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Electrostatic and Magnetostatic Forces That Arise from Electrostatic and Magnetostatic Pressures*

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Abstract

With the insight provided by a balance equation of electromagnetic momentum, we compare the force on a dielectric slab inside a capacitor with the force on a magnetizable rod inside a solenoid. We conclude that these forces are not exactly analogous, as usually thought. We present a device that is a proper analogy of the case of a dielectric slab inside a capacitor. Our analysis shows the significance of the electrostatic and magnetostatic pressures to the understanding of these effects and shows the conceptual differences between both cases.

Keywords

Polarizable Electric and Magnetic Media, Electric and Magnetic Pressures, Electromagnetic Force Densities, Macroscopic Maxwell Equations, Balance Equations of Electromagnetic Momentum

1. Introduction

In the interaction of electromagnetic fields with matter, there appear forces that theory must explain in order to have some control on these forces. Perhaps the simplest of these are the forces that arise from the interaction of electrostatic and magnetostatic fields with polarizable and magnetizable matter.

The electrostatic case may be illustrated with the force exerted on a dielectric slab partially introduced into a charged parallel plate capacitor. In the magne-

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tostatic case, the force that is exerted on a magnetizable bar inside a solenoid, is more familiar and has many applications. It is generally considered that this magnetic effect is analogous to the electrostatic case. We show in this paper that this is not the case, and present a magnetic device really analogous to the electrostatic case.

In the case of a dielectric slab inside a capacitor, the force is usually explained as the action of the non-uniform fringing electrostatic field on the electric dipoles of the dielectric. We have shown elsewhere [1] that this force arises rather from the action of Maxwell's electrostatic stresses at the dielectric-vacuum interface. We have also shown that the above magnetic effect arises from the magnetostatic stresses, but from the tension part of the stress [2]. This interpretation is based on the original conception of Faraday and Maxwell that the electromagnetic forces are transmitted through stresses in matter and vacuum. This view is expressed clearly by Maxwell [3]:

"If we further admit that every part of a dielectric medium through which electric induction is taking place there is a tension, like that of a rope, in the direction of the lines of force, and a pressure in all directions at right angles to the lines of force, we may account for all the mechanical actions which take place between electrified bodies."

Indeed, we have shown elsewhere [1] [2] that in the electric case the force has its origin in the compression around the lines of force, while in the magnetic case the force has its origin in the tension.

2. Theory

In formal terms it is shown in texts [4] [5] [6] that Maxwell's stress tensor can be decomposed in principal axes, with components given by a tension

$$t = \frac{1}{2}\epsilon E^2 \tag{1}$$

along the lines of force, and an orthogonal compression

$$K = -\frac{1}{2}\epsilon E^2 \tag{2}$$

around the lines of force where ϵ is the permittivity and $\boldsymbol{E}$ is the electric field. In spite that these facts are discussed in well-known texts, some authors [7] [8] [9] [10] [11], express surprise at the fact that an electrostatic field can exert forces in a direction orthogonal to the lines of force.

We propose to analyze these forces from the point of view of balance equations that can be derived from Maxwell's equations and therefore are well founded. These balance equations have the structure

$$\nabla \cdot \overleftrightarrow{\boldsymbol{T}} - \partial_t \boldsymbol{g} = \boldsymbol{f} \tag{3}$$

where $\overleftrightarrow{\boldsymbol{T}}$ is Maxwell's stress tensor, $\boldsymbol{g}$ is a momentum density related to Poynting's vector, and $\boldsymbol{f}$ is a force density. In static conditions this balance equation permits to obtain the force as a volume integration of the force density, or as a surface integral of the stress tensor over a surface surrounding the volume of interest. We have done these calculations [1] [2], for the cases of the capacitor and solenoid and we find that they are not exactly analogous. In the first case the force arises from the compression around the electrostatic lines of force, while in the second case the force arises rather from the tension of the magnetostatic lines of force.

In the case of the capacitor, the force accepted as correct is [12] [13],

$$\boldsymbol{F} = \frac{1}{2}\epsilon_0 \chi_e E^2 A_0 \hat{\boldsymbol{k}}. \tag{4}$$

Here A_0 is the cross section of the dielectric slab, and $\boldsymbol{E}$ is the uniform electric field inside the capacitor, taken as the potential difference divided by the separation of the plates.

If we use the method of a balance equation in the calculation of this force, the force density adequate to solve the problem is

$$\boldsymbol{f} = -\frac{1}{2}\boldsymbol{E} \times (\nabla \times \boldsymbol{P}). \tag{5}$$

In the case of the solenoid we have that the force on a magnetizable bar partially introduced in it is [12] [13],

$$\boldsymbol{F} = \frac{1}{2}\mu_0 \chi_m H^2 A_0 \hat{\boldsymbol{k}}. \tag{6}$$

Here $\boldsymbol{H}$ is taken as the uniform field inside the solenoid. No reference is made to the fringing field (We follow the convention proposed by Purcell and Morin [14] and Griffiths [11]) of considering $\boldsymbol{B}$ as the magnetic field and $\boldsymbol{H}$ as an auxiliary field.

The analogy with the capacitor is apparent, with the correspondences

$$\mu_0 \to \epsilon_0, \ \chi_m \to \chi_e, \ H \to E, \tag{7}$$

It is worthwhile to note that in this case the correspondence of fields is

$$\boldsymbol{H} \to \boldsymbol{E}$$

rather than

$$\boldsymbol{B} \to \boldsymbol{E},$$

as frequently proposed [12] [15].

If the analogy is further pursued, and as force density we propose the analogy to "Equation (5)"

$$\boldsymbol{f}_{newM} = -\frac{1}{2}\mu_0 \boldsymbol{H} \times (\nabla \times \boldsymbol{M}), \tag{8}$$

we do not obtain the known result. In this case the adequate force density is rather [2]

$$\boldsymbol{f}_{newM} = -\frac{1}{2}\mu_0 (\boldsymbol{H} \cdot \nabla) \boldsymbol{M}, \tag{9}$$

which is obtained from a magnetic energy density.

It is important to note that this force density is part of a balance equation, and is different from the familiar force density

$$\boldsymbol{f}_M = -\frac{1}{2}\mu_0 (\boldsymbol{M}\cdot\nabla)\boldsymbol{H}, \tag{10}$$

which applies in the case of magnetic dipoles immersed in a non-uniform magnetic field.

The force densities "Equations (5) and (9)" are different, but the forces to which they lead are similar. Then, which is the magnetic system analogous to the capacitor, such that the forces arise from similar force densities? One aim of the present paper is to propose such a magnetic system, another is to explain the origin and relation among different force densities.

The action of stresses across the boundary between two dielectrics may be calculated with the Helmholtz force density [6], which also appears naturally in a balance equation derived from Maxwell's equations [16]. However, the force that arises from the action of these stresses can also be calculated with the force density "Equation (5)"; indeed, this force density is equivalent to the Helmholtz force density as consequence of the discontinuity in the permittivity. The action of the stresses can also be seen in the sucking of a liquid dielectric by a laser that falls orthogonally to the surface of the liquid, since in this case we have an average electric field parallel to interface.

3. Analogy between Capacitor with Dielectric Slab and Solenoid with Magnetizable Bar?

Let us consider the space between the poles of a permanent horseshoe magnet. The field can be assumed almost uniform in middle space, while the fringing field is non-uniform. We define the x direction as that of the uniform field and a magnetizable bar of cross section A_0 and length L is partially introduced a distance l (in the z direction), in the region of the uniform field Figure 1.

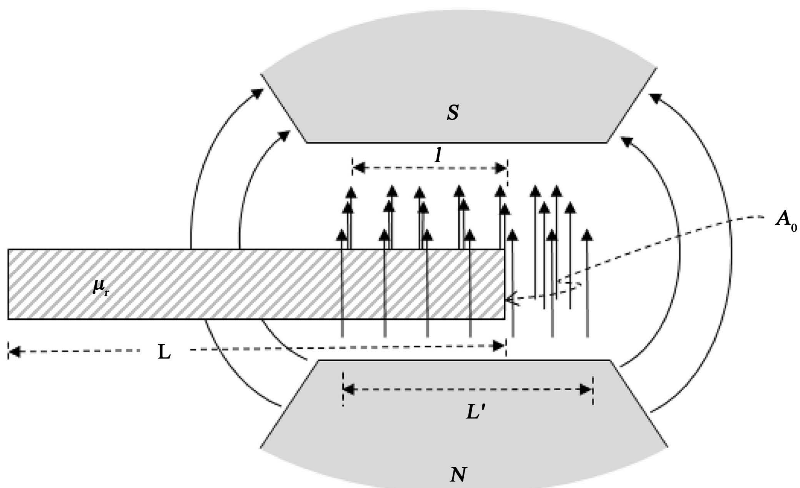

Figure 1. A magnetizable bar of cross section A_0 is partially introduced a distance l between the poles of a permanent magnet. In the region of length L' the magnetic field is approximately uniform.

The magnetic energy is

$$U = \frac{1}{2}\int_{Vol} \mathrm{d}V \boldsymbol{B} \cdot \boldsymbol{H} = \frac{1}{2}\int_{Vol} \mathrm{d}V \, BH, \tag{11}$$

since the magnetic field $\boldsymbol{B}$ and the $\boldsymbol{H}$ field are parallel (we are considering linear media). If the integration volume is $A_0 L'$, where L' is the length over which $\boldsymbol{B}$ and $\boldsymbol{H}$ are approximately uniform, then the energy is

$$U = \frac{1}{2}\left[(BH)_{medium} + (BH)_{vacuum} \right] A_0 L', \tag{12}$$

since $\boldsymbol{B}$ and $\boldsymbol{H}$ are uniform.

The continuity conditions for the fields are

$$B_{vacuum} = \mu_0 H_{vacuum} \tag{13}$$

and

$$B_{medium} = \mu_0 \mu_r H_{medium}. \tag{14}$$

Then, with the constitutive relation,

$$\mu_r = 1 + \chi_m, \tag{15}$$

we can write "Equation (12)" in the form

$$U = \frac{1}{2}\left[\mu_0 \mu_r H^2 A_0 z + \mu_0 H^2 A_0 \left(L' - z \right) \right], \tag{16}$$

which can be rewritten as

$$U = \frac{1}{2} \mu_0 H^2 A_0 \left(\chi_m z + L' \right), \tag{17}$$

leading to the force on the magnetizable bar

$$\boldsymbol{F} = -\frac{\partial U}{\partial l} = \frac{1}{2} \mu_0 \chi_m H^2 A_0 \hat{\boldsymbol{k}}. \tag{18}$$

This has the same structure that the known force, and if the analogy expressed in "Equation (7)" is used, "Equation (18)" is transformed into the known force exerted on a dielectric slab inside a capacitor.

4. The Force Density

The magnetic energy density of linear magnetizable matter when introduced into a magnetic field is [12] [13] [17]

$$u = -\left(\frac{1}{2} \mu_0 \boldsymbol{M} \cdot \boldsymbol{H} \right). \tag{19}$$

Since the field of the permanent magnet can be considered constant, it is equivalent to a constant current system, and the force density can be calculated with the equation

$$\boldsymbol{f} = \left(\nabla u \right)_I . \tag{20}$$

Then,

$$\boldsymbol{f} = -\nabla \left(\frac{1}{2} \mu_0 \boldsymbol{M} \cdot \boldsymbol{H} \right), \tag{21}$$

which by means of the vector identity

$$\nabla(\boldsymbol{a}\cdot\boldsymbol{b})=(\boldsymbol{a}\cdot\nabla)\boldsymbol{b}+(\boldsymbol{b}\cdot\nabla)\boldsymbol{a}+\boldsymbol{a}\times(\nabla\times\boldsymbol{b})+\boldsymbol{b}\times(\nabla\times\boldsymbol{a}), \tag{22}$$

is transformed into

$$\boldsymbol{f}=-\frac{1}{2}\mu_0\left[(\boldsymbol{M}\cdot\nabla)\boldsymbol{H}+(\boldsymbol{H}\cdot\nabla)\boldsymbol{M}+\boldsymbol{M}\times(\nabla\times\boldsymbol{H})+\boldsymbol{H}\times(\nabla\times\boldsymbol{M})\right]. \tag{23}$$

Since there are not free currents, we have

$$\nabla\times\boldsymbol{H}=0. \tag{24}$$

Now, the magnetic susceptibility is different from zero only in matter; therefore the constitutive relation is

$$\boldsymbol{M}=\chi_m\Theta(l-z)\boldsymbol{H}, \tag{25}$$

where χ_m is constant and Θ is the Heaviside distribution.

In this case the $\boldsymbol{H}$ field is in the x direction, so we have

$$\boldsymbol{M}(z)=\chi_m\Theta(l-z)H_x\hat{\boldsymbol{i}}. \tag{26}$$

Therefore,

$$(\boldsymbol{H}\cdot\nabla)\boldsymbol{M}=0. \tag{27}$$

Also, since the interface is in the region where $\boldsymbol{H}$ is uniform,

$$(\boldsymbol{M}\cdot\nabla)\boldsymbol{H}=0. \tag{28}$$

In this way "Equation (23)" reduces to

$$\boldsymbol{f}_{new\,Mag}=-\frac{1}{2}\mu_0\left[\boldsymbol{H}\times(\nabla\times\boldsymbol{M})\right], \tag{29}$$

which is precisely the force density proposed by analogy, "Equation (8)".

In the next section we show that this force density also leads to the known result obtained by conventional methods, that is, from an energy density.

5. Force on a Magnetizable Bar in the Field of a Permanent Magnet Calculated with the Proposed Force Density

With the constitutive relation "Equation (26)" the proposed new force density "Equation (29)" results

$$\boldsymbol{f}_{new\,Mag}=-\frac{1}{2}H_x\left[\partial_z M_x(z)\right]\hat{\boldsymbol{k}}, \tag{30}$$

since the field $\boldsymbol{H}$ is in the *x* direction.

Now, the mathematical result

$$\partial_z\Theta(l-z)=-\delta(l-z) \tag{31}$$

permits to write "Equation (30)" in the form

$$\boldsymbol{f}_{new\,Mag}=\frac{1}{2}\mu_0\chi_m H^2\delta(l-z)\hat{\boldsymbol{k}}. \tag{32}$$

In order to obtain the force on the magnetizable rod it is necessary to integrate the force density over a volume $L'h$ around the interface, that is,

$$\boldsymbol{F} = \int_{vol\ around\ interface} \boldsymbol{f}_{new\,M}\,\mathrm{d}S\mathrm{d}z. \tag{33}$$

If the cross section of the interface is $L'h = A_0$, then with the force density "Equation (32)" the force results

$$\boldsymbol{F} = \int_{vol\ around\ interface} \frac{1}{2}\mu_0\chi_m H^2 h\delta(l-z)\hat{\boldsymbol{k}}\mathrm{d}z. \tag{34}$$

After integration we obtain the force

$$\boldsymbol{F} = \frac{1}{2}\mu_0\chi_m H^2 A_0\hat{\boldsymbol{k}}, \tag{35}$$

which is the result previously obtained with the usual procedure using an energy density, "Equation (18)". This proves the correctness of the proposed force density.

6. Maxwellian Momentum Balance Equation and the Proposed Force Density

We have shown that the force density "Equation (8)" leads to the same result than the method using the energy density "Equation (18)". However, the force density "Equation (8)" was proposed by analogy with the case of a capacitor with dielectric. Then, it is necessary to give this force density a solid foundation. This can be done with a momentum balance equation derived directly from Maxwell's equations. One such equation is [16]

$$\begin{aligned}&\nabla\cdot\left\{\boldsymbol{DE}+\boldsymbol{BH}-\frac{1}{2}\boldsymbol{I}\left(\boldsymbol{D}\cdot\boldsymbol{E}+\boldsymbol{B}\cdot\boldsymbol{H}\right)\right\}-\partial_t\left(\boldsymbol{D}\times\boldsymbol{B}\right)\\&=\rho\boldsymbol{E}+\boldsymbol{J}\times\boldsymbol{B}+\frac{1}{2}\left[\left(\nabla\boldsymbol{E}\right)\cdot\boldsymbol{D}-\left(\nabla\boldsymbol{D}\right)\cdot\boldsymbol{E}+\left(\nabla\boldsymbol{H}\right)\cdot\boldsymbol{B}-\left(\nabla\boldsymbol{B}\right)\cdot\boldsymbol{H}\right].\end{aligned} \tag{36}$$

Our aim is to show that this balance equation contains the proposed force density. Since we are dealing with a magnetostatic case there are not free currents, the balance equation "Equation (36)" reduces to

$$\nabla\cdot\left\{\boldsymbol{BH}-\frac{1}{2}\boldsymbol{I}\left(\boldsymbol{B}\cdot\boldsymbol{H}\right)\right\}=\frac{1}{2}\left[\left(\nabla\boldsymbol{H}\right)\cdot\boldsymbol{B}-\left(\nabla\boldsymbol{B}\right)\cdot\boldsymbol{H}\right]. \tag{37}$$

We then have to show that the right-hand member of "Equation (37)" contains the proposed force density. This can be done as follows.

The identity

$$\left(\nabla\boldsymbol{v}\right)\cdot\boldsymbol{u}=\boldsymbol{u}\times\left(\nabla\times v\right)+\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{v} \tag{38}$$

and the constitutive relation

$$\boldsymbol{B}=\mu_0\left(\boldsymbol{H}+\boldsymbol{M}\right), \tag{39}$$

permits to express the right-hand member of "Equation (37)" as

$$\begin{aligned}&\frac{1}{2}\left[\left(\nabla\boldsymbol{H}\right)\cdot\boldsymbol{B}-\left(\nabla\boldsymbol{B}\right)\cdot\boldsymbol{H}\right]\\&=\frac{1}{2}\mu_0\left[\boldsymbol{M}\times\left(\nabla\times\boldsymbol{H}\right)+\left(\boldsymbol{M}\cdot\nabla\right)\boldsymbol{H}-\boldsymbol{H}\times\left(\nabla\times\boldsymbol{M}\right)-\left(\boldsymbol{H}\cdot\nabla\right)\boldsymbol{M}\right].\end{aligned} \tag{40}$$

In order to see that the force density "Equation (8)" is contained in the right-hand member of "Equation (40)" we must note that there are not free currents and the field $\boldsymbol{H}$ is uniform at the interface, so that "Equations. (24), (27), and (28)" hold, and therefore "Equation (40)" reduces to

$$\frac{1}{2}\left[(\nabla \boldsymbol{H})\cdot \boldsymbol{B}-(\nabla \boldsymbol{B})\cdot \boldsymbol{H}\right]=-\frac{1}{2}\boldsymbol{H}\times(\nabla\times \boldsymbol{M}), \tag{41}$$

which is the result looked for.

In this way, the balance equation for these particular conditions is

$$\nabla\cdot\left\{\boldsymbol{B}\boldsymbol{H}-\frac{1}{2}\boldsymbol{I}(\boldsymbol{B}\cdot\boldsymbol{H})\right\}=-\frac{1}{2}\boldsymbol{H}\times(\nabla\times \boldsymbol{M}). \tag{42}$$

These arguments show that the proposed force density "Equation (8)" has a firm foundation on Maxwell's equations. Additional support for the proposed force density can be obtained by a surface integration of the left-hand member of the balance equation "Equation (42)", that is, of the magnetostatic stress tensor. This is done in the following section.

7. Calculation of the Force by a Surface Integration of the Stress Tensor

The force in terms of the magnetostatic stress tensor is given by

$$\boldsymbol{F}=\oint_{\sigma}\mathrm{d}\boldsymbol{S}\cdot\vec{\boldsymbol{T}}, \tag{43}$$

where

$$\vec{\boldsymbol{T}}=\left[\boldsymbol{B}\boldsymbol{H}-\frac{1}{2}\vec{\boldsymbol{I}}(\boldsymbol{B}\cdot\boldsymbol{H})\right] \tag{44}$$

and σ is a closed surface. The force is the total magnetic force exerted on the matter enclosed in the surface. In this case a convenient surface is one that surrounds the interface.

The constitutive relation

$$\boldsymbol{B}=\mu_0\mu_r H\hat{\boldsymbol{i}} \tag{45}$$

permits to write the tensor $\vec{\boldsymbol{T}}$ as

$$\vec{\boldsymbol{T}}=\mu_0\mu_r H^2\left[\hat{\boldsymbol{i}}\hat{\boldsymbol{i}}-\frac{1}{2}\vec{\boldsymbol{I}}\right]. \tag{46}$$

With this expression "Equation (43)" can be written in the form

$$\boldsymbol{F}=\oint_{\sigma}\mathrm{d}\boldsymbol{S}\cdot\mu_0\mu_r H^2\left[\hat{\boldsymbol{i}}\hat{\boldsymbol{i}}-\frac{1}{2}\vec{\boldsymbol{I}}\right], \tag{47}$$

where surface σ can be taken as a parallelepiped with the surfaces parallel to the interface defined by vectors $\pm\hat{\boldsymbol{k}}$, while the surfaces defined by vectors $\pm\hat{\boldsymbol{i}}$ and $\pm\hat{\boldsymbol{j}}$ do not contribute since they give contributions that cancel each other. Then the surface integral becomes

$$\boldsymbol{F}=\int\mathrm{d}S\left(-\hat{\boldsymbol{k}}\right)\cdot\mu_0\mu_r H_{medium}^2\left[\hat{\boldsymbol{i}}\hat{\boldsymbol{i}}-\frac{1}{2}\vec{\boldsymbol{I}}\right]-\int\mathrm{d}S\hat{\boldsymbol{k}}\cdot\mu_0 H_{vacuum}^2\left[\hat{\boldsymbol{i}}\hat{\boldsymbol{i}}-\frac{1}{2}\vec{\boldsymbol{I}}\right], \tag{48}$$

and the force results

$$\boldsymbol{F} = \frac{1}{2}\mu_0 H^2 \left(\mu_r - 1\right) \int \hat{\boldsymbol{k}} \mathrm{d}S. \tag{49}$$

Now, since

$$\int \hat{\boldsymbol{k}} \mathrm{d}S = \hat{\boldsymbol{k}} A_0, \tag{50}$$

the force can be expressed as

$$\boldsymbol{F} = \frac{1}{2}\mu_0 \left(\mu_r - 1\right) H^2 A_0 \hat{\boldsymbol{k}}, \tag{51}$$

which with the constitutive relation "Equation (15)" takes the form

$$\boldsymbol{F} = \frac{1}{2}\mu_0 \chi_m H^2 A_0 \hat{\boldsymbol{k}}, \tag{52}$$

which is the expected result, "Equation (35)",momentum balance equation give by "Equation (42)".

8. Force Densities for Static Fields and Their Equivalence

Force densities like "Equations (5), (8) and (9)" are unfamiliar, but they are firmly founded on Maxwell's equations, as we proceed to show.

We begin with the momentum balance equation derived from Maxwell's equations, "Equation (36)". This equation is equivalent to Maxwell's equations with linear media, from which it is obtained by means of vector and dyadic identities. For static conditions and in absence of free charge and current densities this balance equation can be separated into two independent equations,

$$\nabla \cdot \left\{ \boldsymbol{DE} - \frac{1}{2}\boldsymbol{I}\left(\boldsymbol{D}\cdot\boldsymbol{E}\right) \right\} = -\frac{1}{2}\left[\left(\nabla \boldsymbol{E}\right)\cdot \boldsymbol{D} - \left(\nabla \boldsymbol{D}\right)\cdot \boldsymbol{E} \right] = \boldsymbol{f}_{Max\ elec\ media}, \tag{53}$$

$$\nabla \cdot \left\{ \boldsymbol{BH} - \frac{1}{2}\boldsymbol{I}\left(\boldsymbol{B}\cdot\boldsymbol{H}\right) \right\} = -\frac{1}{2}\left[\left(\nabla \boldsymbol{H}\right)\cdot \boldsymbol{B} - \left(\nabla \boldsymbol{B}\right)\cdot \boldsymbol{H} \right] = \boldsymbol{f}_{Max\ mag\ media}, \tag{54}$$

where subscripts in the right-hand members indicate the force densities for the electrostatic and magnetostatic cases.

We want now relate the balance "Equations (53) and (54)" with the electric and magnetic Helmholtz force densities, electric and magnetic,

$$\boldsymbol{f}_{Helm\ elect} = -\frac{1}{2}E^2 \nabla \epsilon + \frac{1}{2}\nabla\left[E^2 \rho_m \frac{\partial \epsilon}{\partial \rho_m} \right] \tag{55a}$$

$$\boldsymbol{f}_{Helm\ mag} = -\frac{1}{2}H^2 \nabla \mu + \frac{1}{2}\nabla\left[H^2 \rho_m \frac{\partial \mu}{\partial \rho_m} \right]. \tag{55b}$$

It can be shown that these force densities are contained in "Equations (53) and (54)", but for the present analysis it is enough to consider only the first terms on the right, since the dependence of the permittivity and permeability on mass density is for our purposes irrelevant. Then, using the constitutive relation

$$\boldsymbol{D} = \epsilon_0 \epsilon_r \boldsymbol{E} \tag{56}$$

the right-hand side of "Equation (53)" results

$$\frac{1}{2}\left[(\nabla \boldsymbol{E})\cdot \boldsymbol{D}-(\nabla \boldsymbol{D})\cdot \boldsymbol{E}\right]=\frac{1}{2}E^2\boldsymbol{\nabla}\epsilon. \tag{57}$$

Analogously, for the magnetic case, with the constitutive relation

$$\boldsymbol{B}=\mu_0\mu_r\boldsymbol{H} \tag{58}$$

we obtain

$$\frac{1}{2}\left[(\nabla \boldsymbol{H})\cdot \boldsymbol{B}-(\nabla \boldsymbol{B})\cdot \boldsymbol{H}\right]=\frac{1}{2}H^2\nabla\mu. \tag{59}$$

The force densities "Equations (57) and (59)" look very different from the force densities "Equations (5), (8) and (9)", but now it is evident their relation to Helmholtz's force densities. Let us see how they are related.

The force density "Equation (5)", which has been used to calculate the force on a dielectric slab inside a charged capacitor, is convenient to establish the equivalence among different force densities.

If the slab is in the z direction and has been introduced a distance l, the constitutive relation

$$\boldsymbol{P}=\epsilon_0\chi_e\boldsymbol{E}, \tag{60}$$

can be written as

$$\boldsymbol{P}(z)=\epsilon_0\chi_e\Theta(l-z)E\hat{\boldsymbol{i}}, \tag{61}$$

where Θ is Heaviside's distribution, while $\hat{\boldsymbol{i}}$ is a unit vector in the x-direction, which is the direction of the electrostatic field. Then the force density "Equation (5)" becomes

$$\boldsymbol{f}=-\frac{1}{2}E\left(\partial_z P_x(z)\right)\hat{\boldsymbol{k}}, \tag{62}$$

where $\hat{\boldsymbol{k}}$ is a unit vector in the z direction. With "Equation (60)" written as $P(z)=\epsilon_0\chi_e(z)E_x$ we can express "Equation (62)" in the form

$$\boldsymbol{f}=-\epsilon_0\frac{1}{2}E^2\left(\partial_z\chi_e(z)\right)\hat{\boldsymbol{k}}. \tag{63}$$

If we consider the relation between the relative permittivity and the susceptibility,

$$\epsilon_r=1+\chi_e, \tag{64}$$

we can see that the right-hand member of "Equation (57)" equals "Equation (62)". This validates firmly "Equation (5)".

In the case of a magnetizable bar inside a solenoid, we can proceed analogously to establish that "Equation (8)" is equivalent to the right-hand member of "Equation (59)".

If the axis of the solenoid is in the z direction, then "Equation (9)" turns into.

$$\boldsymbol{f}_{new\,mag}=-\frac{1}{2}\mu_0 H\left(\partial_z\boldsymbol{M}(z)\right). \tag{65}$$

Since the constitutive relation is

$$\boldsymbol{M}=\chi_m\Theta(l-z)\boldsymbol{H}, \tag{66}$$

"Equation (65)" becomes

$$f_{new\,mag} = -\frac{1}{2}\mu_0 H^2 \left(\partial_z \chi_m(z)\right)\hat{k}. \tag{67}$$

Now, the relative permeability can be expressed as

$$\mu_r = 1 + \chi_m, \tag{68}$$

so that "Equation (67)" and the right-hand member of "Equation (59)" are equal, what establishes the validity of "Equation (8)", which seems unfamiliar, but is equivalent to "Equation (59)", a part of the Helmholtz force density.

When we calculate force densities from the gradient of energy densities, the vector identity "Equation (22)" leads in a natural way to force densities like "Equations (5), (8) and (9)". What we have shown is that these force densities are contained in a momentum balance equation derived from Maxwell's equations. The Faraday-Maxwell conception of a tension along the lines of force, and compressions around these lines, permits to interpret these forces as follows.

In the case of a dielectric slab inside a capacitor, the force arises from the difference of compressions at the interface. In the case of a magnetizable bar inside a solenoid, it is the difference in tensions at the interface, while in the device here proposed as a proper analogy of the slab inside a capacitor it is again the difference in compressions what produces the force.

9. Relation with Force Densities Derived as Gradients of Energy Densities

In order to establish the equivalence of force densities obtained as gradients of energy densities and the force densities obtained from momentum balance equations we use the dyadic identity

$$\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\nabla\boldsymbol{a})\cdot\boldsymbol{b} + (\nabla\boldsymbol{b})\cdot\boldsymbol{a}. \tag{69}$$

Then the force density "Equation (53)" can be written as

$$\boldsymbol{f}_{Max\,elec\,media} = (\nabla\boldsymbol{E})\cdot\boldsymbol{D} - \frac{1}{2}\nabla(\boldsymbol{E}\cdot\boldsymbol{D}). \tag{70}$$

Therefore, we have

$$\frac{1}{2}\left[(\nabla\boldsymbol{E})\cdot\boldsymbol{D} - (\nabla\boldsymbol{D})\cdot\boldsymbol{E}\right] = (\nabla\boldsymbol{E})\cdot\boldsymbol{D} - \frac{1}{2}\nabla(\boldsymbol{E}\cdot\boldsymbol{D}). \tag{71}$$

Since the force on a dielectric slab inside a capacitor is given in terms of the polarization $\boldsymbol{P}$, we can use "Equations (56) and (64)", obtaining

$$\frac{1}{2}\left[(\nabla\boldsymbol{E})\cdot\boldsymbol{P} - (\nabla\boldsymbol{P})\cdot\boldsymbol{E}\right] = (\nabla\boldsymbol{E})\cdot\boldsymbol{P} - \frac{1}{2}\nabla(\boldsymbol{E}\cdot\boldsymbol{P}). \tag{72}$$

We have, on the other hand, the dyadic identity

$$\boldsymbol{u}\times(\nabla\times\boldsymbol{v}) = (\nabla\boldsymbol{v})\cdot\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}. \tag{73}$$

If we set $\boldsymbol{v} = \boldsymbol{E}$ and $\boldsymbol{u} = \boldsymbol{P}$, this identity becomes

$$\boldsymbol{P}\times(\nabla\times\boldsymbol{E}) = (\nabla\boldsymbol{E})\cdot\boldsymbol{P} - (\boldsymbol{P}\cdot\nabla)\boldsymbol{E}; \tag{74}$$

but $(\nabla\times\boldsymbol{E})=0$, so that

$$(\nabla\boldsymbol{E})\cdot\boldsymbol{P}=(\boldsymbol{P}\cdot\nabla)\boldsymbol{E}. \tag{75}$$

Then, since the polarization $\boldsymbol{P}(z)$ is in the x direction and the electric field is uniform,

$$(\boldsymbol{P}\cdot\nabla)\boldsymbol{E}=P_x\frac{\partial\boldsymbol{E}}{\partial x}=0, \tag{76}$$

concluding that in the case of the capacitor we have

$$(\nabla\boldsymbol{E})\cdot\boldsymbol{P}=0, \tag{77}$$

so that "Equation (72)" reduces to

$$\frac{1}{2}\left[(\nabla\boldsymbol{E})\cdot\boldsymbol{P}-(\nabla\boldsymbol{P})\cdot\boldsymbol{E}\right]=-\frac{1}{2}\nabla(\boldsymbol{E}\cdot\boldsymbol{P}). \tag{78}$$

We have then that for the electric case the force density expressed in the right-hand member of "Equation (53)", the Helmholtz force density, the force density given as the gradient of an energy density, "Equation (78)", and the unfamiliar force density "Equation (5)" are all equivalent.

In the case of the solenoid, we proceed analogously applying identity "Equation (69)" to the right-hand member of "Equation (54)", obtaining

$$\frac{1}{2}\left[(\nabla\boldsymbol{H})\cdot\boldsymbol{B}-(\nabla\boldsymbol{B})\cdot\boldsymbol{H}\right]=(\nabla\boldsymbol{H})\cdot\boldsymbol{B}-\frac{1}{2}\nabla(\boldsymbol{H}\cdot\boldsymbol{B}). \tag{79}$$

In order to put this result in terms of the magnetization $\boldsymbol{M}$ it is convenient to use the constitutive relation "Equation (39)" in "Equation (79)", obtaining

$$\frac{1}{2}\mu_0\left[(\nabla\boldsymbol{H})\cdot\boldsymbol{M}-(\nabla\boldsymbol{M})\cdot\boldsymbol{H}\right]=\mu_0(\nabla\boldsymbol{H})\cdot\boldsymbol{B}-\frac{1}{2}\mu_0\nabla(\boldsymbol{H}\cdot\boldsymbol{B}). \tag{80}$$

However, the force expressed in "Equation (18)" is obtained assuming, for paramagnetic and diamagnetic media, that the field $\boldsymbol{H}$ is nearly constant [12] so that

$$\mu_0(\nabla\boldsymbol{H})\cdot\boldsymbol{M}=0. \tag{81}$$

The results here obtained give support to this procedure, showing that an elementary problem may have interesting aspects.

Then "Equation (80)" is transformed into

$$\boldsymbol{f}_{Max\ mag\ media}=\frac{1}{2}\mu_0\left[(\nabla\boldsymbol{H})\cdot\boldsymbol{M}-(\nabla\boldsymbol{M})\cdot\boldsymbol{H}\right]=\frac{1}{2}\mu_0\nabla(\boldsymbol{H}\cdot\boldsymbol{M}), \tag{82}$$

which is valid for the magnetic tension.

10. Summary of Results

In order to appreciate better the results obtained with our approach, and facilitate the comparison of electrostatic and magnetosatic effects, we present here a summary. These results are implications of Maxwell's equations and the assumed constitutive relations applied to the electrostatic and magnetostaic conditions.

10.1. Magnetic Force Densities

$$\nabla\cdot\left\{\boldsymbol{BH}-\frac{1}{2}\boldsymbol{I}\left(\boldsymbol{B}\cdot\boldsymbol{H}\right)\right\}=-\frac{1}{2}\left[\left(\nabla\boldsymbol{H}\right)\cdot\boldsymbol{B}-\left(\nabla\boldsymbol{B}\right)\cdot\boldsymbol{H}\right]=\boldsymbol{f}_{Max\ mag\ media}, \quad \text{(Am)}$$

If the permeability does not depend of mass density, then

$$\boldsymbol{f}_{H\ mag}=-\frac{1}{2}H^{2}\nabla\mu \quad \text{(Bm)}$$

Solenoid

$\boldsymbol{H},\boldsymbol{B},\boldsymbol{M}$ in direction *z*, $\boldsymbol{M}$ jumps in interface (*z*)

$$\nabla\times\boldsymbol{H}=0 \quad \text{(C1)}$$

$$\nabla\times\boldsymbol{M}=0 \quad \text{(D1)}$$

$$\left(\boldsymbol{H}\cdot\nabla\right)\boldsymbol{M}\neq 0 \quad \text{(E1)}$$

$$\left(\boldsymbol{M}\cdot\nabla\right)\boldsymbol{H}=0 \quad \text{(F1)}$$

$$\boldsymbol{f}_{new\,M}=-\frac{1}{2}\mu_{0}\left(\boldsymbol{H}\cdot\nabla\right)\boldsymbol{M}, \quad \text{(G1)}$$

$$\boldsymbol{F}=\frac{1}{2}\mu_{0}\chi_{m}H^{2}A_{0}\hat{\boldsymbol{k}}. \quad \text{(H1)}$$

$$\boldsymbol{f}_{Helm\ mag}=-\frac{1}{2}H^{2}\nabla\mu=-\frac{1}{2}H^{2}\nabla\mu_{0}\chi_{m}=-\frac{1}{2}\left(\boldsymbol{H}\cdot\nabla\right)\boldsymbol{M} \quad \text{(K1)}$$

The force arises from the tension part of the stress tensor.

10.2. Horseshoe Permanent Magnet

$\boldsymbol{H},\boldsymbol{B},\boldsymbol{M}$ in direction *x*, $\boldsymbol{M}$ jumps in interface (*x*)

$$\nabla\times\boldsymbol{H}=0 \quad \text{(C2)}$$

$$\nabla\times\boldsymbol{M}\neq 0 \quad \text{(D2)}$$

$$\left(\boldsymbol{H}\cdot\nabla\right)\boldsymbol{M}=0 \quad \text{(E2)}$$

$$\left(\boldsymbol{M}\cdot\nabla\right)\boldsymbol{H}=0 \quad \text{(F2)}$$

$$\boldsymbol{f}_{new\ mag}=-\frac{1}{2}\mu_{0}\boldsymbol{H}\times\left(\nabla\times\boldsymbol{M}\right), \quad \text{(G2)}$$

$$\boldsymbol{F}=\frac{1}{2}\mu_{0}\chi_{m}H^{2}A_{0}\hat{\boldsymbol{k}}, \quad \text{(H2)}$$

The force arises from the compression part of the stress tensor, equivalent to a difference of pressures at the interface.

$$\boldsymbol{f}_{Helm\ mag}=-\frac{1}{2}H^{2}\nabla\mu=-\frac{1}{2}H^{2}\nabla\mu_{0}\chi_{m}=-\frac{1}{2}\mu_{0}\boldsymbol{H}\times\left(\nabla\times\boldsymbol{M}\right), \quad \text{(K2)}$$

Both previous cases

$$\nabla\times\boldsymbol{H}=0$$

$$\left(\boldsymbol{M}\cdot\nabla\right)\boldsymbol{H}=0$$

10.2.1. Electric Force Density

$$\nabla\cdot\left\{\boldsymbol{DE}-\frac{1}{2}\boldsymbol{I}\left(\boldsymbol{D}\cdot\boldsymbol{E}\right)\right\}=-\frac{1}{2}\left[\left(\nabla\boldsymbol{E}\right)\cdot\boldsymbol{D}-\left(\nabla\boldsymbol{D}\right)\cdot\boldsymbol{E}\right]=\boldsymbol{f}_{Max\ elec\ media},\quad\text{(Ae)}$$

If the permittivity does not depend of mass density,

$$\boldsymbol{f}_{Helm\ elect}=-\frac{1}{2}E^2\nabla\epsilon\quad\text{(Be)}$$

10.2.2. Capacitor

$\boldsymbol{E},\boldsymbol{D},\boldsymbol{P}$ in direction *z*, $\boldsymbol{P}$ jumps in interface (*z*)

$$\nabla\times\boldsymbol{E}=0\quad\text{(C3)}$$

$$\nabla\times\boldsymbol{P}\neq 0\quad\text{(D3)}$$

$$\left(\boldsymbol{E}\cdot\nabla\right)\boldsymbol{P}=0\quad\text{(E3)}$$

$$\left(\boldsymbol{P}\cdot\nabla\right)\boldsymbol{E}=0\quad\text{(F3)}$$

$$\boldsymbol{f}=-\frac{1}{2}\boldsymbol{E}\times\left(\nabla\times\boldsymbol{P}\right),\quad\text{(G3)}$$

$$\boldsymbol{F}=\frac{1}{2}\epsilon_0\chi_e E^2A_0\hat{\boldsymbol{k}},\quad\text{(H3)}$$

$$\boldsymbol{f}_{Helm\ elect}=-\frac{1}{2}E^2\nabla\epsilon=-\frac{1}{2}E^2\nabla\epsilon_0\chi_e=-\frac{1}{2}\boldsymbol{E}\times\left(\nabla\times\boldsymbol{P}\right)\quad\text{(K3)}$$

The force arises from the compression part of the stress tensor, equivalent to a difference of pressures at the interface.

It is important to have in mind that

Solenoid: $\left(\boldsymbol{H}\cdot\nabla\right)\boldsymbol{M}\neq 0$.

Horseshoe: $\nabla\times\boldsymbol{M}\neq 0$.

Capacitor: $\left(\boldsymbol{E}\cdot\nabla\right)\boldsymbol{P}\neq 0$.

11. Conclusions

We have shown elsewhere [16] [18] that the macroscopic Maxwell equations can be transformed, by means of vector and dyadic identities, into electromagnetic momentum balance equations. These balance equations involve different momentum flux tensors and force densities.

In the present work we use a particular balance equation to show the similarities and differences between the action of electrostatic and magnetostatic fields on linearly polarizable and magnetizable matter.

As a first point we can conclude that, based on the insights provided by this balance equation, the force exerted on a dielectric slab partially introduced into a charged capacitor is not exactly analogous to the force exerted on a magnetizable rod partially introduced into a solenoid. In the first case the force arises from the orthogonal compressions around the lines of force, equivalent to a difference of pressures at the interface, that is, the force is orthogonal to the electric field $\boldsymbol{E}$, while in the second case, the force arises from the action of the tension along the lines of force, that is, the force is parallel to the magnetic field $\boldsymbol{B}$.

As a second point, we have proposed a device that represents an exact analogy with the case of a dielectric slab inside a capacitor, in the sense that the force also arises from the compression around the magnetic lines of force, implying a difference of magnetic pressures at the interface. In this case the force is orthogonal to the magnetic field $\boldsymbol{B}$. In both cases the fringing fields are irrelevant, though in the electric case it is argued that the fringing field is the cause of the force, while in the magnetic case it is explicitly neglected [12].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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