Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2019, 7, 2632-2649
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Algorithms for Common Solutions to Generalized Mixed Equilibrium Problems and Fixed Point Problems under Nonlinear Transformations in Banach Spaces

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How to cite this paper: Song, Y.L. and Chen, X.H. (2019) Algorithms for Common Solutions to Generalized Mixed Equilibrium Problems and Fixed Point Problems under Nonlinear Transformations in Banach Spaces. *Journal of Applied Mathematics and Physics*, 7, 2632-2649.
https://doi.org/10.4236/jamp.2019.711179

Received: September 30, 2019
Accepted: November 2, 2019
Published: November 5, 2019

Abstract

The purpose of this paper is to present a new iterative scheme for finding a common solution of the generalized mixed equilibrium problems with an infinite family of inverse strongly monotone mappings and the fixed point problems of demimetric mappings under nonlinear transformations in Banach spaces. Applications are also included. The results in this paper are the extension and improvement of the recent results in the literature.

Keywords

Fixed Point, Demimetric Mapping, Generalized Mixed Equilibrium Problems, Banach Space

1. Introduction

Let H be a real Hilbert space, C be a nonempty closed convex subset of H, T be a mapping on C and $F(T) := \{x \in C : Tx = x\}$. Let $A : C \to H$ be a nonlinear mapping, $\varphi : C \to \mathbb{R}$ be a function and F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Then, we consider the following generalized mixed equilibrium problem (for short, GMEP): finding $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x \rangle \geq 0. \tag{1}$$

The set of solutions of the GMEP is denoted by $\text{GMEP}(F, \varphi, A)$ (see [1] and the references therein). Here some special cases of the GMEP are stated as followings:

1) If $A = 0$, then the GMEP becomes the following mixed equilibrium prob-

lem (for short, MEP):

$$\text{finding } x \in C \text{ such that } F(x, y) + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C, \tag{2}$$

which was studied by Ceng and Yao [2]. The set of solutions of the MEP is denoted by $\text{MEP}(F, \varphi)$.

2) If $\varphi = 0$ and $A = 0$, then the GMEP becomes the following equilibrium problem (for short, EP):

$$\text{finding } x \in C \text{ such that } F(x, y) \geq 0, \ \forall y \in C. \tag{3}$$

This general form of the EP was first considered by Nikaido and Isoda [3]. The MEP and EP play an important role in many fields, such as economics, physics, mechanics and engineering sciences. Also, the MEP and EP include many mathematical problems as particular cases, for example, mathematical programming problems, complementary problems, variational inequality problems, Nash equilibrium problems in noncooperative games, minimax inequality problems and fixed point problems. Because of their wide applicability, equilibrium problems and mixed equilibrium problems have been generalized in various directions for the past several years; see, for example, [2] [4]-[9].

3) If $F = 0$ and $\varphi = 0$, then the GMEP reduces to the following classical variational inequality problem (for short, VIP) [10]:

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{4}$$

Since the VIP inception by Stampacchia [10] in 1964, it has received much attention due to its applications in a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences. Using the projection technique, one can easily show that is equivalent to the fixed-point problem; see, [7] [8] [11] [12] and the references therein.

Motivated by Ceng and Yao [2], Nikaido and Isoda [3] and Stampacchia [10], Peng and Yao [1] introduced the GMEP, which can be viewed as development and extension of the MEP, the EP and the VIP. It shows that the GMEP has applications in physics, economics, finance, transportation, network and structural analysis, therapy, image reconstruction, and elasticity. The GMEP includes special cases, MEPs, EPs, VIPs, fixed point problems, complementarity problems, optimization problems, Nash equilibrium problems in noncooperative games, etc (see e.g., [6] [7] [8] and the references contained in them). In other words, the GMEP is a unifying model for several problems arising in several areas of study. In general, the GMEP involves nonlinear equations and there are no known methods to obtain closed form solutions for them. Consequently, several methods are being deployed to approximate their solutions, assuming existence. A number of iterative methods have been utilized to solve equilibrium problems, generalized equilibrium problems and mixed equilibrium problems (see e.g., [2] [4] [5] [13] and the references therein).

Related to the GMEP, the problem of finding the fixed points for nonlinear mappings is the subject of current interest in functional analysis. It turns out that the fixed point theory for nonlinear mappings can be applied to several nonlinear problems such as zero point problems for monotone operators, convex feasibility problems, convex minimization problems, variational inequality and equilibrium problems, and so on; see [14]-[19] for more details.

At the same time, to construct a mathematical model which is as close as possible to a real complex problem, we often have to use more than one constraint. Solving such problems, we have to obtain some solution which is simultaneously the solution of two or more subproblems or the solution of one subproblem on the solution set of another subproblem. These subproblems can be given, for example, by two or more different variational inequality problems or two or more different fixed point problems. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common solution of fixed-point problems for nonlinear mappings, equilibrium problems and variational problems; see, for example, [1] [2] [9] [12] [19] [20] and the references therein.

Recently, Takahashi [21] introduced a broad class of nonlinear mappings in a Banach space called k-demimetric mapping. This class mapping contains the classes of generalized hybrid mappings, k-strict pseudo-contractions, firmly-quasi-nonexpansive mappings, quasi-nonexpansive mappings and demicontractive mappings.

Definition 1.1 Let E be a smooth Banach space and let C be a nonempty, closed and convex subset of E. Let k be a real number with $k \in (-\infty, 1)$. A mapping $T: C \to E$ with $F(T) \neq \emptyset$ is called k-demimetric if, for any $x \in C$ and $q \in F(T)$,

$$\langle x - q, J(x - Tx) \rangle \geq \frac{1-k}{2} \|x - Tx\|^2. \quad (5)$$

We give an example of a k-demimetric mapping which is not pseudo-contractive, hence it is not strictly pseudo-contractive.

Example 1.2 ([22]) Let H be the real line and $C = [-1,1]$. Define T on C by $T(x) = \frac{2}{3} x \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $T(0) = 0$. Clearly, 0 is the only fixed point of T. Also, for $x \in C$,

$$|T(x) - 0|^2 = |T(x)|^2 = \left|\frac{2}{3} x \sin\left(\frac{1}{x}\right)\right|^2 \leq \left|\frac{2x}{3}\right| \leq |x|^2 \leq |x - 0|^2 + k|T(x) - x|^2 \text{ for any}$$

$k \in [0,1)$. Thus T is demimetric.

In order to find a common solution of fixed point problems for an finite family of demimetric mappings and the variational inequality problems for a infinite family of inverse strongly monotone mappings in a Hilbert space, Takahashi [12] recently introduced and studied the following iterative algorithm:

$$\begin{cases} z_n = \sum_{j=1}^{M} \xi_j \left((1-\lambda_n) I + \lambda_n T_j\right) x_n, \\ w_n = \sum_{i=1}^{N} \sigma_i P_C (1 - \eta_n B_i) x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)\left(P_C (\alpha_n x_n + \beta_n z_n + \gamma_n w_n)\right), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{T_j\}_{j=1}^{M}: C \to H$ is a finite family of k_j-demimetric and demiclosed mappings, and $\{B_i\}_{i=1}^{N}: C \to H$ is a finite family of μ_i-inverse strongly monotone mappings. Then he obtained a strong convergence theorem under some mild restrictions on the parameters.

Very recently, Akashi and Takahashi [14] proposed the following Mann's type iteration for finding a common solution of fixed-point problems for an infinite family of demimetric mappings without assuming that demimetric mappings are commutative:

$$\begin{cases} z_n = \sum_{j=1}^{\infty} \xi_j \left((1-\lambda_n) I + \lambda_n T_j \right) x_n, \\ x_{n+1} = P_C\left(\alpha_n x_n + (1-\alpha_n) z_n \right), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{T_j\}_{j=1}^{\infty}: C \to H$ is an infinite family of k_j-demimetric and demiclosed mappings. Then they obtained a weak convergence theorem under certain appropriate assumptions on the parameters.

Most very recently, Takahashi [15] also introduced the following iteration process for finding a common solution of fixed-point problems with an infinite family of demimetric mappings and the variational inequality problems with an infinite family of inverse strongly monotone mappings in a Hilbert space:

$$\begin{cases} z_n = \sum_{j=1}^{\infty} \xi_j \left((1-\lambda_n) I + \lambda_n T_j \right) x_n, \\ w_n = \sum_{i=1}^{\infty} \sigma_i J_{\eta_n} (1-\eta_n B_i) x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)\left(P_C\left(\alpha_n x_n + \beta_n z_n + \gamma_n w_n \right) \right), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{T_j\}_{j=1}^{\infty}: C \to H$ is an infinite family of k_j-demimetric and demiclosed mappings, $\{B_i\}_{i=1}^{\infty}: C \to H$ is an infinite family of μ_i-inverse strongly monotone mappings. Then they obtained a strong convergence theorem under some mild restrictions on the parameters.

On other hand, in order to find a common solution of equilibrium problems and the set of fixed point problems with generalized hybrid mappings, Alizadeh and Moradlou [23] introduced the following Ishikawa-like iteration process by applying the hybrid projection method:

$$\begin{cases} u_n \in H \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in H, \\ y_n = \alpha_n z_n + (1-\alpha_n) S u_n, \\ z_n = \beta_n x_n + (1-\beta_n) S x_n, \\ C_n = \left\{ u \in H : \|y_n - u\|^2 \leq \|x_n - u\|^2 \right\}, \\ Q_n = \left\{ u \in H : \langle x_n - u, x_n - x \rangle \leq 0 \right\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

where S is a generalized hybrid mapping and f is a bifunction satisfying (A1)-(A4). Then they obtained a strong convergence theorem under certain appropriate assumptions on the parameters.

Motivated and inspired by Takahashi [12], Akashi and Takahashi [14], Takahashi [15], Alizadeh and Moradlou [23], we put forward two questions:

1) Can these corresponding results in [12] [14] [15] [23] in Hilbert spaces be extended to the framework of Banach spaces (for example, l_p for $1 < p < \infty$)?

2) Can we extend corresponding results in [12] [14] [15] [23] from finding a solution of the fixed point problems of generalized hybrid mappings or a common solution of the equilibrium problems and fixed point problems of generalized hybrid mappings to the more general and challenging problem for finding a common solution of the generalized mixed equilibrium problems and the fixed point problems of demimetric mappings under nonlinear transformations?

The purpose of this paper is to give the affirmative answers to these questions mentioned above. In this paper, we present a new iterative scheme for finding a common solution of the generalized mixed equilibrium problems and fixed point problems of demimetric mappings under nonlinear transformations in Banach spaces. Applications are also included. Our results improve essentially the corresponding results in [12] [14] [15] [23]. Further, some other results are also improved; see [9] [11] [16] [17] [18] [20] [24] [25].

2. Preliminaries

We denote *E* the real Banach space, E^* the dual of *E*, *I* the identity mapping on *E*, and $\mathbb{N}$ the set of positive integers. The expressions $x_n \to x$ and $x_n \rightharpoonup x$ denote the strong and weak convergence of the sequence $\{x_n\}$, respectively. The (normalized) duality mapping *J* from *E* to E^* is defined by

$$Jx = \left\{x^* \in E^* : \left\langle x, x^* \right\rangle = \|x\|^2 = \left\|x^*\right\|^2\right\}$$

for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the duality product. If *E* is a Hilbert space, then $J = I$, where *I* is the identity mapping on *H*.

The norm of a Banach space *E* is said to be Gâteaux differentiable if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all x, y on the unit sphere $S(E) = \{x \in E : \|x\| = 1\}$. In this case, we say that *E* is smooth.

A Banach space *E* is said to be strictly convex if $\|x - y\| < 2$ whenever $x, y \in S(E)$ and $x \neq y$. It is known that if *E* is strictly convex, then the duality mapping *J* is injective, that is, $x, y \in E$ and $x \neq y$ imply $Jx \cap Jy = \varnothing$. It is known that *E* is reflexive if and only if *J* is surjective. Therefore, if *E* is a smooth, strictly convex and reflexive Banach space, then *J* is a single-valued bijection, see [26] for more details.

Definition 2.1 A mapping $T : C \to H$ is said to be:

1) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$;

2) contractive if there exists a constant $v \in (0,1)$ such that

$$\|Tx - Ty\| \le v\|x - y\|, \ \forall x, y \in C;$$

3) β-demicontractive if there exists a constant $\beta \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \beta \|x - Tx\|^2, \ \forall x \in C, y \in F(T).$$

We use Π_C to denote the collection of mappings T verifying the above inequality. That is

$$\Pi_C = \{T : C \to H : T \text{ is a contraction with constant } v\}.$$

Let D be a nonempty subset of C. A sequence $\{T_n\}$ of mappings of C into H is said to be stable on D (see [27]) if $\{T_n(x) : n \in \mathbb{N}\}$ is a singleton for every $x \in D$. It is clear that if $\{T_n\}$ is stable on D, then $T_n(x) = T_1(x)$ for all $n \in \mathbb{N}$ and $x \in D$.

Lemma 2.2 In a Hilbert space H, it holds for all $x, y \in H$ and $\lambda \in [0,1]$ that

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2,$$

which can be extended to the more general situation: for all $x_1, x_2, \cdots, x_n \in H$, $\lambda_i \in [0,1]$, and $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$\begin{aligned} &\|\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n\|^2 \\ &= \lambda_1 \|x_1\|^2 + \lambda_2 \|x_1\|^2 + \cdots + \lambda_n \|x_n\|^2 - \sum_{1 \le i \le j \le n} \lambda_i \lambda_j \|x_i - x_j\|^2. \end{aligned}$$

Lemma 2.3 ([19]) Let $\{\alpha_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\alpha_{n_i} < \alpha_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied for all (sufficiently large) numbers $k \in \mathbb{N}$:

$$\alpha_{m_k} \le \alpha_{m_k+1} \quad \text{and} \quad \alpha_k \le \alpha_{m_k+1}.$$

In fact, $m_k = \max\{j \le k : \alpha_j < \alpha_{j+1}\}$.

Lemma 2.4 ([28]) Let $\{\alpha_n\}$ be a sequence of nonnegative numbers satisfying the property:

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + b_n + \gamma_n c_n, \quad n \in \mathbb{N},$$

where $\{\gamma_n\}, \{b_n\}, \{c_n\}$ satisfy the restrictions:

1) $\sum_{n=1}^{\infty} \gamma_n = \infty, \lim_{n\to\infty} \gamma_n = 0$,

2) $b_n \ge 0, \sum_{n=1}^{\infty} b_n < \infty$,

3) $\limsup_{n\to\infty} c_n \le 0$.

Then, $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.5 ([21]) Let E be a smooth, strictly convex and reflexive Banach space and let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of E into itself. Then $F(U)$ is closed and convex.

Lemma 2.6 ([16]) Let $P_C : H \to C$ be a metric projection from H on a nonempty closed convex subset C of H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there holds the relation

$$\langle x - z, y - z \rangle \le 0, \quad \forall y \in C.$$

Recall that a mapping $A:C\to H$ is said to be α-inverse-strongly monotone (ism) if there exists a constant $\alpha>0$ such that

$$\langle Ax-Ay,x-y\rangle\ge\alpha\|Ax-Ay\|^2,\quad\forall x,y\in C.$$

Lemma 2.7 ([29]) If $A:C\to H$ is α-ism and λ is any constant in $(0,2\alpha]$, then the mapping $I-\lambda A$ is nonexpansive.

For solving the generalized mixed equilibrium problem, let us assume that the bifunction $F:C\times C\to\mathbb{R}$ and the nonlinear mapping $\varphi:C\to\mathbb{R}$ satisfy the following conditions:

(A1) $F(x,x)=0$ for all $x\in C$;

(A2) F is monotone, *i.e.*, $F(x,y)+F(y,x)\le 0$ for all $x,y\in C$;

(A3) for each fixed $y\in C$, $x\mapsto F(x,y)$ is weakly upper semicontinuous;

(A4) for each fixed $x\in C$, $y\mapsto F(x,y)$ is convex and lower semicontinuous;

(A5) for each $x\in C$ and $r>0$, there exists a bounded subset $D_x\subseteq C$ and $y_x\in C$ such that, for any $z\in C\setminus D_x$,

$$F(z,y_x)-\varphi(y_x)-\varphi(z)+\frac{1}{r}\langle y_x-z,z-x\rangle<0;$$

(A6) C is a bounded set.

Lemma 2.8 [2] Let C be a nonempty, closed and convex subset of H and let $F:C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $\varphi:C\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r>0$ and $x\in H$, define a mapping $K_r:H\to C$ as follows:

$$K_r(x)=\left\{z\in C:F(z,y)+\varphi(y)-\varphi(z)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\forall y\in C\right\}.$$

Then, the following conclusions hold:

1) For each $x\in H$, $K_r(x)\ne\varnothing$ and K_r is single-valued;

2) K_r is a firmly nonexpansive mapping, *i.e.*, for all $x,y\in H$,

$$\|K_rx-K_ry\|^2\le\langle K_rx-K_ry,x-y\rangle;$$

3) $F(K_r)=MEP(F,\varphi)$;

4) $MEP(F,\varphi)$ is closed and convex;

5) $\|K_sx-K_tx\|^2\le\frac{s-t}{s}\langle K_sx-K_tx,K_sx-x\rangle$ for all $s,t>0$ and $x\in H$.

3. Main Results

Throughout the rest of this paper, we always assume the following:

1) H is a real Hilbert space, and C is a nonempty closed subspace of H;

2) E is a smooth, strictly convex and reflexive Banach space, and J is the duality mapping on E;

3) F is a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4);

4) K_r is a mapping defined as in Lemma 2.8;

5) $\{A_j\}_{j=1}^{\infty}:C\to H$ is an infinite family of μ_j-ism mappings with $\mu=\inf\{\mu_j:j\in\mathbb{N}\}$;

6) $\varphi: C \to \mathbb{R}$ is a lower semicontinuous and convex function with restrictions (B1) or (B2);

7) $B: H \to E$ is a bounded linear operator such that $B \neq 0$ and B^* is the adjoint operator of B;

8) $\{T_i\}_{i=1}^{\infty}: E \to E$ is an infinite family of k_i-demimetric and demiclosed mappings with $k = \sup\{k_i : i \in \mathbb{N}\} < 1$;

9) $\Gamma := \bigcap_{i=1}^{\infty} B^{-1}F(T_i) \cap \left(\bigcap_{j=1}^{\infty} GMEP(F, \varphi, A_j)\right) \neq \varnothing$;

10) $\{f_n\} \subset \Pi_C$ is stable on Γ.

Theorem 3.1 For any $x_1 \in C$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = \left(I - \tau \sum_{i=1}^{\infty} \sigma_i B^* J(I - T_i)B\right)x_n, \\ y_n = \sum_{j=1}^{\infty} \delta_j K_{r_n}(I - r_n A_j)x_n, \\ x_{n+1} = P_C(\alpha_n f_n x_n + \beta_n y_n + \gamma_n z_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{6}$$

where $a, b, \tau \in (0, +\infty)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $\{r_n\}, \{\delta_j\}, \{\sigma_i\} \subset (0, +\infty)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$,

(iii) $0 < \tau < \dfrac{1-k}{\|B\|^2}$,

(iv) $\sum_{j=1}^{\infty} \delta_j = 1$ and $\sum_{i=1}^{\infty} \sigma_i = 1$,

(v) $0 < a \leq r_n \leq b < 2\mu$.

Then the sequence $\{x_n\}$ generated by (6) converges strongly to a point $z_0 \in \Gamma$, where $z_0 = P_\Gamma f_1 z_0$.

Proof. Set $Tx = \sum_{i=1}^{\infty} \sigma_i B^* J(I - T_i)Bx$ for all $n \in \mathbb{N}$ and $x \in C$. Then we can prove that T is well defined. In fact, we have, for any $i \in \mathbb{N}$ and $z \in \Gamma$,

$$\begin{aligned} &\left\|\left(I - \tau B^* J(I - T_i)B\right)x - z\right\|^2 = \left\|x - z - \tau B^* J(I - T_i)Bx\right\|^2 \\ &\leq \|x - z\|^2 - 2\tau\left\langle x - z, B^* J(I - T_i)Bx\right\rangle + \left\|\tau B^* J(I - T_i)Bx\right\|^2 \\ &\leq \|x - z\|^2 - 2\tau\left\langle Bx - Bz, J(I - T_i)Bx\right\rangle + \tau^2\|B\|^2\left\|(I - T_i)Bx\right\|^2 \\ &\leq \|x - z\|^2 - \tau(1 - k_i)\|Bx - T_i Bx\|^2 + \tau^2\|B\|^2\|Bx - T_i Bx\|^2 \\ &\leq \|x - z\|^2 - \tau\left(1 - k - \tau\|B\|^2\right)\|Bx - T_i Bx\|^2 \leq \|x - z\|^2, \end{aligned} \tag{7}$$

which implies

$$\left\|\left(I - \tau B^* J(I - T_i)B\right)x - z\right\| \leq \|x - z\|. \tag{8}$$

Thus,

$$\left\|B^* J(I - T_i)Bx\right\| \leq \frac{2}{\tau}\|x - z\|.$$

Then we see the mapping $Tx = \sum_{i=1}^{\infty} \sigma_i B^* J(I - T_i)Bx$ converges absolutely for each x in C.

Furthermore, define $V_n x=\sum_{j=1}^{\infty}\delta_j K_{r_n}\left(I-r_n A_j\right)x$ for all $n\in\mathbb{N}$ and $x\in C$. Then we can prove that V_n is nonexpansive. Indeed, it follows that $K_{r_n}\left(1-r_n A_j\right)$ is nonexpansive from (v), Lemma 2.7 and Lemma 2.8(2). We obtain from Lemma 2.5 that, for any $\hat{x}\in\bigcap_{j=1}^{\infty}\mathrm{GMEP}\left(F,\varphi,A_j\right)$,

$$\left\|K_{r_n}\left(I-r_n A_j\right)x\right\|\le\left\|K_{r_n}\left(I-r_n A_j\right)x-\hat{x}\right\|+\|\hat{x}\|\le\|x-\hat{x}\|+\|\hat{x}\|.$$

Thus $V_n x=\sum_{j=1}^{\infty}\delta_j K_{r_n}\left(I-r_n A_j\right)x$ converges absolutely for each $x\in C$.

Since $K_{r_n}\left(I-r_n A_j\right)$ is nonexpansive, we have that $F\left(K_{r_n}\left(I-r_n A_j\right)\right)=\mathrm{GMEP}\left(F,\varphi,A_j\right)$ is closed and convex. Furthermore, we know from Lemma 2.5 that $F\left(T_i\right)$ is closed and convex for each $i\in\mathbb{N}$. Therefore, we have that $\Gamma=\bigcap_{i=1}^{\infty}B^{-1}F\left(T_i\right)\cap\left(\bigcap_{j=1}^{\infty}\mathrm{GMEP}\left(F,\varphi,A_j\right)\right)$ is nonempty, closed and convex (note that B is linear and continuous). Thus we have that P_Γ is well defined.

We derive from Lemma 2.8 that

$$\left\|y_n-z\right\|=\left\|\sum_{j=1}^{\infty}\delta_j K_{r_n}\left(I-r_n A_j\right)x_n-z\right\|\le\left\|x_n-z\right\|. \tag{9}$$

Noting (8), we have

$$\left\|z_n-z\right\|\le\sum_{i=1}^{\infty}\sigma_i\left\|\left(I-\tau B^*J\left(I-T_i\right)B\right)x_n-z\right\|\le\left\|x_n-z\right\|. \tag{10}$$

It follows from (9), (10) and (v) that

$$\begin{aligned}\left\|x_{n+1}-z\right\|&\le\left\|\alpha_n\left(f_n x_n-z\right)+\beta_n\left(y_n-z\right)+\gamma_n\left(z_n-z\right)\right\|\\&\le\alpha_n\left\|f_n x_n-z\right\|+\left(1-\alpha_n\right)\left\|x_n-z\right\|\\&\le\alpha_n\left\|f_n x_n-f_n z\right\|+\alpha_n\left\|f_n z-z\right\|+\left(1-\alpha_n\right)\left\|x_n-z\right\|\\&\le\alpha_n v\left\|x_n-z\right\|+\alpha_n\left\|f_1 z-z\right\|+\left(1-\alpha_n\right)\left\|x_n-z\right\|\\&=\left(1-\alpha_n\left(1-v\right)\right)\left\|x_n-z\right\|+\alpha_n\left\|f_1 z-z\right\|\\&\le\max\left\{\left\|x_n-z\right\|,\frac{\left\|f_1 z-z\right\|}{1-v}\right\}.\end{aligned}$$

By induction, we obtain

$$\left\|x_n-z\right\|\le\max\left\{\left\|x_1-z\right\|,\frac{\left\|f_1 z-z\right\|}{1-v}\right\},\quad\forall n\in\mathbb{N},$$

which gives that the sequence $\{x_n\}$ is bounded, so are $\{f_n x_n\}$, $\{y_n\}$ and $\{z_n\}$.

We obtain from (7) that

$$\begin{aligned}\left\|z_n-z\right\|^2&=\left\|\left(I-\tau\sum_{i=1}^{\infty}\sigma_i B^*J\left(I-T_i\right)B\right)x_n-z\right\|^2\\&\le\sum_{i=1}^{\infty}\sigma_i\left\|\left(I-\tau B^*J\left(I-T_i\right)B\right)x_n-z\right\|^2\\&\le\sum_{i=1}^{\infty}\sigma_i\left(\|x-z\|^2-\tau\left(1-k-\tau\|B\|^2\right)\left\|Bx_n-T_iBx_n\right\|^2\right)\\&=\left\|x_n-z\right\|^2-\tau\left(1-k-\tau\|B\|^2\right)\sum_{i=1}^{\infty}\sigma_i\left\|Bx_n-T_iBx_n\right\|^2.\end{aligned} \tag{11}$$

It follows from (9), (11) and Lemma 2.2 that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \|\alpha_n(f_nx_n-z)+\beta_n(y_n-z)+\gamma_n(z_n-z)\|^2\\ &\le \alpha_n\|f_nx_n-z\|^2+\beta_n\|y_n-z\|^2+\gamma_n\|z_n-z\|^2\\ &\quad -\beta_n\gamma_n\|y_n-z_n\|^2\\ &\le \alpha_n\|f_nx_n-z\|^2+\beta_n\|x_n-z\|^2-\beta_n\gamma_n\|y_n-z_n\|^2\\ &\quad +\gamma_n\left(\|x_n-z\|^2-\tau\left(1-k-\tau\|B\|^2\right)\sum_{i=1}^{\infty}\sigma_i\|Bx_n-T_iBx_n\|^2\right)\\ &\le \alpha_n\|f_nx_n-z\|^2+\|x_n-z\|^2-\beta_n\gamma_n\|y_n-z_n\|^2\\ &\quad -\gamma_n\tau\left(1-k-\tau\|B\|^2\right)\sum_{i=1}^{\infty}\sigma_i\|Bx_n-T_iBx_n\|^2,\end{aligned}$$

which means that

$$\begin{aligned}&\beta_n\gamma_n\|y_n-z_n\|^2+\gamma_n\tau\left(1-k-\tau\|B\|^2\right)\sum_{i=1}^{\infty}\sigma_i\|Bx_n-T_iBx_n\|^2\\ &\le \alpha_n\|f_nx_n-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2.\end{aligned} \tag{12}$$

Case 1. Assume there exists some integer $m>0$ such that $\{\|x_n-z_0\|\}$ is decreasing for all $n\ge m$. In this case, we deduce that $\lim_{n\to\infty}\|x_n-z_0\|$ exists. From (12), conditions (i), (ii), (iii) and (v), we deduce

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\sigma_i\|Bx_n-T_iBx_n\|^2=0 \tag{13}$$

and

$$\lim_{n\to\infty}\|y_n-z_n\|=0. \tag{14}$$

From (6) and (13), we get that

$$\begin{aligned}\|x_n-z_n\|^2 &= \left\|x_n-\left(I-\tau\sum_{i=1}^{\infty}\sigma_iB^*J(I-T_i)B\right)x_n\right\|^2\\ &\le \tau^2\sum_{i=1}^{\infty}\sigma_i\|Bx_n-T_iBx_n\|^2.\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{15}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_s}\}$ of $\{x_n\}$ satisfying $x_{n_s}\rightharpoonup\tilde{x}\in C$. Without loss of generality, we may also assume

$$\lim_{s\to\infty}\langle f_1z_0-z_0,x_{n_s}-z_0\rangle=\limsup_{n\to\infty}\langle f_1z_0-z_0,x_n-z_0\rangle. \tag{16}$$

Because B is bounded and linear, we see that $Bx_{n_s}\rightharpoonup B\tilde{x}$. This together with (13) implies $\tilde{x}\in B^{-1}F(T_i)$ for each $i\in\mathbb{N}$. And hence, $\tilde{x}\in\bigcap_{i=1}^{\infty}B^{-1}F(T_i)$.

Next let us prove that $\tilde{x}\in\bigcap_{i=1}^{\infty}\mathrm{GMEP}(F,\varphi,A_i)$. Noticing that a nonexpansive mapping T' with $F(T')\ne\varnothing$ is 0-demimetric, then we have

$$\begin{aligned}\left\langle x_n - z_0, x_n - y_n \right\rangle &= \sum_{j=1}^{\infty} \delta_j \left\langle x_n - z_0, x_n - K_{r_n}\left(I - r_n A_j\right) x_n \right\rangle \\ &\geq \frac{1}{2} \sum_{j=1}^{\infty} \delta_j \left\| K_{r_n}\left(I - r_n A_j\right) x_n - x_n \right\|^2 .\end{aligned}$$

This together with (14) and (15) implies, for any $i \in \mathbb{N}$, that

$$\lim_{n\to\infty} \left\| K_{r_n}\left(I - r_n A_j\right) x_n - x_n \right\| = 0. \tag{17}$$

Consider a subsequence $\{r_{n_{s_j}}\}$ of $\{r_{n_s}\}$ corresponding to the sequence $\{x_{n_s}\}$. Since the subsequence $\{r_{n_{s_j}}\}$ of $\{r_{n_s}\}$ is bounded, we have that there exists a subsequence $\{r_{n_{s_k}}\}$ of $\{r_{n_{s_j}}\}$ such that $\lim_{k\to\infty} r_{n_{s_k}} = r$. For such *r*, we have from Lemma 2.8 (5) that

$$\begin{aligned}&\left\| K_r\left(I - rA_j\right) x_{n_{s_k}} - x_{n_{s_k}} \right\| \\ &\leq \left\| K_r\left(I - rA_j\right) x_{n_{s_k}} - K_{r_{n_{s_k}}}\left(I - rA_j\right) x_{n_{s_k}} \right\| \\ &\quad + \left\| K_{r_{n_{s_k}}}\left(I - r_{n_{s_k}} A_j\right) x_{n_{s_k}} - x_{n_{s_k}} \right\| \\ &\quad + \left\| K_{r_{n_{s_k}}}\left(I - rA_j\right) x_{n_{s_k}} - K_{r_{n_{s_k}}}\left(I - r_{n_{s_k}} A_j\right) x_{n_{s_k}} \right\| \\ &\leq \left| \frac{r - r_{n_{s_k}}}{r} \right| \left\| K_r\left(I - rA_j\right) x_{n_{s_k}} - \left(I - rA_j\right) x_{n_{s_k}} \right\| \\ &\quad + \left\| K_{r_{n_{s_k}}}\left(I - r_{n_{s_k}} A_j\right) x_{n_{s_k}} - x_{n_{s_k}} \right\| + \left| r - r_{n_{s_k}} \right| \left\| A_j x_{n_{s_k}} \right\|.\end{aligned}$$

On the other hand, since K_r and $\left(I - rA_j\right)$ are Lipscitz, noting (17), we infer for any $j \in \mathbb{N}$ that

$$\lim_{k\to\infty} \left\| K_r\left(I - rA_j\right) x_{n_{s_k}} - x_{n_{s_k}} \right\| = 0. \tag{18}$$

Therefore, we obtain $\tilde{x} \in \bigcap_{j=1}^{\infty} F\left(K_r\left(I - rA_j\right)\right) = \bigcap_{j=1}^{\infty} \mathrm{GMEP}\left(F, \varphi, A_j\right)$.

It follows from (16) and Lemma 2.6 that

$$\begin{aligned}&\limsup_{n\to\infty} \left\langle f_1 z_0 - z_0, x_n - z_0 \right\rangle \\ &= \lim_{s\to\infty} \left\langle f_1 z_0 - z_0, x_{n_s} - z_0 \right\rangle = \left\langle f_1 z_0 - z_0, \tilde{x} - z_0 \right\rangle \\ &= \left\langle f_1 z_0 - P_\Gamma f_1 z_0, \tilde{x} - P_\Gamma f_1 z_0 \right\rangle \leq 0.\end{aligned} \tag{19}$$

Putting $h_n = \alpha_n f_n x_n + \beta_n y_n + \gamma_n z_n$ for all $n \in \mathbb{N}$, we have from (6) that $x_{n+1} = P_C h_n$. Since $\{f_n\}$ is stable on Γ, we then get by (9), (10) and Lemma 2.6 that

$$\begin{aligned}&\left\| x_{n+1} - z_0 \right\|^2 \\ &= \left\langle P_C h_n - h_n, P_C h_n - z_0 \right\rangle + \left\langle h_n - z_0, P_C h_n - z_0 \right\rangle \\ &\leq \left\langle \alpha_n f_n x_n + \beta_n y_n + \gamma_n z_n - z_0, x_{n+1} - z_0 \right\rangle \\ &\leq \left\| \beta_n\left(y_n - z_0\right) + \gamma_n\left(z_n - z_0\right) \right\| \left\| x_{n+1} - z_0 \right\| + \alpha_n \left\langle f_n x_n - z_0, x_{n+1} - z_0 \right\rangle\end{aligned}$$

$$\begin{aligned}
&\le \beta_n \|y_n - z_0\| \|x_{n+1} - z_0\| + \gamma_n \|z_n - z_0\| \|x_{n+1} - z_0\| \\
&\quad + \alpha_n \langle f_n x_n - f_n z_0, x_{n+1} - z_0 \rangle + \alpha_n \langle f_n z_0 - z_0, x_{n+1} - z_0 \rangle \\
&\le (1-\alpha_n) \|x_n - z_0\| \|x_{n+1} - z_0\| + \alpha_n v \|x_n - z_0\| \|x_{n+1} - z_0\| \\
&\quad + \alpha_n \langle f_1 z_0 - z_0, x_{n+1} - z_0 \rangle \\
&\le (1-\alpha_n(1-v)) \|x_n - z_0\|^2 + \alpha_n \langle f_1 z_0 - z_0, x_{n+1} - z_0 \rangle.
\end{aligned}$$

This together with Lemma 2.4 and (19) implies $x_n \to z_0$ as $n \to \infty$.

Case 2: Suppose that there exists $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - z_0\| < \|x_{n_i+1} - z_0\|$ for all $i \in \mathbb{N}$. Then by Lemma 2.3, there exists a nondecreasing sequence $\{m_j\}$ in $\mathbb{N}$ such that

$$\|x_{m_j} - z_0\| \le \|x_{m_j+1} - z_0\| \text{ and } \|x_j - z_0\| \le \|x_{m_j+1} - z_0\|. \tag{20}$$

Without loss of generality, there exists a subsequence $\{x_{m_{j_k}}\}$ of $\{x_{m_j}\}$ such that $x_{m_{j_k}} \rightharpoonup \tilde{z}$ for some $\tilde{z} \in C$ and

$$\lim_{k\to\infty} \langle f_1 z_0 - z_0, x_{m_{j_k}} - z_0 \rangle = \limsup_{j\to\infty} \langle f_1 z_0 - z_0, x_{m_j} - z_0 \rangle.$$

We show that

$$\limsup_{j\to\infty} \langle f_1 z_0 - z_0, x_{m_j} - z_0 \rangle \le 0, \tag{21}$$

where $z_0 = P_\Gamma f_1 z_0$. To see this, we can first obtain $\tilde{z} \in \Gamma$ by a similar argument as in Case 1. Therefore, we deduce that

$$\begin{aligned}
&\limsup_{j\to\infty} \langle f_1 z_0 - z_0, x_{m_j} - z_0 \rangle \\
&= \lim_{k\to\infty} \langle f_1 z_0 - z_0, x_{m_{j_k}} - z_0 \rangle = \langle f_1 z_0 - P_\Gamma f_1 z_0, \tilde{z} - P_\Gamma f_1 z_0 \rangle \le 0.
\end{aligned} \tag{22}$$

Like in Case 1, we can also get that

$$\lim_{j\to\infty} \|y_{m_j} - z_{m_j}\| = 0 \text{ and } \lim_{j\to\infty} \|x_{m_j} - z_{m_j}\| = 0. \tag{23}$$

Observing that

$$\begin{aligned}
&\|x_{m_j+1} - x_{m_j}\| \\
&\le \|\alpha_{m_j} f_{m_j} x_{m_j} + \beta_{m_j} y_{m_j} + \gamma_{m_j} z_{m_j} - x_{m_j}\| \\
&\le \alpha_{m_j} \|f_{m_j} x_{m_j} - x_{m_j}\| + \beta_{m_j} \|y_{m_j} - x_{m_j}\| + \gamma_{m_j} \|z_{m_j} - x_{m_j}\| \\
&\le \alpha_{m_j} \|f_{m_j} x_{m_j} - x_{m_j}\| + \beta_{m_j} \|y_{m_j} - z_{m_j}\| + \beta_{m_j} \|z_{m_j} - x_{m_j}\| + \gamma_{m_j} \|z_{m_j} - x_{m_j}\| \\
&= \alpha_{m_j} \|f_{m_j} x_{m_j} - x_{m_j}\| + (1-\alpha_{m_j}) \|z_{m_j} - x_{m_j}\| + \beta_{m_j} \|y_{m_j} - z_{m_j}\|,
\end{aligned}$$

we then find from (23) and (i) that

$$\lim_{j\to\infty} \|x_{m_j+1} - x_{m_j}\| = 0. \tag{24}$$

Putting $h_{m_j} = \alpha_{m_j} f_{m_j} x_{m_j} + \beta_{m_j} y_{m_j} + \gamma_{m_j} z_{m_j}$ for all $j \in \mathbb{N}$, we obtain by Lemma 2.6, (9), (10) and (20) that

$$
\begin{aligned}
&\left\|x_{m_j+1}-z_0\right\|^2 \\
&=\left\langle P_C h_{m_j}-h_{m_j}, P_C h_{m_j}-z_0\right\rangle+\left\langle h_{m_j}-z_0, P_C h_{m_j}-z_0\right\rangle \\
&\leq\left\langle \alpha_{m_j} f_{m_j} x_{m_j}+\beta_{m_j} y_{m_j}+\gamma_{m_j} z_{m_j}-z_0, x_{m_j+1}-z_0\right\rangle \\
&\leq\left\|\beta_{m_j}\left(y_{m_j}-z_0\right)+\gamma_{m_j}\left(z_{m_j}-z_0\right)\right\|\left\|x_{m_j+1}-z_0\right\| \\
&\quad+\alpha_{m_j}\left\langle f_{m_j} x_{m_j}-z_0, x_{m_j+1}-z_0\right\rangle \\
&\leq \beta_{m_j}\left\|y_{m_j}-z_0\right\|\left\|x_{m_j+1}-z_0\right\|+\gamma_{m_j}\left\|z_{m_j}-z_0\right\|\left\|x_{m_j+1}-z_0\right\| \\
&\quad+\alpha_{m_j}\left\langle f_{m_j} x_{m_j}-f_{m_j} z_0, x_{m_j+1}-z_0\right\rangle+\alpha_{m_j}\left\langle f_{m_j} z_0-z_0, x_{m_j+1}-x_{m_j}\right\rangle \\
&\quad+\alpha_{m_j}\left\langle f_{m_j} z_0-z_0, x_{m_j}-z_0\right\rangle \\
&\leq\left(1-\alpha_{m_j}\right)\left\|x_{m_j}-z_0\right\|\left\|x_{m_j+1}-z_0\right\|+\alpha_{m_j} v\left\|x_{m_j}-z_0\right\|\left\|x_{m_j+1}-z_0\right\| \\
&\quad+\alpha_{m_j}\left\|f_{m_j} z_0-z_0\right\|\left\|x_{m_j+1}-x_{m_j}\right\|+\alpha_{m_j}\left\langle f_{m_j} z_0-z_0, x_{m_j}-z_0\right\rangle \\
&\leq\left(1-\alpha_{m_j}(1-v)\right)\left\|x_{m_j}-z_0\right\|\left\|x_{m_j+1}-z_0\right\| \\
&\quad+\alpha_{m_j}\left\|f_1 z_0-z_0\right\|\left\|x_{m_j+1}-x_{m_j}\right\|+\alpha_{m_j}\left\langle f_1 z_0-z_0, x_{m_j}-z_0\right\rangle \\
&\leq\left(1-\alpha_{m_j}(1-v)\right)\left\|x_{m_j+1}-z_0\right\|^2+\alpha_{m_j}\left\|f_1 z_0-z_0\right\|\left\|x_{m_j+1}-x_{m_j}\right\| \\
&\quad+\alpha_{m_j}\left\langle f_1 z_0-z_0, x_{m_j}-z_0\right\rangle,
\end{aligned}
$$

which means that

$$\left\|x_{m_j+1}-z_0\right\|^2 \leq \frac{1}{1-v}\left\|f_1 z_0-z_0\right\|\left\|x_{m_j+1}-x_{m_j}\right\|+\frac{1}{1-v}\left\langle f_1 z_0-z_0, x_{m_j}-z_0\right\rangle .$$

Noticing (22) and (24), we deduce

$$\lim_{j \rightarrow \infty}\left\|x_{m_j+1}-z_0\right\|=0.$$

We can also obtain by (20) that

$$\lim_{j \rightarrow \infty}\left\|x_j-z_0\right\| \leq \lim_{j \rightarrow \infty}\left\|x_{m_j+1}-z_0\right\|=0.$$

Consequently, we get $x_j \rightarrow z_0$ as $j \rightarrow \infty$.

Remark 3.2 Theorem 3.1 extends, improves and develops Theorem 3.1 of Takahashi [12], Theorem 3.1 of Akashi and Takahashi [14], Theorem 3.1 of Takahashi [15] and Theorem 3.1 of Alizadeh and Moradlou [23] in the following aspects:

- Theorem 3.1 improves and develops corresponding results in [23] from generalized hybrid mappings to demimetric mappings;
- Theorem 3.1 extends, improves and develops corresponding results in [12] [14] and [15] from finding a common solution of fixed-point problems and the variational inequality problems in Hilbert spaces to the more general and challenging problem for finding a common solution of the generalized mixed equilibrium problems and the null point problems in Banach spaces;
- The proof of our Theorem 3.1 is very different from the proof of the ones

given in [12] [14] [15] [23];

- The algorithm 6 is more advantageous and more flexible than the ones given in [12] [14] [15] [23]. Therefore, the new algorithm is expected to be widely applicable.

4. An Extension of Our Main Results

From Theorem 3.1, we deduce immediately the following results

Corollary 4.1 Suppose $\Gamma_1 := \bigcap_{i=1}^{\infty} B^{-1}F(T_i) \cap \mathrm{MEP}(F,\varphi) \neq \varnothing$. For $x_1 \in C$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = \left(I - \tau \sum_{i=1}^{\infty} \sigma_i B^* J(I - T_i)B\right)x_n, \\ y_n = K_{r_n} x_n, \\ x_{n+1} = P_C(\alpha_n f_n x_n + \beta_n y_n + \gamma_n z_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{25}$$

where $\tau \in (0,+\infty)$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset (0,1)$ and $\{r_n\},\{\sigma_i\} \subset (0,+\infty)$ satisfy the following conditions:

1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$,
3) $0 < \tau < \frac{1-k}{\|B\|^2}$ and $\sum_{i=1}^{\infty} \sigma_i = 1$.

Then the sequence $\{x_n\}$ generated by (25) converges strongly to a point $z_0 \in \Gamma_1$, where $z_0 = P_{\Gamma_1} f z_0$.

Corollary 4.2 Let $\{T_i\}_{i=1}^{\infty} : C \to H$ be an infinite family of directed and demiclosed mappings. For $x_1 \in C$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = \left(I - \tau \sum_{i=1}^{\infty} \sigma_i B^* J(I - T_i)B\right)x_n, \\ y_n = \sum_{i=1}^{\infty} \delta_j K_{r_n}(I - r_n A_j)x_n, \\ x_{n+1} = P_C(\alpha_n f_n x_n + \beta_n y_n + \gamma_n z_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{26}$$

where $\tau \in (0,+\infty)$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset (0,1)$ and $\{r_n\},\{\sigma_i\} \subset (0,+\infty)$ satisfy the following conditions:

1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$,
3) $0 < \tau < \frac{1}{\|B\|^2}$,
4) $\sum_{j=1}^{\infty} \delta_j = 1$ and $\sum_{i=1}^{\infty} \sigma_i = 1$,
5) $0 < a \leq r_n \leq b < 2\mu$.

Then the sequence $\{x_n\}$ generated by (26) converges strongly to a point $z_0 \in \Gamma$, where $z_0 = P_\Gamma f_1 z_0$.

Proof. Noticing that a directed mapping T with $F(T) \neq \varnothing$ is 0-demimetric, then we have the desired result due to Theorem 3.1.

5. Numerical Examples

In this section, we discuss the direct application of Theorem 3.1 on a typical example on a real line.

Example 5.1 Let $C = H = E = \mathbb{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the standard norm $|\cdot|$. Let $T_i : \mathbb{R} \to \mathbb{R}$, $A_i : \mathbb{R} \to \mathbb{R}$, $M : \mathbb{R} \to \mathbb{R}$ and $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_i x = -\frac{1+i}{i}x,\ A_i x = \frac{1+i}{i}x,\ Mx = 4x \text{ and } f_n x = \frac{1}{1+n}x,\ \forall x \in \mathbb{R}, i \in \mathbb{N}.$$

It is easy to check that $\Gamma = \{0\}$, T_i is an infinite family of $\frac{1}{1+2i}$-demimetric and demiclosed mappings, $\{A_i\}_{i=1}^{\infty}$ is an infinite family of $\frac{i}{1+i}$-ism mappings, and f_n is $\frac{1}{2}$-contractive on H and stable on Γ. Let $\varphi = 0$ for all $x \in \mathbb{R}$, then we see φ satisfies (A5).

Letting $Bx = \frac{3}{2}x$ for all $x \in \mathbb{R}$, we then see B is a bounded linear operator with its adjoint $B^* = B$. Note that $\|B\| = \|B^*\| = \frac{3}{2}$. Define a bifunction $F : C \times C \to \mathbb{R}$ by

$$F(z, y) = y^2 + 2zy - 3z^2.$$

We then find that F satisfies (A1)-(A4). So, by Lemma 2.8, we have $K_r(x)$ is nonempty and single-valued for each $x \in C$. Hence, for any $r > 0$, there exists $z \in C$ such that

$$F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\quad \forall y \in C,$$

which is equivalent to

$$ry^2 + (2rz + z - x)y + (zx - 3rz^2 - z^2) \geq 0,\quad \forall y \in C.$$

After solving the above inequality, we get $z = \frac{x}{1+4r}$, *i.e.* $K_r x = \frac{x}{1+4r}$.

Let us choose $\tau = \frac{1}{9}$, $\alpha_n = \frac{1}{6n}$, $\beta_n = \frac{n+1}{3n}$, $\gamma_n = \frac{4n-3}{6n}$, $r_n = \frac{2n-1}{4n}$, and $\delta_i = \sigma_i = \frac{1}{2^i}$ (choosing other values of these variables arbitrarily which satisfy the conditions of Theorem 3.1, the same convergence result also can be obtained). Then τ, α_n, β_n, γ_n, r_n, δ_i and σ_i satisfy all the conditions of Theorem 3.1. Then (6) can be rewrite as

$$\begin{cases} z_n = \left(\frac{1}{9} - \frac{4\ln 2}{9}\right)x_n, \\ y_n = \frac{n}{3n-1}\left[1 - \frac{2n-1}{4n}(1+\ln 2)\right]x_n \\ x_{n+1} = \frac{1}{6n(1+n)}x_n + \frac{n+1}{3n}y_n + \frac{4n-3}{6n}z_n,\quad \forall n \in \mathbb{N}. \end{cases}$$

It is not hard to estimate that

$$|x_{n+1}| \le \frac{1}{2}|x_n| \le \cdots \le \frac{1}{2^n}|x_1|,$$

which shows $x_n \to 0 \in \Gamma$.

6. Conclusion

The present work has been aimed to theoretically establish a new iterative scheme for finding a common solution of the generalized mixed equilibrium problems with an infinite family of inverse strongly monotone mappings and the fixed point problems of demimetric mappings under nonlinear transformations in Banach spaces. Our results can be viewed as improvement, supplementation, development and extension of the corresponding results in some references to a great extent.

Acknowledgements

This research was supported by the Key Scientific Research Projects of Higher Education Institutions in Henan Province (20A110038).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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