

# Checking the Alleged “Incompatibility of the Relic Dark Energy Hypothesis” with Physically Admissible Solutions to the Cosmic Ray Problem of Doubly Special Relativity

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**How to cite this paper:** Beckwith, A.W. (2017) Checking the Alleged “Incompatibility of the Relic Dark Energy Hypothesis” with Physically Admissible Solutions to the Cosmic Ray Problem of Doubly Special Relativity. *Journal of High Energy Physics, Gravitation and Cosmology*, 3, 588-599. <https://doi.org/10.4236/jhepgc.2017.34045>

**Received:** May 8, 2017

**Accepted:** August 30, 2017

**Published:** September 2, 2017

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## Abstract

We offer evidence that the Trans Planckian hypothesis about Dark energy is incompatible with necessary and sufficient conditions for solving the cosmic ray problem along the lines presented by Magueijo *et al.* We can obtain conditions for a dispersion relationship congruent with the Trans Planckian hypothesis only if we cease trying to match cosmic ray data which is important in investigating Doubly Special Relativity. This leads us to conclude that the Trans Planckian hypothesis is inconsistent with respect to current astrophysical data when modeled by Doubly Special Relativity and needs to be seriously revised. Or the Doubly Special Relativity Hypothesis needs to be abandoned.

## Keywords

Doubly Special Relativity, Trans Planckian Physics

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## 1. Introduction

We examine if an alteration of special relativity presented by Magueijo and Smolin [1], assuming joining the speed of light and Planck energy as a new invariant permits a dispersion relationship which will set dark energy [2] from the “tail mode” of ultra high momentum contributions (of the universe) markedly lower than the total energy of the universe. We find that the answer is yes after modifying an energy equation of  $E = MC^2$  to obtain a highly non linear dispersion relationship. However, this dispersion relationship does NOT solve the cosmic

ray problem for low momentum values [1]. Our derived dispersion relationship  $\omega_M(k)$  matches the Epstein function used by Mercini *et al.* [2] only if we cease trying to fit cosmic ray data [5] which lead to Magueijo [3] proposing their alteration of special relativity in the first place. We follow Mersini *et al.* [2] in their derivation of a Trans Planckian dark energy over total energy ratio. Our results argue that we cannot reconcile the requirements of a solution of the “cosmic ray” problem of special relativity in a manner congruent with Mercinis [2] ratios of dark matter energy to total energy being calculated via a Bogoliubov coefficient [4]. The dispersion relationship which we obtained which actually permitted us to calculate the energy of the tail modes of Trans Planckian dark energy [2] vs. total energy ratio [2] to have a value less than ten to the minus 30 power mimics the Epstein function [2] in a manner which contravenes necessary and sufficient conditions [1] for solving the cosmic ray problem of special relativity. Our calculations imply that a Trans-Planckian dark energy depends upon initial conditions which are too specialized and which do not match up with known astrophysical data obtained as of the 1990s. This is in tandem with Lemoine, Martin, and Uzan [5] who dispute on the Trans Planckian hypothesis on different grounds.

## 2. Description of Procedure Used to Obtain Energy Density Ratio

What Mersini [2] did was to use ultra low dispersion relationship values for ultra high momentum values to obtain “ultra low” energy values which were and remain allegedly “frozen” today [2]. They found, using the Epstein function for frequency dispersion relationships a range of frequencies  $\leq H_0$ , where  $H_0$  is the present Hubble rate of expansion. From there, they computed Trans-Planckian dark energy modes which are about 122 to 123 orders of magnitude smaller than the total energy of the universe assumed for their expansion model. Note in this discussion that  $\omega_k(k)$  refers to the dispersion relationship Mercini [2] derived, while  $\omega_M(k)$  will be a dispersion relationship derived from Magueijo and Smolin’s [3] modification of special relativity. Mersini [2] changed a standard linear dispersion relationship to one which has a modified Epstein function with a peak value for frequency given when  $k = k_C$  and where we have if we can set  $k \ll k_C$

$$\omega_k^2(k) \approx k^2 \quad (1)$$

which means for low values of momentum we have a linear relationship for dispersion vs. “momentum” in low momentum situations. In addition we also have that

$$\omega_k^2(k \gg k_C) \approx \exp(-k/k_C) \xrightarrow{k \rightarrow \infty} 0 \quad (2)$$

We also have a specific “tail mode” energy region picked by:

$$\omega_k^2(k_H) \equiv H_0^2 \quad (3)$$

to obtain  $k_H$ . We then have an energy calculation for the “tail” modes:

$$\langle \rho_{TAIL} \rangle_K = \frac{1}{2 \cdot \pi^2} \cdot \int_{K_H}^{\infty} k dk \int \omega_K(k) \cdot d\omega_K \cdot |\beta_k|^2 \tag{4}$$

which is about 122 orders of magnitude smaller than

$$\langle \rho_{TOTAL} \rangle_K = \frac{1}{2 \cdot \pi^2} \cdot \int_0^{\infty} k dk \int \omega_K(k) \cdot d\omega_K \cdot |\beta_k|^2 \tag{5}$$

allowing us to write

$$\frac{\langle \rho_{TAIL} \rangle_K}{\langle \rho_{TOTAL} \rangle_K} \approx \frac{k_H^2}{M_P^4} \cdot \omega_K^2(k_H) \approx \frac{H_0^2}{M_P^2} \approx 10^{-122} \tag{6}$$

Here, the tail modes (of energy) are chosen as “frozen” during any expansion of the universe. This is for energy modes for frequency regions  $\omega_K^2(k) \leq H_0^2$  so that we have resulting ‘tail modes’ of energy obeying Equation (5) above.

### 3. Forming a Dispersion Relationship from Magueijo and Smolins Energy Values and Then Subsequently Modifying It

We shall next determine what sort of dispersion relationship we can obtain by the revision of special relativity Magueijo [3] proposed. Magueijo [3] states that the energy of an independent particle will not exceed  $E_p$  in value, which is the Planck energy. This Planck energy is the inverse of the Planck length defined by  $l_p = \sqrt{\hbar \cdot G / c^3} \approx 10^{-44}$  cm, where  $G$  is the gravitational constant and  $c$  is the speed of light. Specifically, Magueijo and Smolin [3] state that  $E_{PARTICLE} = E_p$  if and only if the rest mass of a particle obtains an infinite value. If we set  $\hbar = c = 1$ , we have  $[M = M_p] = [E_p]$  as an upper bound. This upper bound with respect to particle energy is consistent with respect to four principles elucidated by Magueijo and Smolin [3], which are as follows:

- 1) Assume relativity of inertial frames: When gravitational effects can be neglected, all observers in free, inertial motions are equivalent. This means that there is no preferred state of motion.
- 2) Assume an equivalence principle: Under the effect of gravity, freely falling observers are all equivalent to each other and are equivalent to inertial observers.
- 3) A new principle is introduced: The observer independence of Planck energy. *i.e.* that there exists an invariant energy scale which we shall take to be the Planck energy.
- 4) There exists a correspondence principle: At energy scales much smaller than  $E_p$ , conventional special and general relativity are true: that is that they hold to first order in the ratio of energy scales to  $E_p$ . We ask now how can these principles be fashioned into predictions as to energy values, which we shall use to obtain dispersion relationships. Magueijo and Smolin [3] obtained a modified relationship between energy and mass:

$$E_0 = \frac{m_0 \cdot c^2}{1 + \frac{m_0 \cdot c^2}{E_p}} \tag{7}$$

which if  $m = \gamma \cdot m_0$  and  $c$  set = 1 becomes:

$$E = \frac{m}{1 + \frac{m}{E_p}} \tag{8}$$

We found it useful to work with, instead:

$$E = \frac{m}{\left(1 + \beta \cdot \frac{m}{E_p}\right)^{11}} \left(1 - \frac{m}{E_p}\right) \tag{9}$$

with a power of 11 put in the denominator due to string theory dimensions which gives us preferred numerical values we are seeking for the ratio of dark energy over total cosmological energy. If  $E_{PARTICLE} < E_p$  and  $m = \alpha \cdot k$ , then  $\frac{m}{E_p} \equiv \frac{k}{k_p} < 1$  permits are write of Equation (9) above as (if  $\beta \equiv 1000$ ):

$$\omega_M(k) = \frac{\alpha \cdot k}{\left(1 + \beta \frac{k}{k_p}\right)^{11}} \cdot \left(1 - \frac{k}{k_p}\right) \tag{10}$$

where we used  $\hbar = c = 1$  and  $[E] = [\hbar \cdot \omega] = [\omega_K(k)]$  which if  $k \ll k_p$  will lead to the same result as spoken of with the modified Epstein function [2], assuming that  $|\alpha|^2 \cong 1$ , so:

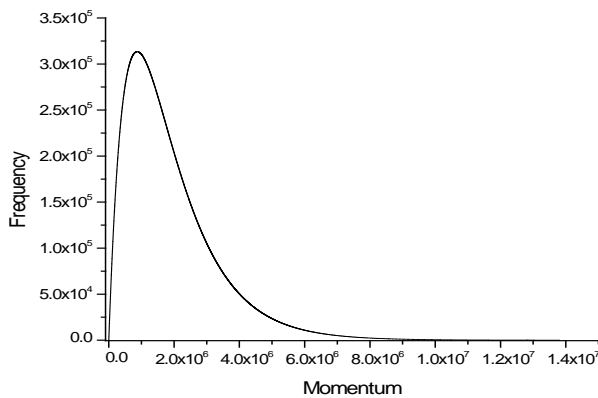
$$\omega_M^2(k) \approx k^2 \tag{11}$$

Furthermore, if  $k \rightarrow k_p - \varepsilon_+$ , Equation (10) will give us

$$\omega_M^2(k_p - \varepsilon_+) \cong \varepsilon_+ \tag{12}$$

which if  $\omega_1(k) \equiv \omega_M(k)$  gives the values seen in **Figure 1** below.

Note how the cut off value of momentum  $k_p$  is due to  $\left(1 - \frac{k}{k_p}\right)$  as a quantity in dispersion behavior leads to the results seen in **Figure 1**.



**Figure 1.** Graph of 1<sup>st</sup> dispersion relationship  $\omega_M(k)$  against momentum. This gives the desired behavior in line with the Trans Planckian dark energy hypothesis. However,  $\beta \equiv 10^3$ !

We can contrast this dispersion behavior with:

$$\omega_1(k) = \frac{\alpha \cdot k}{\left(1 + \beta_1 \frac{k}{k_p}\right)^{11}} \cdot \exp\left(-\beta_2 \cdot \frac{k}{k_p}\right) \tag{13}$$

We set  $\beta_1 \equiv 1$  and  $\beta_2 \equiv 100$ , leading to **Figure 2** as given below. Note, if  $\beta_1 \equiv 1000$  and  $\beta_2 \equiv 0$  we recover Equation (9).

So we used a tail mode energy expressions as given by

$$\langle \rho_{TAIL} \rangle_M = \frac{1}{2 \cdot \pi^2} \cdot \int_{k_H}^{k_p} k dk \int \omega_M(k) \cdot d\omega_M \cdot |\beta_k|^2 \tag{14}$$

and

$$\langle \rho_{TOTAL} \rangle_M = \frac{1}{2 \cdot \pi^2} \int_0^{k_p} k dk \int \omega_M(k) \cdot d\omega_M \cdot |\beta_k|^2 \tag{15}$$

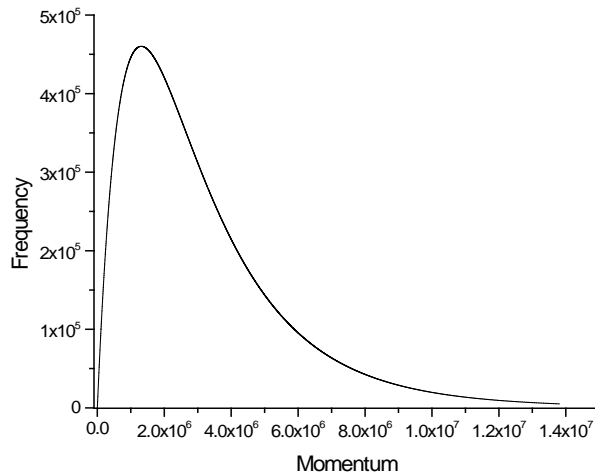
so we obtain [2] a “frozen” tail mode energy vs. total energy ratio of

$$\frac{\langle \rho_{TAIL} \rangle_M}{\langle \rho_{TOTAL} \rangle_M} = \frac{\int_{k_H}^{k_p} k dk \int \omega_M(k) \cdot d\omega_M \cdot |\beta_k|^2}{\int_0^{k_p} k dk \int \omega_M(k) \cdot d\omega_M \cdot |\beta_k|^2} < 10^{-30} \text{ and } \neq 10^{-122} \tag{16}$$

when we are using  $k_H \leq \frac{k_p}{2}$ . Equation (16) has a lower bound  $\approx 10^{-122}$  as stated by Mersini [2] in Equation (6) if we use  $\omega_M(k_H) \approx H_0$ . Detuning the sensitivity of this ratio to exact  $k_H \leq (M) \cdot k_p$  for any  $M < 1$  is extremely important to the viability of our physical theory about how dark matter plays a role in inflationary cosmology.

### 4. The Bogoliubov Function Used in This Paper

We followed Mercinis [4] assumption of negligible deviations from a strictly



**Figure 2.** Graph of 2<sup>nd</sup> dispersion relationship  $\omega_1(k)$  against momentum which has too broad a width to be useful.

thermal universe, and we proved it in our bogoliubov coefficient calculation. This lead to us picking the “thermality coefficient” [4]  $B$  to be quite small. In addition, the ratio of confocal times as given by  $\left| \frac{\eta}{\eta_c} \right|$  had little impact upon Equ-

ation (16). Also,  $x_0 = \frac{k}{k_p} \leq 1$ . Therefore,

$$|\beta_k|^2 \equiv \frac{\sinh^2\left(\pi \cdot \frac{B}{2} \cdot \frac{1}{k} \cdot \left| \frac{\eta}{\eta_c} \right|\right) + \cos^2\left(\frac{\pi}{2} \cdot \sqrt{1 - 4 \cdot B \cdot e^{-x_0}}\right)}{\sinh^2\left(\pi \cdot (2 - B) \cdot \frac{1}{k} \cdot \left| \frac{\eta}{\eta_c} \right|\right) - \sinh^2\left(\pi \cdot \frac{B}{2} \cdot \frac{1}{k} \cdot \left| \frac{\eta}{\eta_c} \right|\right)} \quad (17)$$

We derive this expression in the 1<sup>st</sup> appendix entry. In addition, we note that Bastero-Gil, in 2008 in the IDM conference, of 2008 in Stockholm, brought up a discussion of the results of [6], as useful research results, and then adopted using the results of the document given in [6]. Then, Bastero-Gil when using the results of [6] subsequently delineated the size of tail energy density from Dark matter as  $\rho_x \approx 10^{-122} M_p^4$  which is consistent with our findings that our Bogoliubov function as given by Equation (17) may be often approximated by a constant with small effects on calculating the ratio of energy for the tail vs. total energy [2] given in Equation (6) above.

## 5. Analytical and Numerical Evaluation of Equation (16)

We evaluate  $\omega_M(k) \cdot d\omega_M(k)$  in light of Equation (12) in our Equation (16) integrand. We then obtain:

$$\omega_M(k) \cdot d\omega_M(k) = \left[ \frac{k \cdot \left(1 - \frac{k}{k_p}\right)}{\left(1 + \beta \frac{k}{k_p}\right)^{22}} - \frac{k^2}{k_p} \cdot \frac{\left(1 - \frac{k}{k_p}\right)}{\left(1 + \beta \frac{k}{k_p}\right)^{22}} - 11 \cdot \frac{k^2}{k_p} \cdot \frac{\left(1 - \frac{k}{k_p}\right)^2}{\left(1 + \beta \frac{k}{k_p}\right)^{23}} \right] \cdot dk \quad (18)$$

and set up a numerical parameterization of

$$\int_A^{k_p} k dk \int \omega_M(k) \cdot d\omega_M \cdot |\beta_k|^2 \quad (19)$$

with  $\beta_k$  chosen by considerations presented in Mercini’s [4] 2<sup>nd</sup> paper.

## 6. Why We Still Were Unable to Match Cosmic Ray Data and Found Our Dispersion Relationship Not Physically Tenable

$\beta \equiv 1000$  in Equation (10) was picked so  $k_H$  could have a wide range of values.

This permitted  $\frac{\langle \rho_{TAIL} \rangle_M}{\langle \rho_{TOTAL} \rangle_M}$  to be bounded below by a value  $\leq 10^{-30}$  for

$k_H \leq \frac{k_p}{2}$  in line with de tuning the sensitivity of the ratio results if we use

$\beta \equiv 1000$  in the Equation (10) dispersion relationship. We obtain Mercini’s main result [2] at the expense of not matching cosmic ray data [1]. We should note that Equation (13) lead to a far broader dispersion curve width as given in

**Figure 2**, which also necessitated a far larger  $k_H$  value needed to have the frequency  $\omega_M(k_H) \approx H_0$  as used by Mercini [2]. This in turn leads to a much bigger value for a lower bound for Equation (16) than what would obtain numerically if we used Equation (10) for dispersion. Detuning the sensitivity of this ratio to be  $k_H \leq (M) \cdot k_p$  for any  $M < 1$  is extremely important to the viability of our physical theory about how dark matter plays a role in inflationary cosmology. We find that this result is still not sufficient to match the cosmic ray problem [1] since Equation (10) gives us:

$$\omega_M(k) \xrightarrow{k \ll k_p} \frac{k}{\left(1 + \beta_3 \frac{k}{k_p}\right)} \tag{20}$$

The  $\beta_3 \cong 11 \times 10^{+3}$  whereas we would prefer to find  $\beta_3 \cong 11 \times 10^{-10}$ .

### 7. Can $\beta_3 \cong 11 \times 10^{-10}$ with a Modified Dispersion Relationship?

The answer is no even after a modification of our dispersion relationship:

$$\omega_2(k) = \frac{\alpha \cdot k}{\left(1 + \beta \left(\frac{k}{k_p}\right)^L\right)^{11}} \cdot \left(1 - \left(\frac{k}{k_p}\right)^L\right) \tag{21}$$

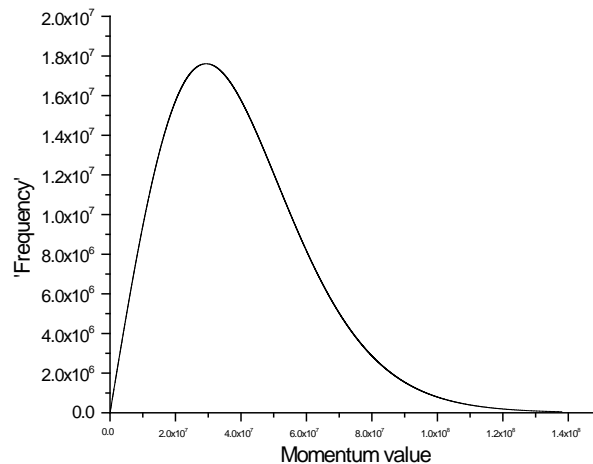
With  $L = 2$ , then 3 put in. However, even with a value of  $L = 2$  put in Equation (21) we obtained, for  $\beta \cong 2.25$  and  $k_H \cong \frac{k_p}{2}$

$$\frac{\langle \rho_{TAIL} \rangle_2}{\langle \rho_{TOTAL} \rangle_2} = \frac{\int_{k_H}^{k_p} k dk \int \omega_2(k) \cdot d\omega_2 \cdot |\beta_k|^2}{\int_0^{k_p} k dk \int \omega_2(k) \cdot d\omega_2 \cdot |\beta_k|^2} \leq 6.425 \times 10^{-3} \tag{22}$$

which has a very different lower bound than the behavior seen in Equation (16). If we pick  $\beta \cong 10^{-10}$  as suggested by T. Jacobson [1] to try to “solve” the cosmic ray problem, we then find that Equation (22) approaches unity which thereby throws into question the Trans-Planckian dark energy hypothesis. Indeed, we believe that the entire Trans-Planckian model of Dark energy makes initial conditions, which contravene known astrophysical cosmic ray data [1] that has been collected in the last decade. Graphically, having even  $\beta \cong 2.25$  for Equation (21) in **Figure 3** creates a dispersion versus momentum graph, which is much greater in width than **Figure 1** which has a much larger  $\beta \cong 10^3$  value. Appendix entry 2 shows us that we still could not match the beta coefficient values [1] needed to solve the cosmic ray problem of special relativity.

### 8. Conclusion

We found that the dispersion relationship given in Equation (10) and its limiting behavior shown in Equation (20) gives the lower bound behavior as noted in Equation (16) above for a wide range of possible  $k_H \leq M \cdot k_p$  values if  $M < 1$



**Figure 3.** Graph of 3<sup>rd</sup> dispersion relationship  $\omega_2(k)$  against momentum which is still too broad in width, and has  $\beta \equiv 2.25$ .

above. This was, however, done for a physically unacceptably large  $\beta \equiv 10^3$  value [1] while we wanted, instead  $\beta \equiv 10^{-10}$  in order to solve the cosmic ray problem [1]. Our additional modifications of dispersion relationships as noted in **Appendix 2** still lead to unacceptably large dark energy versus total energy values. We then conclude that the Trans Planckian dark energy hypothesis contravenes known solutions to the cosmic ray problem of special relativity and is thereby in need of substantial revision. And we think that this document should be compared against the predictions given in [7], since modification of the Dark Energy hypothesis may impact gravity theories. Their confirmation or rejection will be affecting predictions done by Corda, and require careful analysis, and are integral to a fuller understanding of scalar-tensor gravitational alternatives to General Relativity.

## Acknowledgements

This work is supported in part by National Nature Science Foundation of China grant No. 110752.

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## Appendix Entry 1: Deriving the Bogoliubov Coefficient for Section III

### Part I. Initial Assumptions

We derive the Bogoliubov coefficient, which is used in Equation (16) of the main text. We refer to Mersini's article [4] which has a Bogoliubov coefficient which takes into account a deviation function  $\Gamma(k_0, B)$ , which is a measure of deviation from thermality [4] in the spectrum of co moving frequency values  $\Omega_n(k)$  over different momentum values. Note that  $\eta$  is part of a scale factor  $a(\eta) = |\eta_c/\eta|$  and  $k = n/a(\eta)$  so that "momentum"  $k \propto |\eta|$ . Also if we are working with the conformal case of  $\varepsilon = 1/6$  appearing [4] in:

$$\Omega_n^2 = a^2(\eta) \cdot \omega_{NON-LIN}^2(k) - (1 - 6 \cdot \varepsilon) \cdot \frac{a''}{a} = a^2(\eta) \cdot \omega_{NON-LIN}^2(k) = a^2(\eta) \cdot F^2(k) \quad (1)$$

then for small momentum:

$$\omega_{NON-LIN}^2(\tilde{k}_0) \approx \tilde{k}_0^2 \quad (2)$$

if "momentum"  $\tilde{k}_0 \ll k_p$ , where we use the same sort of linear approximation used by Mercini [2], as specified for Equation (17) of their article [2] if the Epstein function specified in Equation (1) of the main text has a linear relationship. We write out a full treatment of the dispersion function  $F(k)$  [4] since it permits a clean derivation of the Bogoliubov coefficient which has the deviation function  $\Gamma(k_0, B)$ . We begin with [4]:

$$|\beta_k|^2 \equiv |\beta_n|^2 = \frac{\sinh^2(2 \cdot \pi \cdot \hat{\Omega}_-) + \Gamma(k_0, B)}{\sinh^2(2 \cdot \pi \cdot \hat{\Omega}_+) - \sinh^2(2 \cdot \pi \cdot \hat{\Omega}_-)} \quad (3)$$

where we get an appropriate value for the deviation function  $\Gamma(k_0, B)$  [4] based upon having the square of the dispersion function  $F(k)$  obey Equations (1) and (2) above for  $\tilde{k}_0 \ll k_p$ . Note,  $k_p$  is a maximum momentum value along the lines Magueijo [3] suggested for an  $E_p$  Plank energy value.

### Part II. Deriving Appropriate $\Gamma(k_0, B)$ Deviation Function Values

We look at how Bastero-Gil [4] obtained an appropriate  $\Gamma(k_0, B)$  value. Bastero-Gil wrote:

$$\Gamma(k_0, B) = \cosh^2\left(\frac{\pi}{2} \cdot \sqrt{4 \cdot B \cdot e^{-x_0} - 1}\right) \quad (4)$$

with

$$x_0 = \frac{\tilde{k}_0}{k_p} \ll 1 \quad (5)$$

and

$$F^2(k) = (k^2 - \tilde{k}_1^2) \cdot V_0(x, x_0) + k^2 \cdot V_1(x - x_0) + \tilde{k}_1^2 \quad (6)$$

where  $\tilde{k}_1 < k_p$  and where  $\tilde{k}_1$  is in the Trans-Planckian regime but is much greater than  $k_0$ . We are determining what  $B$  should be in Equation (16) of the main text provided that  $F(k) \approx k$  as  $x = \frac{\tilde{k}}{k_p} \rightarrow x_0$  which will lead to specific restraints we place upon  $V_0(x, x_0)$  as well as  $V_1(x - x_0)$  above. Following Bastero-Gil [4], we write:

$$V_0(x, x_0) = \frac{C}{1 + e^x} + \frac{E \cdot e^x}{(1 + e^x) \cdot (1 + e^{x-x_0})} \tag{7}$$

and:

$$V_1(x - x_0) = -B \cdot \frac{e^x}{(1 + e^{x-x_0})^2} \tag{8}$$

When  $x = \frac{\tilde{k}}{k_p} \rightarrow x_0 \ll 1$  we get [2] [4]

$$F^2(k_0) \equiv \omega_{NON-LIN}^2(k_0) \cong -k_1^2 \cdot \left(1 - \frac{c}{2} - \frac{E}{4}\right) + k_0^2 \cdot \left(\frac{c}{2} + \frac{E}{4} - \frac{B}{4}\right) \cong k_0^2 \tag{9}$$

which then implies  $0 < B \approx \varepsilon_+ \ll 1$ . Then we obtain:

$$\Gamma(k_0, B \cong \varepsilon_+) \cong \cosh^2\left(\left(\frac{\pi}{2} + \varepsilon_+\right) \cdot i\right) \approx \varepsilon_+ \ll 1 \tag{10}$$

and

$$|\beta_k|^2 \equiv |\beta_n|^2 \cong \frac{\sinh^2(2 \cdot \pi \cdot \hat{\Omega}_-) + \varepsilon_+}{\sinh^2(2 \cdot \pi \cdot \hat{\Omega}_+) - \sinh^2(2 \cdot \pi \cdot \hat{\Omega}_-)} \tag{11}$$

### Part III. Finding Appropriate $\hat{\Omega}_+$ and $\hat{\Omega}_-$ Values

We define, following Bastero-Gil [4]

$$\hat{\Omega}_\pm = \frac{1}{2} \cdot (\hat{\Omega}_{OUT} \pm \hat{\Omega}_{IN}) \tag{12}$$

where we have that

$$\Omega^{OUT} = \xrightarrow{\eta \rightarrow \infty} \Omega_n (\eta \equiv \infty) \tag{13}$$

and

$$\Omega^{IN} = \xrightarrow{\eta \rightarrow -\infty} \Omega_n (\eta \equiv -\infty) \tag{14}$$

whereas we have that

$$\hat{\Omega}_{\tilde{k}} = \frac{\Omega_{\tilde{k}}}{n} \tag{15}$$

where  $\tilde{k}$  denotes either out or in. Also:

$$\Omega^{OUT} \cong \Omega^{IN} \cong 1 \tag{16}$$

which lead to:

$$\hat{\Omega}_+ \cong \left(1 - \frac{B}{2}\right) \cdot \frac{1}{n} = \left(1 - \frac{B}{2}\right) \cdot \frac{1}{k} \cdot \left| \frac{\eta}{\eta_C} \right| \cong \frac{1}{k} \cdot \left| \frac{\eta}{\eta_C} \right| \quad (17)$$

as well as

$$\hat{\Omega}_- \cong \frac{B}{2} \cdot \frac{1}{n} \cong 0 \quad (18)$$

## Appendix Entry 2: How Equation (16) of Text Changes for Varying $\beta$ Values and Different Dispersion Relationships

Starting with Equation (21) of the main text.

$$\text{If } \beta = 1.05 \text{ and } L = 1/2, \left(\frac{k}{k_P}\right) \rightarrow \sqrt{\frac{k}{k_P}}, \text{ then } \frac{\langle \rho_{TAIL} \rangle_M}{\langle \rho_{TOTAL} \rangle_M} \cong 0.371$$

$$\text{If } \beta = 1.05 \text{ and } L = 1, \left(\frac{k}{k_P}\right) \rightarrow \left(\frac{k}{k_P}\right), \text{ then } \frac{\langle \rho_{TAIL} \rangle_M}{\langle \rho_{TOTAL} \rangle_M} \cong 0.263$$

$$\text{If } \beta = 1.05 \text{ and } L = 2, \left(\frac{k}{k_P}\right) \rightarrow \left(\frac{k}{k_P}\right)^2, \text{ then } \frac{\langle \rho_{TAIL} \rangle_M}{\langle \rho_{TOTAL} \rangle_M} \cong 0.115$$

$$\text{If } \beta = 10.5 \text{ and } L = 1/2, \left(\frac{k}{k_P}\right) \rightarrow \sqrt{\frac{k}{k_P}}, \text{ then } \frac{\langle \rho_{TAIL} \rangle_M}{\langle \rho_{TOTAL} \rangle_M} \cong 1.935 \times 10^{-5}$$

$$\text{If } \beta = 10.5 \text{ and } L = 1, \left(\frac{k}{k_P}\right) \rightarrow \left(\frac{k}{k_P}\right), \text{ then } \frac{\langle \rho_{TAIL} \rangle_M}{\langle \rho_{TOTAL} \rangle_M} \cong 7.347 \times 10^{-6}$$

$$\text{If } \beta = 10.5 \text{ and } L = 2, \left(\frac{k}{k_P}\right) \rightarrow \left(\frac{k}{k_P}\right)^2, \text{ then } \frac{\langle \rho_{TAIL} \rangle_M}{\langle \rho_{TOTAL} \rangle_M} \cong 6.7448 \times 10^{-8}$$

We need  $\beta \cong 10^{-10}$  with  $\frac{\langle \rho_{TAIL} \rangle_M}{\langle \rho_{TOTAL} \rangle_M} \leq 10^{-30}$  to get our results via this

Trans-Plankian model to be consistent with physically verifiable solutions to the cosmic ray problem.