Journal of High Energy Physics, Gravitation and Cosmology, 2019, 5, 711-718
http://www.scirp.org/journal/jhepgc
ISSN Online: 2380-4335
ISSN Print: 2380-4327

On an Exact Cylindrically Symmetric Solution in a Born-Infeld Type Theory of Gravity

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How to cite this paper: Rosa, T.O., Guimarães, M.E.X. and Neto, J.L. (2019) On an Exact Cylindrically Symmetric Solution in a Born-Infeld Type Theory of Gravity. *Journal of High Energy Physics, Gravitation and Cosmology*, **5**, 711-718.

https://doi.org/10.4236/jhepgc.2019.53038

Received: January 31, 2019
Accepted: June 14, 2019
Published: June 17, 2019

Abstract

In this work, we derive an exact vacuum solution for a cylindrically symmetric metric in an extended gravity theory developed by Novello, De Lorenci and Luciane (hereafter referred to as the NDL theory) which is inspired in the Born-Infeld theory. The main goal of this paper is to find a cosmic string solution for the NDL theory. However, a careful analysis of the metric shows that it is asymptotically singular and therefore does not represent a cosmic string solution.

Keywords

Modified Theories of Gravity, Topological Defects

1. Introduction

It is well known that several types of topological defects may have been created by the vacuum phase transitions in the early universe. [1, 2] These include domain walls, cosmic strings and monopoles. These topological defects have been extensively studied in many kinds of gravity theories, notably as scalar-tensor and f(R) gravities [3–5] where many aspects and applications were developed.

Although there exist many effective theories of gravity which come from the unification process [6], each of them must, of course, satisfy some predictions. Actually, with the advent of the recent LIGO detections could be, in principle, an important and useful tool to test the alternative theories of gravity as it has been pointed out by Corda in ref. [7]. Here, in this paper, we will deal with one of the extended gravity theory, the so-called NDL theory [8, 9] which is based on a Born-Infeld theory [10].

One important assumption in General Relativity is that all fields interact in an universal way with gravity. This is the so called Strong

Equivalence Principle (SEP). It is well known, with good accuracy, that this is true when it concerns to matter-gravity interaction, *i.e.*, the Weak Equivalence Principle(WEP). But, until now, there is no direct observational confirmation of this assumption in what concerns the gravity-gravity interaction. In [8], an extension of the field theoretical approach of General Relativity built by [11, 12] proposes an alternative field theory of gravity. In this theory, gravitons propagate in a different spacetime of the matter fields. The velocity of propagation of the gravitational waves does not coincide with the General Relativity predictions because of the violation of the SEP and the self-interaction graviton-graviton predicts a massive graviton in this theory.

In this paper, our main purpose is to investigate the properties of a straight cosmic string in the NDL theory. We anticipate, however, that such a solution does not exist. This manuscript is organized as follows. In Section 2, we furnish a brief review of the NDL theory based on the original paper [8]. In Section 3, we write down the detailed calculations to find the exterior metric of a local cosmic string and finally in Section 4 we summarize and discuss our results.

2. The NDL Theory: A Brief Review

In this section we review the NDL theory following the references [8,9]. A more detailed presentation can be find in the ref. [8]. To start with, the main lines of the NDL theory are:

- The gravitational interaction is represented by a symmetric tensor $\varphi_{\mu\nu}$ that obeys a nonlinear equation of motion.
- The matter (but not gravity) couples to gravity through the metric $g_{\mu\nu} = \gamma_{\mu\nu} + \varphi_{\mu\nu}$, where $\gamma_{\mu\nu}$ is the flat background metric.
- The self interaction of the gravitational field breaks the universal modification of the space time geometry, *i.e.*, the gravity couples to gravity in a special way distinct from all different forms of energy.

We begin defining the tensor $F_{\alpha\beta\mu}$, which is anti-symmetric in the two first indices, called the gravitational field:

$$F_{\alpha\beta\mu} := \frac{1}{2}\left(\varphi_{\mu[\alpha;\beta]} + F_{[\alpha}\gamma_{\beta]\mu}\right), \tag{1}$$

where $[x, y] = xy - yx$ and the covariant derivative is constructed with the background metric. Indices are raised and lowered with that metric also, and

$$F_{\alpha} = F_{\alpha\mu\nu}\gamma^{\mu\nu} = \varphi_{,\alpha} - \varphi_{\alpha\mu;\nu}\gamma^{\mu\nu} \tag{2}$$

In order to have a nonlinear theory of the gravitational field $F_{\alpha\beta\mu}$ with the correct weak field limit, we assume that the interaction of gravity with itself is described by a functional of A and B which are invariants built from the gravitational field $F_{\alpha\beta\mu}$:

$$A = F_{\alpha\beta\mu}F^{\alpha\beta\mu} \ \text{ and } \ B = F_{\alpha}F^{\alpha}.$$

In our case we will use the Born-Infeld Lagrangian:

$$\mathcal{L} = \frac{b^2}{k}\sqrt{1 - \frac{A-B}{b^2}} - 1,$$

where k is the Einstein's constant. Thus, the gravitational action will be:

$$\mathcal{S} = \int d^4x \sqrt{\gamma}\mathcal{L}, \tag{3}$$

where γ is the determinant of the Minkowski metric in an arbitrary coordinate system. Taking the variation of the action (3) with respect to the potential $\varphi_{\mu\nu}$, we obtain the following equations of motion:

$$\left(\mathcal{L}_U F^{\lambda(\mu\nu)}\right)_{;\lambda} = -\frac{1}{2}T^{\mu\nu}$$

where $(x, y) = xy + yx$, $U = A - B$ and $\mathcal{L}_U = \frac{\delta\mathcal{L}}{\delta U}$.

3. The Cosmic String Solution in NDL Theory: An Attempt

We consider here the exterior region of a local cosmic string in the NDL theory. In this case our energy momentum tensor will be identically zero in the region outside the string and our source has a cylindrical symmetry. Thus, we begin with the following ansätze:

- The background metric, *i.e.* $\gamma_{\mu\nu}$, will be Minkowski in cylindrical coordinates, following Vilenkin in [2]:

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - dz^2. \tag{4}$$

- The potential $\varphi_{\mu\nu}$ will be :

$$\varphi_{11} = \alpha(r) \ , \ \varphi_{33} = -c(r) \text{ and } \varphi_{44} = -\beta(r). \tag{5}$$

In a way that our cylindrical metric for the straight cosmic string in the NDL theory is:

$$ds^2 = (1 + \alpha(r))dt^2 - dr^2 - (r^2 + c(r))d\theta^2 - (1 + \beta(r))dz^2\,. \tag{6}$$

In order to obtain the equations of motion, first we need to compute some elements. The trace and the covariant derivatives of the potential are, respectively:

$$\begin{aligned}
\varphi &= \varphi_{11}\gamma^{11} + \varphi_{33}\gamma^{33} + \varphi_{44}\gamma^{44} = \alpha + \beta + \frac{c}{r^2},\\
\varphi_{11;2} &= \varphi_{11,2} = \alpha'(r),\\
\varphi_{33;2} &= \varphi_{33,2} - 2\Gamma^3{}_{23} = -c' + 2\frac{c}{r},\\
\varphi_{44;2} &= \varphi_{44,2} = -\beta',\\
\varphi_{32;3} &= \Gamma^3{}_{32}\varphi_{33} = \frac{c}{r}.
\end{aligned}$$

The only non vanishing component of F_α is F_2:

$$F_2 = \varphi_{,2} - \varphi_{23;3}\gamma^{33} = \alpha' + \beta' + \frac{c'}{r^2} - 2\frac{c}{r^3} - \frac{c}{r}\left(-\frac{1}{r^2}\right) = \alpha' + \beta' - \frac{c'}{r^3}.$$

With these, we can start calculating the $F^{\mu\nu\lambda}$ components:

$$\begin{aligned}
F^{211} &= -F_{211} = -\frac{1}{2}\left(\beta' + \frac{c'}{r^2} - \frac{c}{r^3}\right)\\
F^{233} &= -\frac{F_{233}}{r^4} = \frac{1}{2}\left(\frac{\alpha'}{r^2} + \frac{\beta'}{r^2}\right)\\
F^{244} &= -F_{244} = \frac{1}{2}\left(\alpha' + \frac{c'}{r^2} - \frac{c}{r^3}\right).
\end{aligned}$$

To write explicitly the Lagrangian we must find the invariants A and B for our particular metric:

$$A = -\beta'^2 - \alpha'^2 - \frac{c'^2}{r^4} - \frac{c^2}{r^6} - \alpha'\beta' - \frac{\alpha' c'}{r^2} + \frac{\alpha' c'}{r^3} - \frac{\beta' c'}{r^2} + \frac{\beta' c}{r^3} + 2\frac{c' c}{r^5},$$

$$B = -\left(\alpha'^2 + 2\alpha'\beta' + \beta'^2 + \frac{c'^2}{r^4} - 2\frac{cc'}{r^5} + \frac{c^2}{r^6} + 2\frac{\alpha' c'}{r^2} - 2\frac{\alpha' c}{r^3} + 2\frac{\beta' c'}{r^2} - 2\frac{\beta' c}{r^3}\right),$$

$$U = A - B = \alpha'\beta' + \frac{\alpha' c'}{r^2} + \frac{\beta' c'}{r^2} - \frac{\beta' c}{r^3} - \frac{\alpha' c}{r^3}.$$

And, finally, the covariant derivatives of the gravitational tensor $F^{\lambda\mu\nu}$:

$$\begin{aligned}
F^{211}{}_{;2} &= F^{211}{}_{,2} = -\frac{1}{2}\left(\beta'' + \frac{c''}{r^2} - 3\frac{c'}{r^3} + 3\frac{c}{r^4}\right), \\
F^{244}{}_{;2} &= F^{244}{}_{,2} = \frac{1}{2}\left(\alpha'' + \frac{c''}{r^2} - 3\frac{c'}{r^3} + 3\frac{c}{r^4}\right), \\
F^{344}{}_{;3} &= \frac{1}{2r}\left(\alpha' + \frac{c'}{r^2} - \frac{c}{r^3}\right), \\
F^{311}{}_{;3} &= \frac{1}{2}\left(\frac{c}{r^4} - \frac{c'}{r^3} - \frac{\beta'}{r}\right), \\
F^{322}{}_{;3} &= \frac{1}{2r}\left(\alpha' + \beta'\right), \\
F^{233}{}_{;2} &= \frac{1}{2}\left(\frac{\alpha''}{r^2} + \frac{\beta''}{r^2}\right).
\end{aligned}$$

The non vanishing equations of motion are:

$$L'_U F^{211} + L_U F^{211}_{;2} + L_U F^{311}_{;3} = 0, \quad (7)$$

$$L_U F^{322}_{;3} = 0, \quad (8)$$

$$L'_U F^{233} + L_U F^{233}_{;2} = 0, \quad (9)$$

$$L'_U F^{244} + L_U F^{244}_{;2} + L_U F^{344}_{;3} = 0. \quad (10)$$

Knowing that,

$$L_U = -\frac{1}{2}\frac{1}{k}\frac{1}{\sqrt{1 - \frac{U}{b^2}}}$$

$$L'_U = L_U \frac{U'}{2b^2}\frac{1}{1 - \frac{U}{b^2}}$$

where prime means $\frac{\partial}{\partial r}$ these equations can be written by:

$$L_U \frac{U'}{2b^2}\left(1 - \frac{U}{b^2}\right)^{-1}\left(-\frac{1}{2}\right)\left(\beta' + \frac{c'}{r^2} - \frac{c}{r^3}\right) + L_U\left(-\frac{1}{2}\right)\left(\beta'' + \frac{c''}{r^2} - \frac{2c'}{r^3} + \frac{2c}{r^4} + \frac{\beta'}{r}\right) = 0, \quad (11)$$

$$L_U \frac{1}{2}\left(\frac{\alpha' + \beta'}{r}\right) = 0, \quad (12)$$

$$L_U \frac{U'}{2b^2}\left(1 - \frac{U}{b^2}\right)^{-1}\left(\frac{1}{2}\right)\left(\frac{\alpha' + \beta'}{r^2}\right) + L_U\left(\frac{1}{2}\right)\left(\frac{\alpha'' + \beta''}{r^2}\right) = 0, \quad (13)$$

$$L_U \frac{U'}{2b^2}\left(1-\frac{U}{b^2}\right)^{-1}\left(\frac{1}{2}\right)\left(\alpha' + \frac{c'}{r^2} - \frac{c}{r^3}\right) + L_U \frac{1}{2}\left(\alpha'' + \frac{c''}{r^2} - \frac{2c'}{r^3} + \frac{2c}{r^4} + \frac{\alpha'}{r}\right) = 0. \tag{14}$$

From (12) we have $\alpha' = -\beta'$ and Equation (13) is identically satisfied.

If we add Equation (11) to the Equation(14) and use $\beta' = -\alpha'$ we have:

$$L'_U \alpha' + L_U\left(\alpha'' + \frac{\alpha'}{r}\right) = 0. \tag{15}$$

Since

$$L'_U = L_U \frac{U'}{2b^2}\frac{1}{1-\frac{U}{b^2}} = -L_U \frac{\alpha'\alpha''}{b^2}\frac{1}{1+\frac{(\alpha')^2}{b^2}}$$

we finally have:

$$\frac{\alpha'}{r} + \alpha'' + \frac{\alpha'^3}{b^2 r} = 0. \tag{16}$$

The solution of this equation is

$$\alpha(r) = \pm b\sqrt{k_1}\ln\frac{k_1 r + \sqrt{-(-r^2+k_1)k_1}\sqrt{k_1}}{\sqrt{k_1}} + k_2, \tag{17}$$

where k_1 is a positive constant and k_2 is an arbitrary real constant. We have to find the equation for $c(r)$. Making now the difference (14)-(11) we have:

$$-\frac{\alpha'\alpha''}{b^2}\left(1+\frac{(\alpha')^2}{b^2}\right)^{-1}\left(\frac{c'}{r^2} - \frac{c}{r^3}\right) + \left(\frac{c''}{r^2} - \frac{2c'}{r^3} + \frac{2c}{r^4}\right) = 0. \tag{18}$$

Using the fact that $\alpha'' = -\left(\frac{\alpha'}{r} + \frac{\alpha'^3}{r}\right)$ we have:

$$c''r^4 - 2c'r^3 + 2cr^2 - k_1 c'' r^2 + 3k_1 c' r - 3k_1 c = 0, \tag{19}$$

which give the solution:

$$c(r) = k_3 r + k_4 r\sqrt{r^2 - k_1}. \tag{20}$$

We know that $\beta' = -\alpha'$. Thus:

$$\beta(r) = -\alpha(r) + k_5, \tag{21}$$

where k_3, k_4 and k_5 are constants to be determined. k_5 can be incorporated into a new coordinate z' in the metric (6) by a straightforward reparametrization of the coordinate z. Then,

$$\beta(r) = -\alpha(r),$$

which indicates that our local string is invariant under Lorentz boost. However, the other integration constants can be determined only after relating them with the internal structure of the cosmic string and making a proper match between the internal and the exterior solutions of the metric.

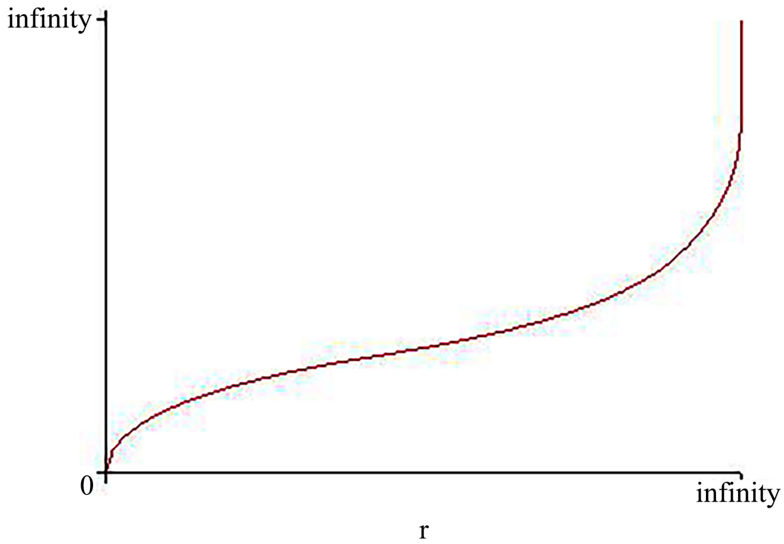

Figure 1. $\alpha(r)$ for $k_1 = 1$.

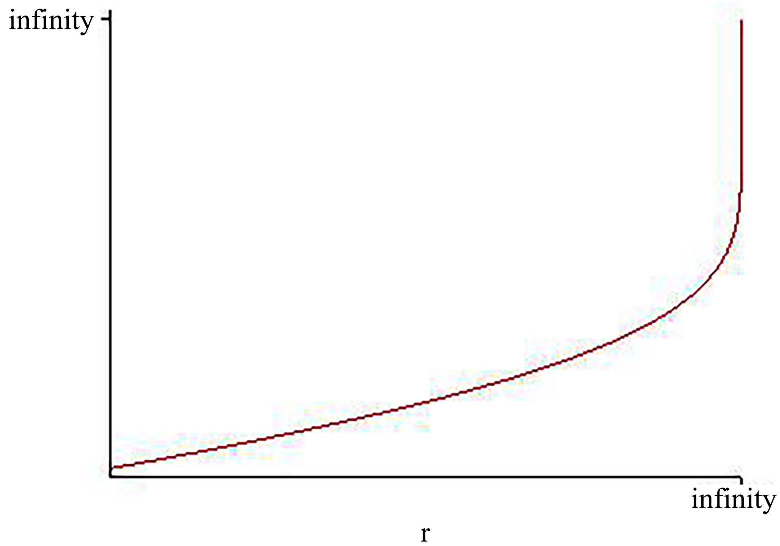

Figure 2. $\alpha(r)$ for $k_1 = 10$.

4. Summary and Analysis of the Solution

Since the expressions (17) and (20) are complicated, we will make some numerical analysis in order to understand their behavior for various values of the parameter k_1. These are represented in Figures 1-3 and we labeled k_1 as $k_1 = 1, 10$ *and* 100, respectively. These graphs show that the solution is not regular and is singular when r tends to infinity.

We remind the reader that the main goal of this paper was to consider a straight and neutral cosmic string in the NDL theory. This result has been presented in Section 3. At a first sight, it could appear that we reached our goal. However, a careful look at the behavior of the functions $\alpha(r)$ (it is straightforward to do the same with the function $c(r)$) shows that this metric is singular for large r. It could represent a string if one introduces a cut-off in the parameter k_1 but there is no reason, *a priori*, to do that, unless we include here also the internal structure of the string. Therefore, what we got here is

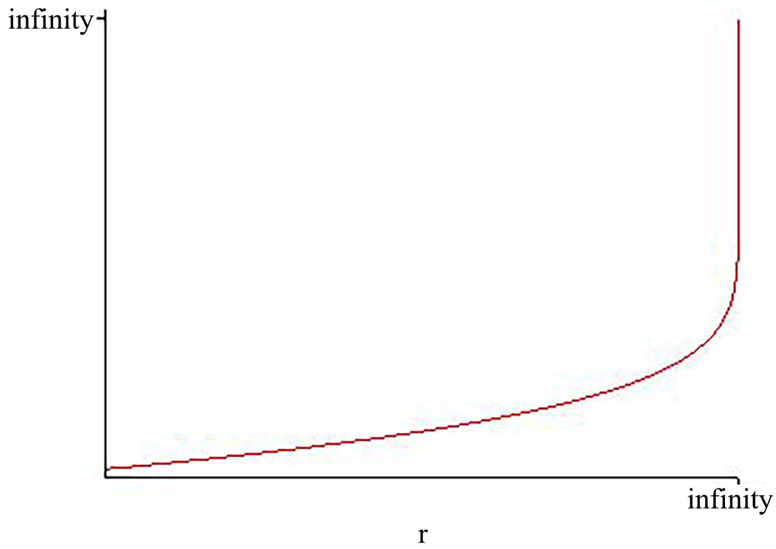

Figure 3. $\alpha(r)$ for $k_1 = 100$.

an exact cylindrically symmetric vacuum metric in the NDL theory but this metric definitively does not represent the exterior region of a cosmic string and it is singular at infinity.

Finally, we must stress that, looking at the expressions for the solutions of $\alpha(r)$ and $c(r)$, we conclude that r must satisfy $r > k_1$, otherwise we obtain imaginary values for the position r. We conjecture that it might be possible to separate the exterior metric into two regions, the $r < k_1$ and the $r > k_1$ in an analogous way as in the Rindler space. We plan to return to this point in a forthcoming paper.

Acknowledgements

The authors would like to thank Profs. M. Novello, J. A. Helayel-Neto, V. C. de Andrade and D. Muller for profite discussions and support. T. O. Rosa acknowledges the hospitality of the Intituto de Física of Universidade Federal Fluminense where part of this work has been developed. The authors would like to thank the anonymous referee for very constructive suggestions which improve the original manuscript.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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