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Rearrangement Invariant, Coherent Risk Measures on L^0

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Abstract

By this paper, we give an answer to the problem of definition of coherent risk measures on rearrangement invariant, solid subspaces of L^0 with respect to some atom less probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This problem was posed by F. Delbaen, while in this paper we proposed a solution via ideals of L^0 and the class of the dominated variation distributions, as well.

Keywords

Rearrangement Invariance, Dominated Variation, Moment-Index

1. Introduction

In (Delbaen, 2009), the problem of defining a risk measure on a *solid*, *rearrangement invariant* subspace of $L^0(\Omega, \mathcal{F}, \mathbb{P})$-space of random variables with respect to some atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We recall that a vector space E, being a vector subspace of L^0 is called *rearrangement invariant* if for random rariables $y, x \in L^0$, which have the same distribution, $x \in E$ implies $y \in E$. Also, the space E is *solid* if for andom viariables $y, x \in L^0$, $|y| \leq |x|, x \in E$, implies $y \in E$. In (Delbaen, 2009), there is an extensive treatment of this problem, related to the role of the spaces L^∞ and L^1, compared to E, especially in (Delbaen, 2009). On the other hand, the whole paper (Delbaen, 2002) is devoted to the difficulties of defining coherent risk measures on subspaces of L^0, while it is proved that if the probability space is atomless, no coherent risk measure is defined all over L^0 (Delbaen, 2002). Of course these attempts of moving from L^∞ to appropriately defined subspaces of L^0, are related to the tail propertes of the random variables in actuarial science and finance and more specifically to heavy-tailed distributed random variables. The actual problem behind these seminal article by F. Delbaen is since we cannot define a coherent risk measure on the entire L^0, whether subspaces of L^0 which

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are both alike L^{∞} and preserve nice distributional properties (from the aspect of heavy-tails). Especially, we treat the rearrangement invariance in the sense of remaining in the same *class of distributions* and not by requiring distributional invariance. This is the topic of our paper.

2. Ideals of L^0 and Heavy-Tailed Distributions

It is well-known that since $L^0(\Omega, \mathcal{F}, \mathbb{P})$ is a Riesz space, being ordered by the pointwise-$\mathbb{P}$-a.e. partial ordering ≥, it would be taken as a Riesz subspace of $\mathbb{R}^{\Omega}$. Hence, it may be considered to be an order-complete Riesz space. Let us take an element y of L^0, which corresponds to a *heavy-tailed* random variable. This indicates that either for $y^+ = y \vee 0$ for $y^- = (-y) \vee 0$,

$$\mathbb{E}\left(\mathrm{e}^{\varepsilon \cdot y^*}\right) = \infty,$$

for any real number $\varepsilon > 0$, where $* = +$ or $* = -$. Heavy-tailed random variables may not have even a finite moment $\mathbb{E}(|y|)$. On the other hand, according to (Aliprantis & Border, 1999), the principal ideal E_y generated by y in E, endowed by the norm

$$\|z\|_{\infty} = \inf\{\lambda > 0 \,|\, |z| \le \lambda |y|\},$$

is an AM-space with *order unit* $|y|$. We also have to mention the following relevant.

Lemma 2.1 *If* $y > 0, (y \ge 0, y \ne 0)$ *and* y *is a heavy-tailed random variable, then every* $z \in [-y, y]$ *is a heavy-tailed random variable.*

Proof. Since $y \ge z, z \in [-y, y]$, we get that for the sets $y(t) = \{\omega \in \Omega \,|\, y(\omega) \le t\}, z(t) = \{\omega \in \Omega \,|\, z(\omega) \le t\}$, the inclusion $y(t) \subseteq z(t)$ holds, which implies $F_y(t) \le F_z(t), t \in \mathbb{R}$ for the corresponding cumulative distribution functions. Since for the integral

$$\int_0^{\infty} \mathrm{e}^{\varepsilon \cdot t} \mathrm{d}F_y(t) = \infty,$$

holds for any $\varepsilon > 0$, this implies

$$\int_0^{\infty} \mathrm{e}^{\varepsilon \cdot t} \mathrm{d}F_z(t) = \infty,$$

for any $\varepsilon > 0$.

We recall the class $\mathcal{D}$ of *dominated variation* distributions:

$$F \in \mathcal{D} \Leftrightarrow \limsup_{t \to +\infty} \frac{\bar{F}(tu)}{\bar{F}(t)} < \infty, \; u \in (0,1).$$

This class is a sub-class of heavy -tailed distributions, see (Cai & Tang, 2004).

Theorem 2.2 *If* $F_y \in \mathcal{D}$, *where* $\mathcal{D}$ *denotes the class of dominated variation distributions respectively, then for every* $z \in E_y$, $F_z \in \mathcal{D}$.

Proof. According to what is proved in (Cai & Tang, 2004), the class $\mathcal{D}$ is *convolution-closed*, namely if $F_x, F_y \in \mathcal{D}$, then $F_{x+y} \in \mathcal{D}$. First, we have to prove that if $F_y \in \mathcal{D}$, then $F_z \in \mathcal{D}$, for any $z \in E_y$. Since $z \in E_y$, there exists some $\lambda > 0$ such that $z \in [-\lambda \cdot y, \lambda \cdot y]$. But $F_{\lambda \cdot y} \in \mathcal{D}$. This is easy to prove, since if

$$\limsup_{t \to +\infty} \frac{\bar{F}_y(ut)}{\bar{F}_y(t)} < \infty,$$

for any $u \in (0,1)$, then in order to prove that

$$\limsup_{t \to +\infty} \frac{\bar{F}_{\lambda \cdot y}(ut)}{\bar{F}_{\lambda \cdot y}(t)} < \infty,$$

for any $u \in (0,1)$, then we get that the above limsup is equal to

$$\limsup_{r\to+\infty}\frac{\bar{F}_y(ru)}{\bar{F}_y(r)}<\infty,$$

for any $u\in(0,1)$. Hence, $F_{\lambda\cdot y}\in\mathcal{D}$. Moreover, we have to prove that if $z\in[-\lambda\cdot y,\lambda\cdot y]$, then $F_z\in\mathcal{D}$. From the previous Lemma,

$$\bar{F}_z(t)\le\bar{F}_y(t),\ \bar{F}_z(tu)\le\bar{F}_y(tu),\ u\in(0,1).$$

From the properties of the tail function of *z* we also have that since $ut<t$ for $u\in(0,1)$, then

$$\bar{F}_z(ut)\ge\bar{F}_z(t).$$

Hence,

$$1\le\frac{\bar{F}_z(ut)}{\bar{F}_z(t)}\le\frac{\bar{F}_y(ut)}{\bar{F}_z(t)}.$$

Since $\frac{1}{\bar{F}_y(t)}\le\frac{1}{\bar{F}_z(t)}$,

$$\limsup_{t\to+\infty}\frac{\bar{F}_z(tu)}{\bar{F}_z(t)}<\infty,$$

which is the desired conclusion.

Hence we obtain subspaces *E* of L^0, which are actually the ideals E_y which satisfy the *rearrangement invariance* property, while they contain non-integrable distributions, in the sense that for any $z\in E_y$ there is a maximum *p* for which the moment $\mathbb{E}\left(|z|^p\right)$ exists in $\mathbb{R}$. Let us discuss more this question. A notion which is very important is the one of the *moment index*. We recall that the *moment index* for a non-negative random variable *x* is equal to

$$\mathbb{I}(x)=\sup\left\{v\,|\,\mathbb{E}\left(x^v\right)<\infty\right\}.$$

We also recall that if $F_x\in\mathcal{D}$, then $\mathbb{I}(x)<\infty$, see in (Seneta, 1976), (Tang & Tsitsiashvili, 2003). The use of the moment index in the specific case is that despite the validity of the (Delbaen, 2002), due to the fact that the elements of E_y distributions lie in the class $\mathcal{D}$, we assure that at least in the ideal E_y, we assure a general level of non-integrability of $z\in E_y$, given by a finite $\mathbb{I}(z)$. About the question whether the class $\mathcal{D}$ is the greatest in which the specific Theorem holds, we have to mention that if we move up to the class of the *subexponential* distributions, it is not convolution-closed, see for example in (Leslie, 1989). As it is also well-known from (Aliprantis & Border, 1999), the dual space E_y^* of E_y is an AL-space, since the ideal E_y is an AM-space with unit $|y|$, as mentioned above. Hence, we keep the dual pair

$$\langle E_y,E_y^*\rangle,$$

for any of the *y* described above.

3. Expected-Shortfall on Ideals of L^0

Taking any $y>0$ whose $F_y\in\mathcal{D}$, and defining the corresponding dual pair

$$\langle E_y,E_y^*\rangle,$$

we may define an Expected Shortfall-form risk measure on E_y. We have to notice that E_y satisfies both the order and the distributional rearrangement property, as a subspace of L^0. This is due to the properties of the class $\mathcal{D}$ of the dominated variation distributions. Hence we use (Kaina & Rüschendorf, 2009) of the dual (robust) representation of the usual Expected Shortfall in order to prove the following.

Theorem 3.1 *The functional* $R:E_y\to\bar{\mathbb{R}}$, *where*

$$R(x) = \sup_{f \in \left[0, \frac{1}{a}e\right]} f(-x)$$

is an $(E_{y,+}, y)$-coherent risk measure, where $e \in E_{y,+} \setminus \{0\}$ is such that $\langle y, e \rangle = a, a > 0$.

Proof.

1)

$$R(x + t \cdot y) = \sup_{f \in \left[0, \frac{1}{a}e\right]} f(-x - t \cdot y) = \sup_{f \in \left[0, \frac{1}{a}e\right]} f(-x) - t,$$

for any $t \in \mathbb{R}$, due to the order completeness of the ideal E_y (*y*-Translation Invariance).

2)

$$R(x + z) = \sup_{f \in \left[0, \frac{1}{a}e\right]} f(-x - z) \leq \sup_{f \in \left[0, \frac{1}{a}e\right]} f(-x) + \sup_{f \in \left[0, \frac{1}{a}e\right]} f(-z) = R(x) + R(z),$$

for any $x, z \in E_y$ (Subadditivity).

3)

$$R(\lambda \cdot x) = \sup_{f \in \left[0, \frac{1}{a}e\right]} f(-\lambda \cdot x) = \lambda \cdot \sup_{f \in \left[0, \frac{1}{a}e\right]} f(-x),$$

for any $\lambda \in \mathbb{R}_+$ and $x \in E_y$ (Positive Homogeneity).

4) If $x \geq z$ then for any $f \in \left[0, \frac{1}{a}e\right]$ we get $f(-z) \geq f(-x)$. Hence by taking suprema all over $f \in \left[0, \frac{1}{a}e\right]$, we ger $R(z) \geq R(x)$ $x, z \in E_y$ ($E_{y,+}$-Monotonicity).

Finally, if we suppose that the dual pair $\langle E_y^*, E_y \rangle$ is a symmentric Riesz pair, or else that E_y^* has order-continuous norm (see also (Aliprantis & Border, 1999)), then the values of R are finite since they represent the supremum value of a weak-star continuous linear functional on a weak-star compact set, which is the box of functionals $\left[0, \frac{1}{a}e\right]$. Otherwise, the infinity of the values of R may be excused by the presence of heavy-tailed distributions.

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