculate the degree of risk aversion maximizing the investor indirect mean-variance utility function in either return expansion or risk contraction. An empirical illustration is provided and reveal ranking of preferred risks aversion for some "CAC40" assets.

1. Introduction

Distance functions, have been introduced by Shephard to measure by Shephard [5] for efficiency measurement either in input or output orientation. At the same time, Markowitz [6,7] has formulated the mean-variance model, a mathematical approach for determining the optimal riskreturn trade-off for portfolio selection. This approach is based upon quadratic programming. However, its computational cost was very high. Hence, Sharpe [8,9] had developed the simplified diagonal model and later formulated the capital asset pricing model (CAPM) with Lintner [10]. Markowitz [11] criticized the relation between risk and excess returns described by the linear model due to Sharpe and Lintner. He argued that different expected returns might surely be obtained from the same risk structure.

Nevertheless, the mean-variance approach is the cornerstone of portfolio management and risk assessment. The purpose of this paper is to provide a general taxonomy of ratio-based performance indicator for risk management. This contribution extends the analysis proposed by Morey and Morey [1] and also provides a new look at some more recent contributions. In [2,3] a general

framework was introduced that is based upon the *shortage function* a concept introduced by Luenberger [12] in microeconomic analysis. Transposed in a portfolio optimization context, this function looks for possible simultaneous improvement of return and reduction of risk in the direction of a vector g. Though this approach generalizes that of Morey and Morey in the mean-variance space, the choice of a direction remains much arbitrary. In this paper, we make other investigations about measures involving a proportionate improvement of risk and return. It is shown that the measure proposed by Morey and Morey satisfies a special type of duality, termed "fractional duality". It is also established that one can obtain a duality result linking the indirect mean-variance utility both to the hyperbolic measure and the McFadden gauge function [4]. Some mathematical programs are also proposed and we propose a procedure to measure the risk aversion from the Khum and Tucker multipliers.

Among other things we propose a procedure to compute the distance functions introduced by Morey and Morey in the case where short sales are allowed. Hence, it is then possible to measure the impact the budget constraint has on performance. This we do by introducing a special version of the Thomson metric.

This paper is organized as follow. In Section 2, succinctly presents the basic tools of the portfolio management approach proposed in Briec *et al.* [2]. Section 3 focusses on the distance function proposed by Morey and Morey. We then study some of their more appealing properties. Duality results are analyzed in Section 4 and allow us to decompose efficiency following the Farrell approach [13]. Hence, the preferred risk aversion in input or output orientation can be computed. Sources of performances change are discussed in Section 5. Section 6 introduces an indicator based under the Thomson metric to measure the impact on the performance of managerial constraint. In the next section the dual properties of hyperbolic measures and McFadden gauge are analyzed. They are compared to the return oriented measure. The last section provides an empirical illustration with some "CAC40". A concludingsection outlines conclusions and possible extensions.

2. Efficient Frontier and Portfolio Management

This section introduces main ideas of the portfolio selection problem. Let us consider a market with n financial assets. Note $E[R_i]$ for $i = 1, \cdots, n$ the expected return of the asset i and the covariance matrix of these assets such that $\Omega_{i,j} = Cov\left[R_i, R_j\right]$ for $i, j \in \{1, \cdots, n\}$. A portfolio is an combination of one or more of these assets. Their proportions may be represented by the vector $x = (x_1, x_2, \cdots, x_n)$ with $\sum_{i=1}^{n} x_i = 1$ and $x_i > 0$ if short sales is not allowed.

It is assumed throughout the paper that economic constraints (Pogues [14], Rudd and Rosenberg [15]) are linear functions of the asset weights. Thus, the set of the admissible portfolios may be written as follow :

$$\mathfrak{I}_A = \left\{ x = \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, Ax \geq 0 \right\}, \qquad (2.1)$$

where A is an affine $m \times n$ map whose the range on $\mathbb{R}^n$ is a subset of $\mathbb{R}^m$. If Ax is null for all $x \in \mathbb{R}^n$ then we say that short-sales are allowed. In such a case, the set of admissible portfolios is extended from the unit simplex to an unbounded hyperplane. In general, the constraint $Ax \geq 0$ represents the economic and managerial constraints the manager must deal with. The return of portfolio x is $R(x) = \sum_{i=1}^{n} x_i R_i$. The expected return and its variance can be defined as follows: $E\left[R(x)\right] = \sum_{i=1}^{n} x_i E\left[R_i\right]$ and $Var\left[R(x)\right] = \sum_{i,j}^{n} x_i x_j Cov\left[R_i, R_j\right]$, respectively. For the sake of simplicity, let us denote

$$\mu(x) = E\left[R(x)\right] \text{ and } \rho(x) = Var\left[R(x)\right]. \quad (2.2)$$

In addition, we consider the map $\Phi : \mathfrak{I}_A \to \mathbb{R}^2$ defined by

$$\Phi(x) = \left(\rho(x), \mu(x)\right) \qquad (2.3)$$

See for instance [20] for more details about mean-variance approaches and stochastic dominance. The return and the variance are continuous in x. Hence $\Phi(\mathfrak{I}_A)$ is a compact subset of $\mathbb{R}^2$. Following the Markowitz approach, the subset $\Phi(\mathfrak{I}_A)$ is important to identify the efficient portfolios. However, it is not convex and, consequently, this subset cannot be used for a quadratic programming approach. Briec, Lesourd and Kerstens [2] have extended the subset $\Phi(\mathfrak{I}_A)$ as follows:

$$DR_A = \Phi(A) + \left(\mathbb{R}_+ \times \mathbb{R}_-\right) \qquad (2.4)$$

It is important to notice that DR_A is convex. Equivalently, one has:

$$DR_A = \left\{(r, m) \in \mathbb{R}^2 : \right.$$
$$\left. \exists x \in \mathfrak{I}_A, (-r, m) \leq \left(-\rho(x), \mu(x)\right)\right\} \qquad (2.5)$$

This set is not only compatible with the definition in Markowitz [6], it guarantees a minimum variance of the feasible portfolios analogously to a "Free-Disposal Hull". The subset of the all the mean-variance points that are not strictly dominated is termed the "efficient frontier". It is useful to define the efficient portfolios from the above definition.

Definition 2.1. *The set of the weakly efficient portfolios n the simplex is defined as*:

$$\Lambda^M\left(\mathfrak{I}_A\right) = \left\{ x \in \mathfrak{I} : \left(-\rho(x), \mu(x)\right) < (-r, m) \right.$$
$$\left. \Rightarrow (r, m) \notin DR_A \right\}$$

For a given degree of risk aversion, Markowitz [6] defined the following utility function to compute the corresponding efficient portfolio.

$$U_{\alpha,\beta}^A(x) = \beta\mu(x) - \alpha\rho(x)$$

where $\alpha \geq 0$ and $\beta \geq 0$. The following program maximize this mean-variance utility function.

$$\max U_{\alpha,\beta}^A(x) = \beta\mu(x) - \alpha\rho(x)$$
$$s.t. \; Ax \geq 0$$
$$\sum_{i-1}^{n} x_i = 1, \; x \geq 0$$

where the ratio $\varphi = \alpha/\beta \in [0, +\infty]$ stands for risk aversion.

3. Portfolio Efficiency Measures

Measuring efficiency in a portfolio context accounts usually the possible return improvements and/or risk contractions. In this paper, we propose some class of measures which consider investor preferences for risk.

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3.1. The Morey and Morey Distance Functions

Two distance functions were introduced by Morey and Morey to gauge portfolio performance. The first function computes the maximum expansion of the mean of return for a given level of risk. The map $D_{RE}^A(x) : \Im \to [1, +\infty]$ defined as:

$$D_{RE}^A(x) = \sup\left\{\theta : (\rho(x), \theta\mu(x)) \in DR_A\right\} \quad (3.1)$$

is called the Morey return expansion distance function. It is easy to see that this measure has some drawbacks for portfolios whose the return is not positive. Thus, we shall restrict its domain to the subset of $\Im$ defined by:

$$\Im_A^+ = \left\{x \in \Im_A : \mu(x) > 0\right\}. \quad (3.2)$$

Focusing on the risk contraction, the map $D_{RC}^A : \Im \to [0,1]$ defined by:

$$D_{RC}^A(x) = \inf\left\{\lambda : (\lambda\rho(x), \mu(x)) \in DR_A\right\} \quad (3.3)$$

is called the Morey risk contraction distance function. Notice that this function may be zero valued when there is a riskless asset whose the variance is 0. We propose some of their elementary properties which are essentials in portfolio performance gauging. To simplify the notations, we introduce the partial order $\succeq$ defined by:

$$x \succeq y \Leftrightarrow (-\rho(x), \mu(x)) \ge (-\rho(y), \mu(y)). \quad (3.4)$$

Proposition 3.1. Let D_{RE}^A be the mean expansion return distance defined in (3.1). D_{RE}^A has the following properties:

i) $x \in \Im_A^+ \Rightarrow 1 \le D_{RE}^A(x) \le +\infty$

ii) $D_{RC}^A(x) = 1 \Leftrightarrow \Lambda^M(\Im_A)$ (*Weak efficiency*).

iii) $\forall x, y \in \Im$, if $x \succeq y$ then $D_{RE}^A(y) \succeq D_{RE}^A(x)$ (*Weak monotonicity*).

iv) D_{RE}^A is continuous on $\Im_A$.

Proof. Let us prove *i*). The first inequality is immediate. From the definition of the representation set, if $x \in \Im_+$ then the subset

$$C(x) = \left\{(v, e) \in DR_A : (v, -e) \le (\rho(x), -\mu(x))\right\}$$

is bounded. It trivially follows that $D_{RE}(x) < +\infty$. To prove *ii*), assume that $x \notin \wedge^M(\Im_A)$. In such a case, there exists some $(v, e) \in DR_A$ such that $(-v, e) \le (-\rho(x), \mu(x))$. But, from Definition (3.1), it immediately follows that $D_{RE}^A(x) > 1$. Consequently, it can be deduced that $D_{RE}^A(x) = 1 \Rightarrow x \in \Lambda^M(\Im_A)$. To prove the converse, assume that $D_{RE}^A(x) > 1$. We get:

$$(-\rho(x), D_{RE}^A(x)\mu(x)) \le (-\rho(x), \mu(x)).$$

It can immediately be deduced that $x \notin \wedge^M(\Im_A)$. Let us prove *iii*). $\forall x, y \in \Im$, if $\rho(y) \ge \rho(x)$ and $\mu(y) \le \mu(x)$ then $D_{RE}^A(y) \le D_{RE}^A(x)$. From the notations

above, we have $C(x) \subset C(y)$, for all $x, y \in \Im_A^+$. Consequently,

$$\left\{\theta : (\rho(x), \theta\mu(x)) \in DR_A\right\} \subset \left\{\theta : (\rho(y), \theta\mu(y)) \in DR_A\right\},$$

and the result follows. We end by proving iv). Let $T : DR_A \to R_+$ be the function defined by

$$T(r, m) = \sup\left\{\theta : (r, \theta m) \in DR_A\right\}$$

Since DR_A is convex and satisfies the free disposal rule, it is easy to show the continuity of T (see Shephard [5]). Hence $D_{RE}^A(x)$ is continuous on $\Im_A$.

The next result analyzes the case where the distance function involves a risk contraction of portfolios.

Proposition 3.2. Let D_{RC}^A be the risk contraction distance function defined in (3.3). We have the following properties.

i) If there is no riskless asset then $0 < D_{RC}^A(x) \le 1$

ii) For all $x \in \Im_A$, $D_{RC}^A(x) = 1 \Leftrightarrow D_{RE}^A(x) = 1 \Leftrightarrow x \in \wedge^M(\Im_A)$ (*Weak efficiency*)

iii) $\forall x, y \in \Im_A$, $x \succeq y \Leftrightarrow D_{RC}^A(y) \ge D_{RC}^A(x)$ (*Weak monotinicity*).

iv) D_{RC}^A is continuous on $\Im_A$.

Proof. With obvious changes, the proofs are similar to those of Proposition (3.1).

3.2. The Efficiency Improvement Possibility Function

To gauge portfolio efficiency, Briec, Lesourd and Kerstens [2] have introduced a variation of the shortage function which computes simultaneously risk reduction and expected return improvement. For a portfolio x in DR, the direction of the shortage function is determined by the vector g = $(-g_V, g_E)$. Formally, this efficiency improvement possibility function is defined by:

$$S_g(x) = \sup\left\{\delta; (\rho(x) - \delta g_V, \mu(x) + \delta g_E) \in DR\right\} \quad (3.5)$$

The following quadratic program computes the maximum percentage improvement of the portfolio y_k:

$$\max \delta$$

$$s.t. \ \mu(y_k) + \delta g_E \le \sum_{i-1}^n x_i E[R_i]$$

$$\rho(y_k) - \delta g_V \ge \sum_{i=1}^n x_i V[R_i]$$

$$Ax \le b, \ \sum_{i=1 \cdots n} x_i = 1, \ x_i \ge 0, \ i = 1 \cdots n$$

For more details of the basic properties about this function, see for instance Briec *et al.* [2].

3.3. Hyperbolic and McFadden Distance Function

We introduce two distance functions defined in the mean-

variance graph. We analyze the basic properties and establish a duality result. Measuring efficiency in a portfolio context accounts the possible return improvements and/or risk contractions. In the following, we introduce two specific measures. The first one, called hyperbolic distance function, computes the maximum simultaneous shrinkage and expansion of the risk and expected return respectively. It is defined as:

$$D_H^A(x) = \sup\left\{\theta : (\frac{1}{\theta}\rho(x), \theta\mu(x)) \in DR_A\right\} \qquad (3.6)$$

We also introduce the McFadden gauge that computes the maximum proportionate expansion of the risk and expected return respectively:

$$G_{MF}^A(x) = \sup\{\theta : (\theta\rho(x), \theta\mu(x)) \in DR_A\} \qquad (3.7)$$

We propose some of their elementary properties which are essentials in portfolio performance gauging. **Figure 1** illustrate the basic ideas behind the definitions above. The "hyperbolic distance function" simultaneously involves a contraction of the risk and an expansion of the expected return. The "McFadden gauge function" is very different because it computes simultaneously the maximum expansion of the risk and the expected return of investors.

Proposition 3.3. *The map* D_H^A *defined in* (3.1) *has the following properties*:

i) $x \in \mathfrak{I} \Rightarrow D_H^A(x) < +\infty$

ii) if $\mu(x) \geq 0$ *, then* $D_H^A(x) = 1 \Leftrightarrow x \in \partial^M(\mathfrak{I}_A)$ *(Weak efficiency).*

iii) $\forall x, y \in \mathfrak{I}$ *, if* $x \succeq y$ *then* $D_H^A(y) \leq D_H^A(x)$ *(Weak monotonicity).*

iv) D_H^A *is continuous on* $\mathfrak{I}_A$ *.*

Proof. The proofs are very similar to those concerning the maximum risk expansion distance function and thus it is omitted.

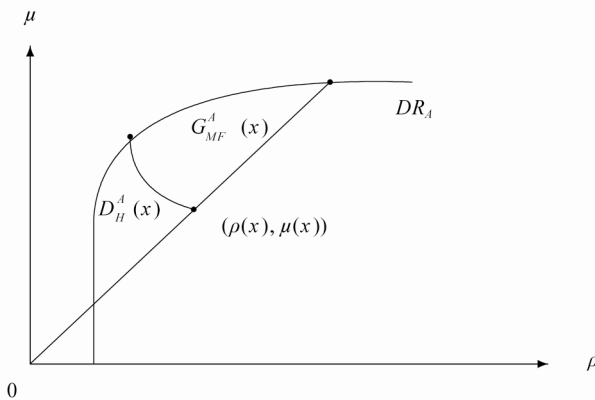The mathematical program, one need to solve is the following:

Figure 1. Hyperbolic Distance Function and McFadden Gauge.

$$D_H^A(y) = \max \lambda$$

$$s.t. \ \theta\mu(y) \leq \sum_{i=1}^n x_i E[R_i]$$

$$\frac{1}{\theta}\rho(y) \geq \sum_{i,j=1}^n \Omega_{i,j} x_i x_j \qquad (P_H)$$

$$\sum_{i=1}^n x_i = 1, \ Ax \geq 0.$$

The next result analyzes the case where the distance function involves a risk contraction of portfolios. In the following we denote $(G_{MF}^A) = \{x \in \mathfrak{I}_A : G_{MF}^A < +\infty\}$.

Proposition 3.4. *Let* G_{MF}^A *be the map defined in* (3.3). *We have the following properties.*

i) If $\mu(x) > 0$ *then* $G_{MF}^A(x) < +\infty$ *.*

ii) $G_{MF}^A(x) = 1 \Rightarrow x \in \partial^M(\mathfrak{I}_A)$ *(Weak efficiency)*

iii) $\forall x, y \in \mathfrak{I}_A$,
$(\rho(y), \mu(y)) \leq (\rho(x), \mu(x)) \Leftrightarrow G_{MF}^A(y) \leq G_{MF}^A(x)$.

Proof. i) Since $\mathfrak{I}_A$ is a compact set it follows that $\Phi(\mathfrak{I}_A)$ is a compact subset of $\mathbb{R}^2$. Therefore it is norm bounded. Consequently, by definition, DR_A is bounded in its return dimension. Thus $G_{MF}^A(x) < +\infty$. *ii)* Clearly, if $G_{MF}^A(x) = 1$ then the point $(\rho(x), \mu(x))$ lies on the upper part of the mean-variance frontier. However, all these frontier points are efficient which ends the proof. The proof of *iii)* is immediate.

Notice that, in general, the gauge function is not continuous on $\mathfrak{I}_A$. Moreover, it does not characterize the weak efficient frontier. The next result offers a comparison between all the distance functions introduced in the paper.

Lemma 3.5. *For all* $x \in \mathfrak{I}_A$ *and such that* $\mu(x) > 0$ *, we have:* $D_H^A(x) \leq D_{RE}^A(x) \leq G_{MF}^A(x)$ *.*

Proof. Clearly, we have the inclusion

$$\left\{\left(\theta : \frac{1}{\theta}\rho(x), \theta\mu(x)\right) \in DR_A\right\} \subset \{\theta : (\rho(x), \theta\mu(x)) \in DR_A\} ,$$

hence we obtain the first inequality. To prove the second one, fix $\bar\theta = D_{RE}^A(x)$. By definition $(\rho(x), \bar\theta\mu(x)) \in DR_A$. However, by construction $DR_A = DR_A + (\mathbb{R}_+ \times \mathbb{R}_-)$. Since $\bar\theta \geq 1$, we then deduce that $(\bar\theta\rho(x), \theta\mu(x)) \in DR_A$. Thus, $G_{MF}^A(x) \geq \bar\theta$ which ends the proof.

Suppose that $\mu(y) > 0$, the mathematical program one should compute is the following:

$$G_{MF}^A(y) = \max \theta$$

$$s.t. \ \theta\mu(y) \leq \sum_{i=1}^n x_i E[R_i]$$

$$\theta\rho(y) \geq \sum_{i,j=1}^n \Omega_{i,j} x_i x_j \qquad (P_{MF})$$

$$\sum_{i=1}^n x_i = 1, \ Ax \geq 0.$$

4. Duality and Graph Distance Functions

Following our earlier results, we establish a link between these distance functions and the indirect mean-variance utility function, except for the shortage function which has already been discussed in [2]. The indirect mean-variance utility gives the portfolio which achieves a maximal utility of an agent given his (her) risk aversion. Thus, fixing some parameters α, β, it is well known that an efficient portfolio maximizing the utility can be calculated using standard procedures of quadratic optimization. The mean-variance indirect utility function is defined by $V^A : \mathbb{R}_+^2 \to \mathbb{R}$ by:

$$V^A(\alpha, \beta) = \sup\{\beta\mu(x) - \alpha\rho(x) : x \in \mathfrak{I}_A\} \quad (4.1)$$

This function associates to the pair (ρ, μ), which stands for the risk aversion, the maximal utility level a fund manager can expect.

4.1. Morey and Morey Fractional Duality

Morey and Morey's approach allows to distinguish two dual relations with the indirect mean-variance utility functions. They are expressed in term of return expansion and risk contraction respectively. We show that the distance functions we have introduced can be related to the maximisation of this utility functions with an optimal degree of risk aversion. Following Briec, Kerstens and Lesourd [2], duality results allows a decomposition of efficiency measures. This is done paralleling an earlier concept due to Farrell [13] in a production theory context. However, in view of the nature of the tools they used, these duality results have an additive structure, while the measure proposed by Morey and Morey has a multiplicative one. In this subsection, we shall prove that the distance functions proposed by Morey and Morey also have a dual interpretation based upon the indirect mean-variance utility function.

4.1.1. Efficiency Decompositions
Given portfolio, the overall return expansion is the ratio computed as its maximum return by its return value independently of the asset price information. Suppose that $\mu(x) > 0$, *i.e.* $x \in \mathfrak{I}_A^+$. The overall return expansion (O_{RE}) index is then defined as the quantity:

$$O_{RE}(x, \alpha, \beta)$$
$$= \sup\{\theta : \theta\beta\mu(x) - \alpha\rho(x) \le V^A(\alpha, \beta)\}. \quad (4.2)$$

Hence, one can equivalently write

$$O_{RE}(x, \alpha, \beta) = \frac{V^A(\alpha, \beta) + \alpha\rho(x)}{\beta\mu(x)} \quad (4.3)$$

The allocative return expansion A_{RE} index corres-

ponds to the portfolio reallocation required to achieve the maximum of the the indirect mean-variance utility. It is defined as follow:

$$A_{RE}(x, \alpha, \beta) = O_{RE}(x, \alpha, \beta) / D_{RE}^A(x) \quad (4.4)$$

The portfolio return expansion (P_{RE}) index t is the quantity:

$$P_{RE}(x) = D_{RE}^A(x). \quad (4.5)$$

The Farrell decomposition is then, by definition:

$$O_{RE}(x, \alpha, \beta) = A_{RE}(x, \alpha, \beta) \cdot P_{RE}(x). \quad (4.6)$$

Using a symmetrical approach, one can introduce an overall risk contraction (O_{RC}) index, an allocative risk contraction (A_{RE}) index and a portfolio risk contraction (P_{RC}) index. The overall risk contraction (O_{RC}) index defined as follows:

$$O_{RC}(x, \alpha, \beta)$$
$$= \inf\{\lambda : \beta\mu(x) - (\lambda\alpha)\rho(x) \le V^A(\alpha, \beta)\}. \quad (4.7)$$

Equivalently, one has:

$$O_{RC}(x, \alpha, \beta) = \frac{\beta\mu(x) - V^A(\alpha, \beta)}{\alpha\rho(x)} \quad (4.8)$$

The allocative risk contraction index is

$$A_{RC}(x, \alpha, \beta) = O_{RC}(x, \alpha, \beta) / D_{RC}^A(x), \quad (4.9)$$

and the portfolio risk-contraction index is:

$$P_{RC}(x) = D_{RC}^A(x). \quad (4.10)$$

By definition, we have:

$$O_{RC}(x, \rho, \mu) = A_{RC}(x, \alpha, \beta) \cdot P_{RC}(x) \quad (4.11)$$

If the overall risk contraction $O_{RC} = P_{RC} = 1$ then we have an efficient portfolio and the allocative risk contraction $A_{RC} = 1$ is certainly equal to one.

4.1.2. Implicit Risk Aversion
Duality between indirect utility functions and Morey and Morey's distance functions involves an implicit risk aversion which makes the selected portfolio optimal regarding to the mean-variance utility function.

Allocative return expansion may change with respect to the risks aversion parameter. Given portfolio, one can calculate by how much the level of $\phi = \rho / \mu$ needs to be increased to optimize overall portfolio efficiency. The earlier decompositions show that Allocative return expansion is greater than zero excepted whenever the selected portfolio lies on the indifference curve of the mean-variance utility function at its optimal level. Hence, if $O_{MRE} = \theta^*$, we have:

$$V^A(\alpha, \beta) = \beta\theta^*\mu(x) - \alpha\rho(x) \quad (4.12)$$

It follows that:

$$\theta^* = \frac{V^A(\alpha,\beta) + \alpha\rho(x)}{\beta\mu(x)} \qquad (4.13)$$

It follows that $\theta^* \geq D_{RE}^A(x)$. Hence, we have:

$$D_{RE}^A(x) \leq \frac{V^A(\alpha,\beta) + \alpha\rho(x)}{\beta\mu(x)}. \qquad (4.14)$$

Proposition 4.1. *For all portfolio* $x \in \mathfrak{I}_A$ *such that* $\mu(x) > 0$, *we have*

$$D_{RE}(x) = \inf\left\{\frac{V^A(\alpha,\beta) + \alpha\rho(x)}{\beta\mu(x)} : (\alpha,\beta) \geq 0\right\}.$$

Proof. From its definition, the representation set that is convex, then it is the intersection of all its supporting hyperplanes (see [2]). Hence, we have:

$$DR_A = \bigcap_{(\alpha,\beta)\geq 0}\{(v,e) \in \mathbb{R}^2 : \beta e - \alpha v \leq V^A(\alpha,\beta)\}.$$

We can equivalently write

$$D_{RE}^A(x) = \inf\{\theta : (\rho(x),\theta\mu(x)) \notin DR_A\}.$$

Let us denote $H(\alpha,\beta) = \{(v,e) \in \mathbb{R}^2 : \beta e - \alpha v \leq V^A(\alpha,\beta)\}$ for all $(\alpha,\beta) \geq 0$. It follows that

$$D_{RE}^A(x) = \inf\left\{\theta : (\rho(x),\theta\mu(x)) \in \mathbb{R}^2 \setminus \bigcap_{(\alpha,\beta)\geq 0} H(\alpha,\beta)\right\}.$$

Since

$$\inf\{\theta : (\rho(x),\theta\mu(x)) \in H(\alpha,\beta)\} = \frac{V^A(\alpha,\beta) + \alpha\rho(x)}{b\mu(x)},$$

we immediately deduce the result.

Paralleling the approach above, one can consider the situation where the risk of a given portfolio is contracted while fixing the returns at an arbitrary level. Following, Farrell decompositions, Allocative risk contraction is equal to one if a portfolio maximizes the mean variance-utility function. In such a case the Overall risk contraction is equal to one. Hence, in general, if we have $\lambda^* = O_{RC}$, we have

$$V^A(\alpha,\beta) = \beta\mu(x) - \alpha\lambda^*\rho(x)$$
$$\Leftrightarrow \lambda^* = \frac{\beta\mu(x) - V^A(\alpha,\beta)}{\alpha\rho(x)}. \qquad (4.15)$$

Therefore, we deduce that

$$D_{RC}^A(x) \geq \frac{\beta\mu(x) - V^A(\alpha,\beta)}{\alpha\rho(x)} \qquad (4.16)$$

In the next result, we show that one can go a bit further by establishing the following duality result:

Proposition 4.2. *For all portfolio* $x \in \mathfrak{I}_A$, *we have*

$$D_{RC}^A(x) = \sup\left\{\frac{\beta\mu(x) - V^A(\alpha,\beta)}{\alpha\rho(x)} : (\alpha,\beta) \geq 0\right\}.$$

Proof. From the definition of the representation set we have: $DR_A = \bigcap_{(\alpha,\beta)\geq 0}\{(v,e) \in \mathbb{R}^2 : \beta e - \alpha v \leq V^A(\alpha,\beta)\}$. We can equivalently write

$$D_{RC}^A(x) = \sup\{\lambda : (\lambda\rho(x),\mu(x)) \notin DR_A\}.$$

It follows that

$$D_{RC}^A(x) = \sup\left\{\lambda : (\rho(x),\theta\mu(x)) \in \mathbb{R}^2 \setminus \bigcap_{(\alpha,\beta)\geq 0} H(\alpha,\beta)\right\},$$

where $H(\alpha,\beta) = \{(v,e) \in \mathbb{R}^2 : \beta e - \alpha v \leq V^A(\alpha,\beta)\}$ for all $(\alpha,\beta) \geq 0$. Since

$$\sup\{\lambda : (\lambda\rho(x),\mu(x)) \in H(\alpha,\beta)\} = \frac{\beta\mu(x) - V^A(\alpha,\beta)}{\alpha\rho(x)},$$

we immediately deduce the result.

4.1.3. Computing the Implicit Risk Aversion Degree

In this subsection we show how to compute the implicit risk aversion. First, we define the mathematical programs one should deal with to compute the distance functions. Notice that though these programs have some analogies to those proposed in [16] and its subsequent development, they are not non-parametric [17,18]. In fact the frontier has a quadratic functional representation and, consequently the model is parametric. An illustration is proposed in [19,20], where a stochastic frontier approach is proposed. However, the piecewise approximation obtained from the projection onto the frontier is a non-parametric estimation of the disposal representation set (see [2]). The following programs where first proposed by [1]. In the risk-oriented case we have, for all portfolio y:

$$D_{RC}^A(y) = \min \lambda$$
$$s.t. \ \mu(y) \leq \sum_{i=1}^{n} x_i E[R_i]$$
$$\lambda\rho(y) \geq \sum_{i,j=1}^{n} \Omega_{i,j} x_i x_j \qquad (P_{RC})$$
$$\sum_{i=1}^{n} x_i = 1, \ Ax \geq 0.$$

In the return-oriented case, if $\mu(y) > 0$, we have:

$$D_{RE}^A(y) = \max \theta$$
$$s.t. \ \theta\mu(y) \leq \sum_{i=1}^{n} x_i E[R_i]$$
$$\rho(y) \geq \sum_{i,j=1}^{n} \Omega_{i,j} x_i x_j \qquad (P_{RE})$$
$$\sum_{i=1}^{n} x_i = 1, \ Ax \geq 0.$$

It will be proven in the remainder that the implicit risk

aversion can be deduced from the Kuhn-Tucker multipliers of the above programs. Two set valued maps taking into account either a risk-contraction or a return-expansion distance function are now introduced.

Definition 4.3. *The set-valued map* $\gamma : \mathfrak{I}_A \to 2^{\mathbb{R}_+^2}$ *defined by*

$$\gamma_{RC}(x) = \text{argmax}\left\{\frac{\beta\mu(x) - V^A(\alpha,\beta)}{\beta\rho(x)} : (\alpha,\beta) \geq 0\right\}$$

is called the adjusted risk-contraction function. The set-valued map $\gamma : \mathfrak{I}_A^+ \to 2^{\mathbb{R}_+^2}$ defined by

$$\gamma_{RE}(x) = \text{argmin}\left\{\frac{V^A(\alpha,\beta) + \alpha\rho(x)}{\beta\mu(x)} : (\alpha,\beta) \geq 0\right\}$$

is called the adjusted return-expansion function.

Notice that, in the return oriented case, we have limited our definition to $\mathfrak{I}_A^+$ that is a portfolio subset of $\mathfrak{I}$. These definitions allow to provide a formal definition of the implicit risk aversion.

Definition 4.4. *Suppose that the maps* φ_{RC} *and* φ_{RE} *are single-valued at* x. *The implicit risk-aversion degree in the risk oriented and return oriented case are defined respectively by*:

$$RA_{RC}(x) = \frac{\gamma_{RC,1}(x)}{\gamma_{RC,2}(x)} \text{ and } RA_{RE}(x) = \frac{\gamma_{RE,1}(x)}{\gamma_{RE,2}(x)}.$$

In the case where φ_{RC} *and* φ_{RE} *are not single-valued at* x, *we say the the risk aversion is undefined.*

The next result shows that the Kunh-Tucker Multipliers of the mathematical programs above can be used to find the implicit risk aversion.

Proposition 4.5. *Suppose that the maps* γ_{RC} *is single-valued at* x. *Let* (α_{RC},β_{RC}) *denotes the Kuhn-Tucker multipliers of the first and second constraints of Program* (P_{RC}), *respectively. We have*

$$\gamma_{RC}(x) = (\alpha_{RC},\beta_{RC}) \text{ and } RA_{RC}(x) = \frac{\alpha_{RC}}{\beta_{RC}}.$$

Proof. To prove this result, we use an earlier result due to Briec *et al.* [2] who used the shortage function S_g^A that is defined for a given portfolio x by:

$$S_g^A(x) = \sup\{\delta : (\rho(x) - \delta g_v, \mu(x) + \delta g_e) \in DR_A\}$$

Setting, $g_v = \rho(x)$ and $g_e = 0$, we obtain:

$$S_g^A(x) = 1 - D_{RC}^A(x).$$

However, from [2], we have:

$$S_g^A(x) = \inf_{(\alpha,\beta)\ngeq 0}\left\{\frac{V^A(\alpha,\beta) - \beta\mu(x) + \alpha\rho(x)}{\alpha g_v + \beta g_e}\right\}.$$

Since $g_v = \rho(x)$ and $g_e = 0$, we have

$$S_g^A(x) = \inf_{(\rho,\mu)\geq 0}\left\{\frac{V^A(\alpha,\beta) - \beta\mu(x) + \alpha\rho(x)}{\alpha\rho(x)}\right\}. \quad (4.17)$$

Making an elementary transformation, we obtain:

$$S_g^A(x) = 1 - \sup\left\{\frac{\beta\mu(x) - V^A(\alpha,\beta)}{\alpha\rho(x)} : (\alpha,\beta) \geq 0\right\}.$$

Hence, from Proposition 4.2 the solution of the dual optimization program in (4.17) is identical to $\gamma_{RC}(x)$. However, from [2], this solution coïncides with the Kuhn and Tucker multiplier. If the dual solution is unique the result follows.

Proposition 4.6. *Suppose that the maps* γ_{RE} *is single-valued at* $x \in \mathfrak{I}_A^+$. *Let* (α_{RE},β_{RE}) *denotes the Kuhn-Tucker multipliers of the first and second constraints of Program* (P_{RE}), *respectively. We have*

$$\gamma_{RE}(x) = (\alpha_{RE},\beta_{RE}), \text{ and } RA_{RE}(x) = \frac{\alpha_{RE}}{\beta_{RE}}.$$

Proof. The proof is similar to that of Proposition using [2] and setting, $g_v = 0$ and $g_e = \mu(x)$.

4.2. Hyperbolic and McFadden Distance Functions and Duality Result

We can also establish duality between these distance functions and the indirect mean-variance utility function.

Proposition 4.7. *For all portfolio* $x \in \mathfrak{I}_A$ *such that* $\mu(x) > 0$, *we have*

$$D_H^A(x) =$$
$$\inf_{(\alpha,\beta)\geq 0}\left\{\frac{[V^A(\alpha,\beta)^2 + 4\alpha\beta\mu(x)\rho(x)]]^{1/2} - V^A(\alpha,\beta)}{2\beta\mu(x)}\right\}.$$

Proof. From the definition of the representation set we have:

$$DR_A = \bigcap_{(v,e)\geq 0}\{(v,e) \in \mathbb{R}^2 : \beta e - \alpha v \leq V^A(\alpha,\beta)\}.$$

We can equivalently write

$$D_H^A(x) = \inf\left\{\theta : (\frac{1}{\theta}\rho(x), \theta\mu(x)) \notin DR_A\right\}.$$

Let us denote $H(\alpha,\beta) = \{(v,e) \in \mathbb{R}^2 : \beta e - \alpha v \leq V^A(\alpha,\beta)\}$ for all $(\alpha,\beta) \geq 0$. It follows that

$$D_H^A(x) = \inf\left\{\theta : (\frac{1}{\theta}\rho(x), \theta\mu(x)) \in \mathbb{R}^2 \setminus \bigcap_{(\rho,\mu)\geq 0} H(\alpha,\beta)\right\}.$$

Solving a second order equation yields:

$$\inf\left\{\theta : (\frac{1}{\theta}\rho(x), \theta\mu(x)) \in \mathbb{R}^2 \setminus H(\alpha,\beta)\right\}$$
$$= \frac{[V^A(\alpha,\beta)^2 + 4\beta\alpha\mu(x)\rho(x)]^{1/2} - V^A(\alpha,\beta)}{2\beta\mu(x)}.$$

Since $\mathbb{R}^2 \setminus \bigcap_{(\alpha,\beta) \geq 0} H(\alpha,\beta) = \bigcup_{(\alpha,\beta) \geq 0} \mathbb{R}^2 \setminus H(\alpha,\beta)$, we immediately deduce the result.

A simpler duality result can be established regarding to the McFadden Gauge function [4].

Proposition 4.8. *For all portfolio* x *such that* $\mu(x) > 0$, *we have*

$$G_{MF}^A(x) = \inf_{(\alpha,\beta) \geq 0} \frac{V^A(\alpha,\beta)}{U_{\alpha,\beta}^A(x)}.$$

Proof. The result is immediate using the fact that, if $x \in dom(G_{MF}^A)$, there is at least some halfspace $H(\rho, \mu)$ such that

$$\inf\{\theta : (\theta\rho(x), \theta\mu(x)) \in \mathbb{R}^2 \setminus H(\alpha,\beta)\} = \frac{V^A(\alpha,\beta)}{U_{\alpha,\beta}^A(x)}.$$

5. Impact of Managerial Constraints on Portfolio Selection and Short Sales

This section analyzes how performance measurement varies regarding to the managerial constraints summarized by the constraint $Ax \geq 0$. First, notice that if $Ax = 0$ for all $x \in \mathbb{R}^n$, then the set of portfolio is defined by

$$\mathfrak{I}_0 = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1 \right\}. \qquad (5.1)$$

Since in such a case the map A is identically null, we replace the subscript "A", with "0". Suppose B is an affine map defined on $\mathbb{R}^n$. Let $\mathcal{A}$ be the set of all the affine maps defined on $\mathfrak{I}_A$. In the following, for all $A, B \in \mathcal{A}$, we denote $A \geq B$ if $Ax \geq Bx$ for all $x \in \mathbb{R}^n$. Next, we introduce a specific measure inspired from the Thomson metric that has some similarities to the well known Hilbert projective metric.

Definition 5.1. *The map* $\mathcal{M}_{RC} : \mathcal{A} \times \mathcal{A} \to [1, +\infty]$ *defined by*

$$\mathcal{M}_{RC}(A,B) = \sup_{x \in \mathfrak{I}_A \cap \mathfrak{I}_B} d_x(A,B) \text{ where}$$

$$d_x(A,B) = \max\left\{ \frac{D_{RC}^A(x)}{D_{RC}^B(x)}, \frac{D_{RC}^B(x)}{D_{RC}^A(x)} \right\}$$

is called the risk oriented distance between A and B.

Typically, d_x is inspired from the Thomson metric, because it computes from a radial projection the maximum difference at a between A and B. The main difference comes from the fact that the Thomson metric computes the distance between two points. Hence, d_x allows to define the map $\mathcal{M}$ that measures some kind of distance between two sets of managerial constraints. By virtue of its nature, this index takes values greater than one for portfolios whose the efficiency score is affected by the managerial constraints. If the metric is equal to one for each portfolio of a sample, then the efficiency scores are not affected by the managerial constraints. However, this does not mean that the efficient frontiers respectively obtained from A and B is identical. An interesting case arise in the situation where A is the identity map (that is $A = I$) and B is identically null. In such a case this measure allows to compare the situations with and without short sale and we not $B = 0$. The following properties are trivial:

Lemma 5.2. *Suppose that* $\mathfrak{I}_A \cap \mathfrak{I}_B \neq \varnothing$.
i) If $B \geq A$ *then*

$$\mathcal{M}_{RC}(A,B) = \inf_{x \in \mathfrak{I}_A \cap \mathfrak{I}_B} \max\left\{ \frac{D_{RC}^A(x)}{D_{RC}^B(x)} \right\}$$

ii) If $A = B$ *then* $\mathcal{M}_{RC}(A,B) = 1$.
iii) If $\mathcal{M}_{RC}(A,B) > 1$ *then* $A \neq B$.

Figure 2 depicts the idea the measure is based on. In line with [14,15], the efficient frontier is modified according to the managerial constraints an investor is dealing with. Hence, the performance of portfolios is also modified by shift of the frontier. In particular, this measure is useful to test the impact of short sales impact on the performance of a given portfolio x. In such a case, we have:

$$d_x(I,0) = \max\left\{ \frac{D_{RC}^I(x)}{D_{RC}^0(x)}, \frac{D_{RC}^0(x)}{D_{RC}^I(x)} \right\}, \qquad (5.2)$$

and

$$\mathcal{M}_{RC}(I,0) = \inf_{x \in \mathfrak{I}_I \cap \mathfrak{I}_0} d_x(I,0). \qquad (5.3)$$

Since, $\mathfrak{I}_I \subset \mathfrak{I}_0$, we have

$$\mathcal{M}_{RC}(I,0) = \sup_{x \in \mathfrak{I}_I} d_x(I,0). \qquad (5.4)$$

If there exists some portfolio $y \in \mathfrak{I}_0$ such that $\mu(y) = \mu(x)$ and $\rho(y) < D_{RC}^I(x)\rho(x)$ then it is easy to see that $\mathcal{M}_{RC}(I,0) > 1$. Hence, in this situation, the possibility of short sales enables decision makers to find a risk-
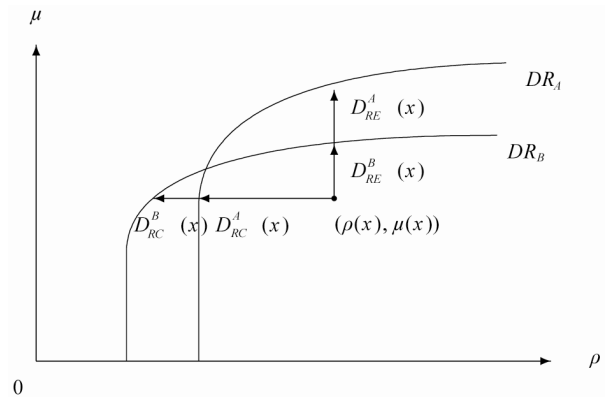

Figure 2. Impact of Managerial Constraints.

less portfolio. A similar approach is possible when one looks for possible improvement of the return.

Definition 5.3. *The map* $\mathcal{M}_{RC} : \mathcal{A} \times \mathcal{A} \to [1, +\infty]$ *defined by*

$$\mathcal{M}_{RE}(A, B) = \sup_{x \in \mathfrak{I}_A^+ \cap \mathfrak{I}_B^+} \delta_x(A, B) \text{ where}$$

$$\delta_x(A, B) = \max \left\{ \frac{D_{RE}^A(x)}{D_{RE}^B(x)}, \frac{D_{RE}^B(x)}{D_{RE}^A(x)} \right\} \quad (5.5)$$

is called the risk oriented distance between A and B.

Lemma 5.4. *The following properties are also trivial. Suppose that* $\mathfrak{I}_A^+ \cap \mathfrak{I}_B^+ \neq \varnothing$.

i) If $B \geq A$ then

$$\mathcal{M}_{RE}(A, B) = \sup_{x \in \mathfrak{I}_A \cap \mathfrak{I}_B} \max \left\{ \frac{D_{RE}^B(x)}{D_{RE}^A(x)} \right\}$$

ii) If $A = B$ then $\mathcal{M}_{RE}(A, B) = 1$.

iii) If $\mathcal{M}_{RE}(A, B) > 1$ then $A \neq B$.

This measure is useful to test whether or not the short sales impact the return expansion given some portfolio x. In such a case, we have:

$$\delta_x(I, 0) = \max \left\{ \frac{D_{RE}^I(x)}{D_{RE}^0(x)}, \frac{D_{RE}^0(x)}{D_{RE}^I(x)} \right\}, \quad (5.6)$$

and since, $\mathfrak{I}_I \subset \mathfrak{I}_0$, we have

$$\mathcal{M}_{RE}(I, 0) = \sup_{x \in \mathfrak{I}_I} \delta_x(I, 0). \quad (5.7)$$

If there exists some portfolio $y \in \mathfrak{I}_0$ such that $\rho(y) = \rho(x)$ and $D_{RE}^I(x)\mu(x) < \mu(y)$, then it is easy to see that $\mathcal{M}_{RC}(I, 0) > 1$. Hence, in this situation, the possibility of short sales enables decision makers to find a riskless portfolio.

In general, it is difficult to compute the map $\mathcal{M}_{RE}(A, B)$ and $\mathcal{M}_{RC}(A, B)$. However, one can compute an approximation based upon each specific asset. By construction, we have $R(e_i) = R_i$, for $i = 1, \cdots, n$ where $\{e_i\}_{i=1,\cdots,n}$ is the canonical basis of $\mathbb{R}^n$. This we do by defining

$$\tilde{\mathcal{M}}_{RC}(I, 0) = \max_{i=1,\cdots,n} d_{e_i}(I, 0) \text{ and} \quad (5.8)$$

$$\tilde{\mathcal{M}}_{RE}(I, 0) = \max_{i=1,\cdots,n} \delta_{e_i}(I, 0). \quad (5.9)$$

To analyse the case where there are short sales, an approach based upon quadratic programming may be used. However, since the optimization constraints are binding, it is possible to give a solution in closed form for γ^*, β^* and x^*.

6. Empirical Illustration

This section presents a numerical example. It is shown

that some key contributions of this paper can be easily implemented using standard methods of quadratic programming. The data are obtained from the CAC 40 monthly stocks over a period running from January 1984 to December 2008. For each of the 38 stocks, we have calculated monthly expected returns, covariances, variances with a monthly percentage return. After computing the efficient frontier, we set parameters $\mu = 1$ and $\rho = 2$. Below, **Tables 1** and **2** respectively summarize in the risk oriented and return oriented cases the decomposition of monthly performances for each asset. Remember that overall efficiency looks for a global improvement of a title following a chosen direction. Hence, it can be decomposed in two parts reflecting portfolio efficiency and allocative efficiency.

As it is shown in **Table 1**, asset 1 (Accor) has a low return and therefore it is too risky. The investor can

Table 1. Decomposition of "CAC 40" performance (Risk Oriented).

Asset	O_{RC}	D_{RC}	A_{RC}	γ_{RC}
Accor	2.9767	0.0358	2.9409	0.0000
Air france	0.8191	0.0291	0.7900	0.0140
Air liquide	9.6218	0.0935	9.5283	0.0000
Alcaltel	−0.1972	0.2086	−0.4058	0.0133
Alstom	−0.6221	1.0000	−1.6221	0.0196
Arcelor mittal	3.6585	0.0438	3.6147	0.0000
Axa	1.8391	0.0225	1.8166	0.0000
Bnp	2.5783	0.0336	2.5447	0.0000
Bouygues	2.6120	0.0269	2.5851	0.0000
Cap gemini	0.5518	0.0147	0.5372	0.0230
Carrefour	5.0090	0.0489	4.9601	0.0000
Credit agricole	2.1384	0.1024	2.0360	0.0097
Danone	6.1840	0.0623	6.1218	0.0000
Dexia	−0.1622	0.3111	−0.4733	0.0128
Eads	1.2664	0.0392	1.2272	0.0175
Edf	17.4122	0.1721	17.2401	0.0000
Essilor intl	9.2642	0.0716	9.1925	0.0000
France telecom	0.1872	0.0333	0.1539	0.0107
Gdf suez	41.1744	0.3918	40.7825	0.0000
Lafarge	2.5799	0.0362	2.5437	0.0000
Lagardere	2.3142	0.0272	2.2869	0.0000
Loreal	6.2259	0.0522	6.1737	0.0000
Lvmh	2.4824	0.0286	2.4539	0.0000
Michelin	2.6201	0.0354	2.5847	0.0000
Pernod ricard	6.5931	0.0642	6.5289	0.0000
Peugeot	1.8569	0.0284	1.8285	0.0000
Ppr	2.1359	0.0258	2.1101	0.0000
Renault	0.9004	0.0209	0.8794	0.0319
Saint gobain	2.7787	0.0324	2.7463	0.0000
Sanofi aventis	9.4330	0.1144	9.3185	0.0000
Schneider	3.0940	0.0356	3.0584	0.0000
Societe generale	2.2684	0.0269	2.2415	0.0000
Stmicroelectronics	0.9322	0.0161	0.9161	0.1430
Suez E.	15.3026	0.9597	14.3429	0.0071
Total	7.2215	0.0662	7.1553	0.0000
Unibail	5.4427	0.0559	5.3868	0.0000
Vallourec	2.1729	0.0181	2.1548	0.0000
Veolia environ	2.1113	0.0484	2.0628	0.0337
Vinci	3.9696	0.0393	3.9303	0.0000
Vivendi	1.8895	0.0288	1.8607	0.0000

Table 2. Decomposition of "CAC 40" performance (Return Oriented).

Asset	O_{RE}	D_{RE}	A_{RE}	γ_{RE}
Accor	33.6665	33.6640	0.0025	0.0133
Air france	2.5886	2.5561	0.0325	0.0135
Air liquide	−5.8125	−5.5131	−0.2994	0.0128
Alcaltel	1.8827	1.5611	0.3215	0.0170
Alstom	1.3252	1.0000	0.3252	0.0196
Arcelor mittal	37.7219	37.5822	0.1397	0.0132
Axa	27.2942	26.9360	0.3582	0.0135
Bnp	11.8982	11.8983	−0.0001	0.0133
Bouygues	−11.7382	−11.6888	−0.0494	0.0134
Cap gemini	3.7739	3.4874	0.2865	0.0142
Carrefour	−6.6492	−6.5984	−0.0508	0.0132
Credit agricole	1.7816	1.7288	0.0528	0.0130
Danone	−7.8429	−7.6831	−0.1599	0.0131
Dexia	1.3169	1.3140	0.0030	0.0134
Eads	2.4821	2.4815	0.0005	0.0134
Edf	−6.1239	−5.4436	−0.6804	0.0122
Essilor intl	−2.3266	−2.2569	−0.0697	0.0130
France telecom	2.3786	2.1740	0.2046	0.0147
Gdf suez	−4.6683	−3.7145	−0.9538	0.0103
Lafarge	7.5510	7.5500	0.0010	0.0133
Lagardere	77.9918	77.6968	0.2950	0.0134
Loreal	−3.2041	−3.1702	−0.0339	0.0132
Lvmh	−307.6488	−306.9223	−0.7265	0.0134
Michelin	9.2097	9.2095	0.0001	0.0133
Pernod ricard	−6.2128	−6.0738	−0.1390	0.0130
Peugeot	5.7738	5.7592	0.0146	0.0134
Ppr	33.6970	33.4996	0.1974	0.0134
Renault	3.4948	3.4060	0.0888	0.0135
Saint gobain	162.9716	162.9394	0.0322	0.0134
Sanofi aventis	25.2989	23.5278	1.7711	0.0126
Schneider	−252.3979	−252.3878	−0.0100	0.0133
Societe generale	56.2202	55.9829	0.2373	0.0134
Stmicroelec-tronics	5.2064	4.9761	0.2303	0.0136
Suez environnement	1.4416	1.0284	0.4132	0.0037
Total	−4.4442	−4.3357	−0.1085	0.0130
Unibail	−9.4252	−9.2927	−0.1325	0.0131
Vallourec	−4.2453	−4.1137	−0.1315	0.0135
Veolia environ	2.7819	2.7700	0.0119	0.0132
Vinci	−7.6228	−7.6144	−0.0085	0.0133
Vivendi	5.8051	5.7926	0.0125	0.0134

Table 3. Implicit Risk Aversion.

Asset	Risk Oriented			Return Oriented		
	α_{RC}	β_{RC}	γ_{RC}	α_{RE}	β_{RE}	γ_{RE}
Accor	0.000	1.000	0.000	0.006	0.482	0.013
Air france	0.000	0.011	0.014	0.008	0.613	0.013
Air liquide	0.000	1.000	0.000	0.003	0.284	0.012
Alcaltel	0.006	0.462	0.013	0.020	1.197	0.017
Alstom	0.030	1.534	0.019	0.030	1.534	0.019
Arcelor mittal	0.000	1.000	0.000	0.005	0.433	0.013
Axa	0.000	1.000	0.000	0.008	0.616	0.013
Bnp	0.000	1.000	0.000	0.006	0.499	0.013
Bouygues	0.000	1.000	0.000	0.007	0.561	0.013
Cap gemini	0.000	0.008	0.023	0.012	0.846	0.014
Carrefour	0.000	1.000	0.000	0.005	0.408	0.013
Credit agricole	0.000	0.014	0.009	0.004	0.332	0.013
Danone	0.000	1.000	0.000	0.004	0.358	0.013
Dexia	0.003	0.287	0.012	0.007	0.542	0.013
Eads	0.000	0.010	0.017	0.006	0.509	0.013
Edf	0.000	1.000	0.000	0.002	0.195	0.012
Essilor intl	0.000	1.000	0.000	0.004	0.331	0.013
France telecom	0.001	0.105	0.010	0.013	0.890	0.014
Gdf suez	0.000	1.000	0.000	0.001	0.095	0.010
Lafarge	0.000	1.000	0.000	0.006	0.480	0.013
Lagardere	0.000	1.000	0.000	0.007	0.557	0.013
Loreal	0.000	1.000	0.000	0.005	0.394	0.013
Lvmh	0.000	1.000	0.000	0.007	0.543	0.013
Michelin	0.000	1.000	0.000	0.006	0.485	0.013
Pernod ricard	0.000	1.000	0.000	0.004	0.352	0.013
Peugeot	0.000	1.000	0.000	0.007	0.545	0.013
Ppr	0.000	1.000	0.000	0.007	0.573	0.013
Renault	0.000	0.006	0.031	0.009	0.668	0.013
Saint gobain	0.000	1.000	0.000	0.006	0.509	0.013
Sanofi aventis	0.000	1.000	0.000	0.003	0.252	0.012
Schneider	0.000	1.000	0.000	0.006	0.484	0.013
Societe generale	0.000	1.000	0.000	0.007	0.561	0.013
Stmicroelec-tronics	0.000	0.001	0.143	0.009	0.733	0.013
Suez E.	0.000	0.017	0.007	0.000	0.021	0.003
Total	0.000	1.000	0.000	0.004	0.346	0.013
Unibail	0.000	1.000	0.000	0.005	0.380	0.013
Vallourec	0.000	1.000	0.000	0.009	0.689	0.013
Veolia environ	0.000	0.006	0.033	0.005	0.429	0.013
Vinci	0.000	1.000	0.000	0.006	0.459	0.013
Vivendi	0.000	1.000	0.000	0.007	0.541	0.013

choose an efficient portfolio that is 0,036% less risky and does not require risk-taking ($\gamma_{RC} = 0$). Considering only risk aversion, it is possible to pursue reduction of risk to maximize the utility of a portfolio but, due to the market imperfections, this result cannot be achieved without short sales.

In the return expansion case, the first asset is less efficient (**Table 2**) than is the risk oriented case. Given the same level of risk, it should be 50 times more profitable ($O_{RE} = 33.6665$). However, technically, we can only select a portfolio that produces 33 times more important return ($D_{RE} = 33.6641$). The market imperfections again limits this return improvement ($A_{RE} = 0.0025$). Here, the investor must take 6 units of risk to increase his (or her) return of a one unit amount ($\gamma_{RE} = 0.0133$). **Table 3** details implicit risk aversion of risk reduction

and return expansion for each asset. Note that some assets have a negative performance indicator. This can be explained by the fact that they have a negative return. The portfolio may have a greater risk than their expected returns and their respective utility functions measures the potential loss of each invested euro. This is not consistent with the objective of maximizing behaviour the investors have and, in such a case, allocative efficiency makes no sense.

Improvements both based on increasing expected returns and risk contraction are presented in **Table 4**. In general, the shortage function and the hyperbolic function give almost the same improvement of performance (0.008 for asset 1). The Hyperbolic functions provide a seven times greater improvement than those obtained from these two functions. Hence, much more return, but

Table 4. "CAC 40" Efficiencies.

Asset	S_g	D_{RC}	D_{RE}	D_H	G_{MF}
Accor	0.001	0.036	33.664	17.796	55.920
Air france	0.006	0.029	2.556	2.109	3.759
Air liquide	0.001	0.093	−5.513	10.691	
Alcaltel	0.007	0.208	1.561	1.410	1.695
Alstom	0.000	1.000	1.000	1.000	1.000
Arcelor mittal	0.001	0.044	37.582	17.622	65.610
Axa	0.001	0.022	26.936	15.078	39.517
Bnp	0.002	0.033	11.898	7.769	19.444
Bouygues	0.002	0.026	−11.688	37.171	
Cap gemini	0.008	0.014	3.487	2.646	4.330
Carrefour	0.002	0.048	−6.598	20.440	
Credit agricole	0.001	0.102	1.728	1.574	2.045
Danone	0.002	0.062	−7.683	16.060	
Dexia	0.001	0.311	1.314	1.242	1.458
Eads	0.004	0.039	2.481	2.082	3.583
Edf	0.001	0.172	−5.443	5.809	
Essilor intl	0.002	0.071	−2.256	13.958	
France telecom	0.008	0.033	2.174	1.820	2.652
Gdf suez	0.000	0.391	−3.714	2.552	
Lafarge	0.003	0.036	7.550	5.304	12.570
Lagardere	0.000	0.027	77.696	30.111	120.149
Loreal	0.003	0.052	−3.170	19.150	
Lvmh	0.000	0.028	−306.922	35.013	
Michelin	0.002	0.035	9.209	6.271	15.251
Pernod ricard	0.002	0.064	−6.073	15.570	
Peugeot	0.004	0.028	5.759	4.160	9.004
Ppr	0.001	0.025	33.499	18.262	51.040
Renault	0.007	0.021	3.406	2.643	4.776
Saint gobain	0.000	0.032	162.939	30.859	263.763
Sanofi aventis	0.001	0.114	23.527	8.552	50.616
Schneider	0.000	0.035	−252.387	28.082	
Societe generale	0.001	0.027	55.983	25.723	86.276
Stmicroelec-tronics	0.006	0.016	4.976	3.564	6.620
Suez E.	0.000	0.959	1.028	1.017	1.057
Total	0.003	0.066	−4.335	15.103	
Unibail	0.002	0.056	−9.293	17.886	
Vallourec	0.007	0.018	−4.114	55.354	
Veolia environ	0.004	0.048	2.770	2.306	4.102
Vinci	0.003	0.039	−7.614	25.461	
Vivendi	0.004	0.029	5.793	4.184	9.091

also much more risk (56 more times for asset 1) are involved with the McFadden Gauge [4].

Intuitively, this last measure may be use for a risk-lover manager. The mean return expansion function is suitable for risk-neutral investor. Risk averse manager may be appreciate by risk contraction, hyperbolic or shortage functions.

Finally, regarding to all the results calculated in **Table 4**, il clearly appears that the efficient frontier is entirely characterized by two funds: Asset 5 (Alsthom) and Asset 34 (Suez environment). This is an illustration, of the two funds theorem.

7. Conclusions

This paper has analyzed duality relations between the indirect mean-variance utility function and a broad class

of portfolio efficiency measures. It has been shown that such approaches are useful to measure the impact of managerial constraints on the performance. In addition, the implicit risk aversion of an optimal solution can be deduced from the Kuhn-Tucker optimality conditions.

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Appendix: The Short Sale Case Computations

It is possible to give a solution of the case where there are short sales with an approach based upon quadratic programming.

Computation of Risk Oriented Measure

In the following, we assume that $\mu(x) \geq \mu(x^{\#})$ where $x^{\#} \in \operatorname{argmin}\{\rho(x) : x \in \mathfrak{I}_0\}$. Under such an assumption, we know that $(D_{RC}^0(x)\rho(x), \mu(x))$ lies on the curve representing the hyperbola relating the expected returns and the variance. We first write the Lagrangian of the optimization problem we need to solve. It is defined by

$$L(x,\lambda,\alpha,\beta,\gamma) = \lambda - \alpha\left(\sum_i x_i E[R_i] - E[R_p]\right)$$
$$- \beta\left(\lambda Var[R_p] - \sum_{i,j} x_i x_j \Omega_{i,j}\right) - \gamma\left(\sum_i x_i - 1\right) \quad (7.1)$$

Suppose that $(x^*, \alpha^*, \beta^*, \gamma^*, \lambda^*)$ is solution. The first order conditions yield:

$$\frac{\partial}{\partial x_i} L(x,\delta,\alpha,\beta,\gamma) = -\alpha E[R_i] + \beta \sum_j x_j \Omega_{i,j} - \gamma = 0 \quad (7.2)$$

$$\frac{\partial}{\partial \lambda} L(x,\delta,\alpha,\beta,\gamma) = 1 - \beta Var[R_p] = 0 \quad (7.3)$$

Following (7.2); we have

$$x = \Omega^{-1}\left(\frac{\gamma}{\beta}1_n + \frac{\alpha}{\beta}E[R]\right) \quad (7.4)$$

Since $E[R(x)] \geq E[R(x^{\#})]$ the constraints are binding. Hence, substituting in the first order conditions, we have a three-dimensional system to solve, with:

$$^t E[R]\left(\frac{\gamma}{\beta}1_n + \frac{\alpha}{\beta}E[R]\right) - E[R_p] = 0 \quad (7.5)$$

$$^t\left(\frac{\gamma}{\beta}1_n + \frac{\alpha}{\beta}E[R]\right)\Omega^{-1}\left(\frac{\gamma}{\beta}1_n + \frac{\alpha}{\beta}E[R]\right) - \lambda V[R_p] = 0$$
$$(7.6)$$

$$^t 1_n \Omega^{-1}\left(\frac{\gamma}{\beta}1_n + \frac{\alpha}{\beta}E[R]\right) = 1 \quad (7.7)$$

The last equation yields:

$$\frac{\gamma}{\beta} = \left(\frac{1}{^t 1_n \Omega^{-1} 1_n}\right) - \frac{\alpha}{\beta}\left(^t 1_n \Omega^{-1}\right)E[R] \quad (7.8)$$

Substituting in 7.5, we obtain:

$$^t E[R]\left[\left(\frac{1}{^t 1_n \Omega^{-1} 1_n}\right)1_n - \frac{\alpha}{\beta}\left(^t 1_n \Omega^{-1}\right)E[R]1_n \right.$$
$$\left. + \frac{\alpha}{\beta}\left(\Omega^{-1}\right)E[R]\right] - E[R_p] = 0.$$

Hence, we obtain the optimal values:

$$\frac{\alpha^*}{\beta^*} = [^t E[R]\left(^t 1_n \mathbf{\Omega}^{-1}\right) E[R] 1_n -^t E[R] \mathbf{\Omega}^{-1} E[R]]^{-1}$$

$$\times [^t E[R]\left(\frac{1}{^t 1_n \mathbf{\Omega}^{-1} 1_n}\right) 1_n + E[R_p]] \tag{7.9}$$

and γ^* is then obtained from equation 46. From 52, we deduce x^*. The distance function is given by λ^*

$$\lambda^* = Var[R_p]^{-1} \times \left[\left(\frac{1}{^t 1_n \mathbf{\Omega}^{-1} 1_n}\right) 1_n\right.$$

$$-\frac{\alpha^*}{\beta^*}\left(^t 1_n \mathbf{\Omega}^{-1}\right) E[R] 1_n + \frac{\alpha^*}{\beta^*} \mathbf{\Omega}^{-1} E[R]\right]$$

$$\times \mathbf{\Omega}\left[\left(\frac{1}{^t 1_n \mathbf{\Omega}^{-1} 1_n}\right) 1_n - \frac{\alpha^*}{\beta^*}\left(^t 1_n \mathbf{\Omega}^{-1}\right) E[R] 1_n \right. \tag{7.10}$$

$$\left. + \frac{\alpha^*}{\beta^*} \mathbf{\Omega}^{-1} E[R]\right].$$

Computation of Return Oriented Measure

In the following, we suppose that $E[R_p] > 0$. The Lagrangian of the optimization problem we need to optimize is:

$$L(x, \lambda, \alpha, \beta, \gamma) = -\theta - \alpha\left(\sum_i x_i E[R_i] - \theta E[R_p]\right)$$

$$-\beta\left(Var[R_p] - \sum_{i,j} x_i x_j \Omega_{i,j}\right) - \gamma\left(\sum_i x_i - 1\right) \tag{7.11}$$

Suppose that $(x^*, \alpha^*, \beta^*, \gamma^*, \lambda^*)$ is solution. The first order conditions yield:

$$\frac{\partial}{\partial x_i} L(x, \delta, \alpha, \beta, \gamma) = -\alpha E[R_i] + \beta\sum_j x_j \Omega_{i,j} - \gamma = 0 \tag{7.12}$$

$$\frac{\partial}{\partial x_i} L(x, \delta, \alpha, \beta, \gamma) = -\alpha E[R_i] = 0 \tag{7.13}$$

$$\frac{\partial}{\partial \theta} L(x, \delta, \alpha, \beta, \gamma) = 1 - \alpha E[R_p] = 0 \tag{7.14}$$

At the optimum, we clearly have $\alpha^* = 1/E[R_p]$. Following (7.2); we have

$$x = \mathbf{\Omega}^{-1}\left(\frac{\gamma}{\beta} 1_n + \frac{\alpha}{\beta} E[R]\right) \tag{7.14}$$

Since the constraints are binding, substituting in the first order conditions, we have a three-dimensional system to solve, with:

$$^t E[R]\left(\frac{\gamma}{\beta} 1_n + \frac{\alpha}{\beta} E[R]\right) - \theta E[R_p] = 0 \tag{7.15}$$

$$^t\left(\frac{\gamma}{\beta} 1_n + \frac{\alpha}{\beta} E[R]\right) \mathbf{\Omega}^{-1}\left(\frac{\gamma}{\beta} 1_n + \frac{\alpha}{\beta} E[R]\right)$$

$$- Var[R_p] = 0 \tag{7.16}$$

$$^t 1_n \mathbf{\Omega}^{-1}\left(\frac{\gamma}{\beta} 1_n + \frac{\alpha}{\beta} E[R]\right) = 1 \tag{7.17}$$

Combining 7.15 and 7.16 we obtain

$$^t\left(\theta E[R_p]\right) \mathbf{\Omega}^{-1}\left(\theta E[R_p]\right) - \|E[R]\|^2 Var[R_p] = 0 \tag{7.18}$$

It follows that

$$\theta^* = [^t E[R_p] \mathbf{\Omega}^{-1/2} E[R_p]]^{-1/2}$$

$$\|E[R]\|[Var[R_p]]^{1/2} \tag{7.19}$$

Using 7.17 and 7.15 we obtain:

$$^t E[R]\left[\left(\frac{1}{^t 1_n \mathbf{\Omega}^{-1} 1_n}\right) 1_n - \frac{\alpha}{\beta}\left(^t 1_n \mathbf{\Omega}^{-1}\right) E[R] 1_n + \right.$$

$$\left. \frac{\alpha}{\beta}\left(\mathbf{\Omega}^{-1}\right) E[R]\right] - \theta^* E[R_p] = 0$$

Hence, we obtain the optimal values:

$$\frac{\alpha^*}{\beta^*} = \left[^t E[R]\left(^t 1_n \mathbf{\Omega}^{-1}\right) E[R] 1_n -^t E[R] \mathbf{\Omega}^{-1} E[R]\right]^{-1}$$

$$\times\left[^t E[R]\left(\frac{1}{^t 1_n \mathbf{\Omega}^{-1} 1_n}\right) 1_n + \theta^* E[R_p]\right]. \tag{7.20}$$

Therefore γ^* and β^* are then obtained from equation 7.20. From 7.14, we deduce x^*.