

The Average Errors for Linear Combinations of Bernstein Operators on the Wiener Space*

Yanjie Jiang[#], Ziqing Zhang

Department of Mathematics and Physics, North China Electric Power University, Baoding, China
 Email: [#]jiangyj@126.com

Received September 18, 2013; revised October 20, 2013; accepted November 4, 2013

Copyright © 2013 Yanjie Jiang, Ziqing Zhang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

ABSTRACT

In this paper, we discuss the average errors of function approximation by linear combinations of Bernstein operators. The strongly asymptotic orders for the average errors of the combinations of Bernstein operators sequence are determined on the Wiener space.

Keywords: Linear Combinations; Bernstein Operators; Weighted L_p -Norm; Average Error; Wiener Space

1. Introduction

Let F be a real separable Banach space equipped with a probability measure μ on the Borel sets of F . Let X be another normed space such that F is continuously embedded in X . By $\|\cdot\|_X$ we denote the norm in X . Any $T: F \rightarrow X$ such that $f \mapsto \|f - T(f)\|_X$ is a measurable mapping is called an approximation operator. The p -average error of T is defined as

$$e_p(T, F, \|\cdot\|_X, \mu) := \left(\int_F \|f - T(f)\|_X^p \mu(df) \right)^{1/p}.$$

Let

$$F_0 := \{f \in C[0,1] \mid f(0) = 0\}.$$

For every $f \in F_0$ set

$$\|f\|_C := \max_{0 \leq t \leq 1} |f(t)|.$$

Then $(F_0, \|\cdot\|_C)$ becomes a separable Banach space. Denote by $\mathfrak{B}(F_0)$ the Borel class of $(F_0, \|\cdot\|_C)$ and by ω_0 the Wiener measure on $\mathfrak{B}(F_0)$ (see [1]). From [1, p. 70] we know

$$\int_{F_0} f(s)f(t)\omega_0(df) = \min\{s,t\} = \frac{1}{2}(s+t-|s-t|), \quad \forall s,t \in [0,1]. \quad (1)$$

*Supported by National Natural Science Foundation of China (Project no 10871132 and 11271263) and by a grant from Hebei province higher school science and technology research (Z2010160).

[#]Corresponding author.

The Bernstein operator on $C[0,1]$ defined by

$$B_n(f, x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}(x),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \dots, n.$$

This operator turned out to be a very interesting operator, easy to deal with and having many applications in approximation theory and practice.

Since Bernstein operators cannot be used in the investigation of higher orders of smoothness, Butzer [2] introduced combinations of Bernstein operators. Ditzian and Totik [3, p. 116] extended this method and defined the combinations as

$$L_{n,m}(f, x) := \sum_{i=0}^{m-1} C_i(n) B_{n_i}(f, x), \quad (2)$$

where n_i and $C_i(n)$ satisfy the following conditions:

- (a) $n = n_0 < \dots < n_{m-1} \leq Cn$;
- (b) $\sum_{i=0}^{m-1} |C_i(n)| \leq C$;
- (c) $\sum_{i=0}^{m-1} C_i(n) = 1$;
- (d) $\sum_{i=0}^{m-1} C_i(n) n_i^{-\rho} = 0, \quad \rho = 1, 2, \dots, m-1.$

Throughout this paper, C denotes a positive constant

independent of n and x , which may be a different constant in different cases.

For

$$\rho \in L_1[0,1], \rho \geq 0, 1 \leq p < \infty,$$

the weighted L_p -norm of $f \in C[0,1]$ is defined by

$$\|f\|_{p,\rho} := \left(\int_0^1 |f(t)|^p \cdot \rho(t) dt \right)^{\frac{1}{p}}.$$

2. Main Result

Recently G. Q. Xu [4] studied the average errors of Bernstein operators approximation on the Wiener space. Motivated by [4], we considered the average errors of function approximation by linear combinations of Bernstein operators. The strongly asymptotic orders for the average errors of the linear combinations of Bernstein operators sequence are determined on the Wiener space. We obtain:

Theorem 1. Let $1 \leq p < \infty$, $L_{n,m}(f, x)$ be given by (2), $\rho \in L_1[0,1], \rho(x) > 0$ and $\rho(x)$ is continuous on $(0,1)$. Then we have

$$e_p(L_{n,m}, F_0, \|\cdot\|_{p,\rho}, \omega_0) = \left[\sum_{i=0}^{m-1} \sum_{j=0}^{m-1} C_i(n) C_j(n) \left(\sqrt{\frac{2}{\pi n_i}} - \frac{\sqrt{\frac{n_i}{n_j}} + \sqrt{\frac{n_j}{n_i}}}{\sqrt{2\pi(n_i + n_j)}} \right) \right]^{\frac{1}{2}} \cdot \left(v_p \int_0^1 (x(1-x))^{\frac{p}{4}} \rho(x) dx \right)^{\frac{1}{p}} + o(n^{-\frac{1}{4}}),$$

where

$$v_p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |x|^p e^{-\frac{x^2}{2}} dx.$$

Here and in the following the notation $a_n = o(b_n)$ for sequences $\{a_n\}$ and $\{b_n\}$ means that

$$\lim_{n \rightarrow \infty} a_n/b_n = 0.$$

3. Proof of Theorem 1

To prove Theorem 1 we need the following two lemmas.

Lemma 1([5, p. 15]). If

$$0 < \alpha < \frac{1}{2},$$

then

$$\sum_{\substack{k \\ |\frac{k}{n} - x| \geq n^{-\alpha}}} p_{n,k}(x) \leq Cn^{-k}$$

for each $k > 0$, the constant C depending only on α

and k .

Lemma 2 ([5, p. 15]). For fixed

$$0 < x < 1, \delta > \frac{1}{3},$$

the asymptotic relation

$$\begin{aligned} p_{n,k}(x) &= \binom{n}{k} x^k (1-x)^{n-k} \\ &\cong (2\pi x(1-x)n)^{-\frac{1}{2}} \exp \left\{ -\frac{n}{2x(1-x)} \left(\frac{k}{n} - x \right)^2 \right\} \\ &= P_{n,k}(x) \end{aligned}$$

holds uniformly for all values of k satisfying the inequality

$$\left| \frac{k}{n} - x \right| \leq n^{-\delta}. \tag{4}$$

In other words,

$$\lim_{n \rightarrow \infty} p_{n,k}(x)/P_{n,k}(x) = 1$$

uniformly for all k satisfying (4).

Proof of Theorem 1. From [1, p.107] we have

$$e_p(L_{n,m}, F_0, \|\cdot\|_{p,\rho}, \omega_0) = v_p \int_0^1 \left(\int_{F_0} |f(x) - L_{n,m}(f, x)|^2 \omega_0(df) \right)^{\frac{p}{2}} \rho(x) dx. \tag{5}$$

By (2),

$$\begin{aligned} &\int_{F_0} |f(x) - L_{n,m}(f, x)|^2 \omega_0(df) \\ &= \int_{F_0} f^2(x) \omega_0(df) - 2 \sum_{i=0}^{m-1} C_i(n) \sum_{k=0}^{n_i} p_{n_i,k}(x) \\ &\quad \cdot \int_{F_0} f(x) f\left(\frac{k}{n_i}\right) \omega_0(df) + \sum_{i=0}^{m-1} C_i(n) \sum_{j=0}^{m-1} C_j(n) \\ &\quad \cdot \sum_{k=0}^{n_i} p_{n_i,k}(x) \sum_{s=0}^{n_j} p_{n_j,s}(x) \int_{F_0} f\left(\frac{k}{n_i}\right) f\left(\frac{s}{n_j}\right) \omega_0(df) \\ &= A_1(x) - 2A_2(x) + A_3(x). \end{aligned} \tag{6}$$

On using (1), we obtain

$$A_1(x) = \int_{F_0} f^2(x) \omega_0(df) = x. \tag{7}$$

Note that

$$\begin{aligned} \sum_{k=0}^n p_{n,k}(x) &= 1, \quad \sum_{k=0}^n k p_{n,k}(x) = nx, \\ \sum_{k=0}^n k^2 p_{n,k}(x) &= n^2 x^2 + nx(1-x), \end{aligned} \tag{8}$$

by (1), we have

$$\begin{aligned}
 A_2(x) &= \sum_{i=0}^{m-1} C_i(n) \sum_{k=0}^{n_i} p_{n_i,k}(x) \int_{F_0} f(x) f\left(\frac{k}{n_i}\right) \omega_0(df) \\
 &= \sum_{i=0}^{m-1} C_i(n) \sum_{k=0}^{n_i} p_{n_i,k}(x) \frac{1}{2} \left(x + \frac{k}{n_i} - \left| x - \frac{k}{n_i} \right| \right) \quad (9) \\
 &= x - \frac{1}{2} \sum_{i=0}^{m-1} C_i(n) \sum_{k=0}^{n_i} p_{n_i,k}(x) \left| x - \frac{k}{n_i} \right|.
 \end{aligned}$$

From [4,(3.24)], we know

$$\sum_{k=0}^{n_i} p_{n_i,k}(x) \left| x - \frac{k}{n_i} \right| = \frac{\sqrt{2x(1-x)}}{\sqrt{\pi n_i}} + o\left(n_i^{-\frac{1}{2}}\right).$$

Combining(3) and (9) we get

$$\begin{aligned}
 A_2(x) &= x - \frac{1}{2} \sum_{i=0}^{m-1} C_i(n) \left(\frac{\sqrt{2x(1-x)}}{\sqrt{\pi n_i}} + o\left(n_i^{-\frac{1}{2}}\right) \right) \quad (10) \\
 &= x - \sqrt{\frac{x(1-x)}{2\pi}} \sum_{i=0}^{m-1} \frac{C_i(n)}{\sqrt{n_i}} + o\left(n^{-\frac{1}{2}}\right).
 \end{aligned}$$

Now, we estimate the term $A_3(x)$. From (2) and (8),

$$\begin{aligned}
 A_3(x) &= \sum_{i=0}^{m-1} C_i(n) \sum_{j=0}^{m-1} C_j(n) \sum_{k=0}^{n_i} p_{n_i,k}(x) \sum_{s=0}^{n_j} p_{n_j,s}(x) \\
 &\quad \cdot \int_{F_0} f\left(\frac{k}{n_i}\right) f\left(\frac{s}{n_j}\right) \omega_0(df) \\
 &= \sum_{i=0}^{m-1} C_i(n) \sum_{j=0}^{m-1} C_j(n) \sum_{k=0}^{n_i} p_{n_i,k}(x) \sum_{s=0}^{n_j} p_{n_j,s}(x) \\
 &\quad \cdot \frac{1}{2} \left(\frac{k}{n_i} + \frac{s}{n_j} - \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \right) \quad (11) \\
 &= x - \frac{1}{2} \sum_{i=0}^{m-1} C_i(n) \sum_{j=0}^{m-1} C_j(n) \sum_{k=0}^{n_i} p_{n_i,k}(x) \\
 &\quad \cdot \sum_{s=0}^{n_j} p_{n_j,s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right|.
 \end{aligned}$$

Using Lemma 1 and (3), we have

$$\sum_{\left| \frac{s}{n_j} - x \right| > n_j^{-5/12}} p_{n_j,s}(x) \leq Cn^{-2}.$$

Note that

$$0 \leq \frac{k}{n_i}, \frac{s}{n_j} \leq 1,$$

we get

$$\sum_{\left| \frac{s}{n_j} - x \right| > n_j^{-5/12}} p_{n_j,s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \leq Cn^{-2}. \quad (12)$$

By (8) and (12), we obtain

$$\begin{aligned}
 &\sum_{k=0}^{n_i} p_{n_i,k}(x) \sum_{s=0}^{n_j} p_{n_j,s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \\
 &= \sum_{k=0}^{n_i} p_{n_i,k}(x) \sum_{\left| \frac{s}{n_j} - x \right| > n_j^{-5/12}} p_{n_j,s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \\
 &\quad + \sum_{\left| \frac{k}{n_i} - x \right| > n_i^{-5/12}} p_{n_i,k}(x) \sum_{\left| \frac{s}{n_j} - x \right| \leq n_j^{-5/12}} p_{n_j,s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \\
 &\quad + \sum_{\left| \frac{k}{n_i} - x \right| \leq n_i^{-5/12}} p_{n_i,k}(x) \sum_{\left| \frac{s}{n_j} - x \right| \leq n_j^{-5/12}} p_{n_j,s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \\
 &= o\left(n^{-\frac{23}{12}}\right) + \sum_{\left| \frac{k}{n_i} - x \right| \leq n_i^{-5/12}} p_{n_i,k}(x) \sum_{\left| \frac{s}{n_j} - x \right| \leq n_j^{-5/12}} p_{n_j,s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right|. \quad (13)
 \end{aligned}$$

For

$$\left| \frac{k}{n_i} - x \right| \leq n_i^{-5/12} \quad \text{and} \quad \left| \frac{s}{n_j} - x \right| \leq n_j^{-5/12},$$

by Lemma 2,

$$\begin{aligned}
 &\left| \frac{k}{n_i} - \frac{s}{n_j} \right| p_{n_i,k}(x) p_{n_j,s}(x) \\
 &= \frac{1 + o(1)}{2\pi x(1-x)\sqrt{n_i n_j}} \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \quad (14) \\
 &\quad \cdot \exp \left\{ \frac{-n_i}{2x(1-x)} \left(x - \frac{k}{n_i} \right)^2 + \frac{-n_j}{2x(1-x)} \left(x - \frac{s}{n_j} \right)^2 \right\}.
 \end{aligned}$$

Set

$$F(u_1, u_2) := (u_1 - u_2) \exp \left\{ \frac{-n_i}{2x(1-x)} u_1^2 + \frac{-n_j}{2x(1-x)} u_2^2 \right\}.$$

For

$$(v_1, v_2) \in \left[\frac{k-1}{n_i}, \frac{k+1}{n_i} \right] \times \left[\frac{s-1}{n_j}, \frac{s+1}{n_j} \right]$$

by the differential mean value theorem we have

$$\begin{aligned}
 &\left(\left(x - \frac{k}{n_i} \right) - \left(x - \frac{s}{n_j} \right) \right) \exp \left\{ \frac{-n_i}{2x(1-x)} \left(x - \frac{k}{n_i} \right)^2 \right. \\
 &\quad \left. + \frac{-n_j}{2x(1-x)} \left(x - \frac{s}{n_j} \right)^2 \right\} - ((x - v_1) - (x - v_2)) \\
 &\quad \cdot \exp \left\{ \frac{-n_i}{2x(1-x)} (x - v_1)^2 + \frac{-n_j}{2x(1-x)} (x - v_2)^2 \right\} \quad (15) \\
 &= F_1'(\xi, \eta) \left(v_1 - \frac{k}{n_i} \right) + F_2'(\xi, \eta) \left(v_2 - \frac{s}{n_j} \right),
 \end{aligned}$$

where

$$(\xi, \eta) \in \left(x - \frac{k+1}{n_i}, x - \frac{k-1}{n_i}\right) \times \left(x - \frac{s+1}{n_j}, x - \frac{s-1}{n_j}\right).$$

By a simple computation we know

$$|F_1'(\xi, \eta)| \leq C, \quad |F_2'(\xi, \eta)| \leq C.$$

From (15),

$$\begin{aligned} & \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \exp \left\{ \frac{-n_i}{2x(1-x)} \left(x - \frac{k}{n_i}\right)^2 + \frac{-n_j}{2x(1-x)} \left(x - \frac{s}{n_j}\right)^2 \right\} \\ &= |(x - v_1) - (x - v_2)| \exp \left\{ \frac{-n_i}{2x(1-x)} (x - v_1)^2 + \frac{-n_j}{2x(1-x)} (x - v_2)^2 \right\} + \left(n^{-\frac{11}{12}}\right). \end{aligned} \tag{16}$$

Integrating two side of (16) about (v_1, v_2) in

$$\left[\frac{k}{n_i}, \frac{k+1}{n_i}\right] \times \left[\frac{s}{n_j}, \frac{s+1}{n_j}\right]$$

we get

$$\begin{aligned} & \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \exp \left\{ \frac{-n_i}{2x(1-x)} \left(x - \frac{k}{n_i}\right)^2 + \frac{-n_j}{2x(1-x)} \left(x - \frac{s}{n_j}\right)^2 \right\} \\ &= n_i n_j \int_{\frac{k}{n_i}}^{\frac{k+1}{n_i}} dv_1 \int_{\frac{s}{n_j}}^{\frac{s+1}{n_j}} |(x - v_1) - (x - v_2)| \cdot \exp \left\{ \frac{-n_i}{2x(1-x)} (x - v_1)^2 + \frac{-n_j}{2x(1-x)} (x - v_2)^2 \right\} dv_2 + \left(n^{-\frac{11}{12}}\right). \end{aligned} \tag{17}$$

From (14)-(17), we have

$$\begin{aligned} & \sum_{\left|\frac{k}{n_i} - x\right| \leq n_i^{-5/12}} p_{n_i, k}(x) \sum_{\left|\frac{s}{n_j} - x\right| \leq n_j^{-5/12}} p_{n_j, s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \\ &= \sum_{\left|\frac{k}{n_i} - x\right| \leq n_i^{-5/12}} \sum_{\left|\frac{s}{n_j} - x\right| \leq n_j^{-5/12}} \left| \frac{k}{n_i} - \frac{s}{n_j} \right| p_{n_i, k}(x) p_{n_j, s}(x) \\ &= \sum_{\left|\frac{k}{n_i} - x\right| \leq n_i^{-5/12}} \sum_{\left|\frac{s}{n_j} - x\right| \leq n_j^{-5/12}} \frac{(1 + o(1)) \sqrt{n_i n_j}}{2\pi x(1-x)} \int_{\frac{k}{n_i}}^{\frac{k+1}{n_i}} dv_1 \int_{\frac{s}{n_j}}^{\frac{s+1}{n_j}} \\ & \quad |(x - v_1) - (x - v_2)| \cdot \exp \left\{ \frac{-n_i}{2x(1-x)} (x - v_1)^2 + \frac{-n_j}{2x(1-x)} (x - v_2)^2 \right\} dv_2 + o\left(n^{-\frac{23}{12}}\right) \\ &= \sum_{\left|\frac{k}{n_i} - x\right| \leq n_i^{-5/12}} \sum_{\left|\frac{s}{n_j} - x\right| \leq n_j^{-5/12}} \frac{\sqrt{n_i n_j}}{2\pi x(1-x)} \int_{\frac{k}{n_i}}^{\frac{k+1}{n_i}} dv_1 \int_{\frac{s}{n_j}}^{\frac{s+1}{n_j}} |(x - v_1) - (x - v_2)| \exp \left\{ \frac{-n_i}{2x(1-x)} (x - v_1)^2 + \frac{-n_j}{2x(1-x)} (x - v_2)^2 \right\} dv_2 \\ & \quad + o\left(n^{-1}\right) \\ &= \frac{\sqrt{n_i n_j}}{2\pi x(1-x)} \int_{x - n_i^{-5/12}}^{x + n_i^{-5/12} + \frac{1}{n_i}} dv_1 \int_{x - n_j^{-5/12}}^{x + n_j^{-5/12} + \frac{1}{n_j}} |(x - v_1) - (x - v_2)| \exp \left\{ \frac{-n_i}{2x(1-x)} (x - v_1)^2 + \frac{-n_j}{2x(1-x)} (x - v_2)^2 \right\} dv_2 + o\left(n^{-1}\right). \end{aligned} \tag{18}$$

Let

$$w_1 = \sqrt{\frac{n_i}{2x(1-x)}} (x - v_1), w_2 = \sqrt{\frac{n_j}{2x(1-x)}} (x - v_2),$$

by (18), we get

$$\begin{aligned} & \left| \sum_{\left| \frac{k-x}{n_i} \right| \leq n_i^{-5/12}} p_{n_i, k}(x) \sum_{\left| \frac{s-x}{n_j} \right| \leq n_j^{-5/12}} p_{n_j, s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \right. \\ & \left. = \frac{\sqrt{2x(1-x)}}{\pi} \int_{\frac{-n_i^{1/12}}{\sqrt{2x(1-x)}}}^{\frac{n_i^{1/12}}{\sqrt{2x(1-x)}}} dw_1 \int_{\frac{-n_j^{1/12}}{\sqrt{2x(1-x)}}}^{\frac{n_j^{1/12}}{\sqrt{2x(1-x)}}} dw_2 \left| \frac{w_1}{\sqrt{n_i}} - \frac{w_2}{\sqrt{n_j}} \right| \cdot \exp\{-w_1^2 - w_2^2\} dw_2 + o(n^{-1}). \right. \end{aligned} \quad (19)$$

By (3), suppose that

$$\frac{n}{n_i} = c_i^2, \quad \frac{n}{n_j} = c_j^2,$$

from (19) and the convergence of the improper integral

$$\int_{-\infty}^{+\infty} dw_1 \int_{-\infty}^{+\infty} |c_i w_1 - c_j w_2| \exp\{-w_1^2 - w_2^2\} dw_2,$$

we have

$$\begin{aligned} & \left| \sum_{\left| \frac{k-x}{n_i} \right| \leq n_i^{-5/12}} p_{n_i, k}(x) \sum_{\left| \frac{s-x}{n_j} \right| \leq n_j^{-5/12}} p_{n_j, s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \right. \\ & = \frac{\sqrt{2x(1-x)}}{\pi} \int_{-\infty}^{+\infty} dw_1 \int_{-\infty}^{+\infty} \left| \frac{w_1}{\sqrt{n_i}} - \frac{w_2}{\sqrt{n_j}} \right| \cdot \exp\{-w_1^2 - w_2^2\} dw_2 + o\left(n^{-\frac{1}{2}}\right) \\ & = \frac{\sqrt{2x(1-x)}}{\pi} \int_{-\infty}^{+\infty} dw_1 \int_{-\infty}^{+\infty} \left(\frac{w_1}{\sqrt{n_i}} - \frac{w_2}{\sqrt{n_j}} \right) \cdot \exp\{-w_1^2 - w_2^2\} dw_2 + 2 \frac{\sqrt{2x(1-x)}}{\pi} \int_{-\infty}^{+\infty} dw_1 \int_{\frac{n_j}{\sqrt{n_i} w_1}}^{+\infty} \left(\frac{w_2}{\sqrt{n_j}} - \frac{w_1}{\sqrt{n_i}} \right) \\ & \quad \cdot \exp\{-w_1^2 - w_2^2\} dw_2 + o\left(n^{-\frac{1}{2}}\right) \\ & = \frac{\sqrt{2x(1-x)}}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-\left(1 + \frac{n_j}{n_i}\right) w_1^2}}{\sqrt{n_j}} dw_1 + \frac{\sqrt{2x(1-x)}}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-\left(1 + \frac{n_i}{n_j}\right) w_2^2}}{\sqrt{n_i}} dw_2 + o\left(n^{-\frac{1}{2}}\right) \\ & = \frac{\sqrt{2x(1-x)}}{\pi} \left(\frac{1}{\sqrt{n_j}} \sqrt{\frac{n_i \pi}{n_i + n_j}} + \frac{1}{\sqrt{n_i}} \sqrt{\frac{n_j \pi}{n_i + n_j}} \right) + o\left(n^{-\frac{1}{2}}\right) = \frac{\sqrt{2x(1-x)}}{\sqrt{\pi(n_i + n_j)}} \left(\sqrt{\frac{n_i}{n_j}} + \sqrt{\frac{n_j}{n_i}} \right) + o\left(n^{-\frac{1}{2}}\right). \end{aligned} \quad (20)$$

Combining (11) and (20), we obtain

$$\begin{aligned} A_3(x) &= x - \frac{1}{2} \sum_{i=0}^{m-1} C_i(n) \sum_{j=0}^{m-1} C_j(n) \cdot \sum_{k=0}^{n_i} p_{n_i, k}(x) \sum_{s=0}^{n_j} p_{n_j, s}(x) \left| \frac{k}{n_i} - \frac{s}{n_j} \right| \\ &= x - \sqrt{\frac{2x(1-x)}{2\pi}} \sum_{i=0}^{m-1} C_i(n) \sum_{j=0}^{m-1} C_j(n) \cdot \frac{\sqrt{n_i/n_j} + \sqrt{n_j/n_i}}{\sqrt{n_i + n_j}} + o\left(n^{-\frac{1}{2}}\right). \end{aligned} \quad (21)$$

From (5)-(7), (10), and (21), we complete the proof of Theorem 1.

REFERENCES

- [1] K. Ritter, "Average-Case Analysis of Numerical Problems," Springer-Verlag, Berlin, 2000.
- [2] P. L. Butzer, "Linear Combinations of Bernstein Polynomials," *Canadian Journal of Mathematics*, Vol. 5, 1953, pp. 559-567. <http://dx.doi.org/10.4153/CJM-1953-063-7>
- [3] Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer-Verlag, Berlin, 1987. <http://dx.doi.org/10.1007/978-1-4612-4778-4>
- [4] G. Q. Xu, "The Simultaneous Approximation Average Errors for Bernstein Operators on the R-Fold Integrated Wiener Space," *Numerical Mathematics Theory Methods and Applications*, Vol. 5, No. 3, 2012, pp. 403-422.
- [5] G. G. Lornetz, "Bernstein Polynomials," University of Toronto, Toronto, 1953.