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Local Strong Solutions for the Cauchy Problem of 2D Density-Dependent Boussinesq Equations with Vacuum

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Abstract

The main goal of the paper is to obtain the local strong solution of the Cauchy problem of the nonhomogeneous incompressible Boussinesq equation in two-dimension space. Especially, when the far-field density is vacuum, we make a priori estimate in a bound ball and prove the existence and uniqueness of the local strong solution of the Boussinesq equation.

Keywords

Non-Homogeneous Incompressible Boussinesq Equation, Strong Solution, Vacuum, Cauchy Problem

1. Introduction

The Boussinesq equation is an important class of equations in fluid equations. We consider the Cauchy problem of two-dimensional nonhomogeneous incompressible Boussinesq equations:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = \mu \Delta u, \\ \theta_t + u \cdot \nabla \theta - \kappa \Delta \theta = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{1}$$

for viscous incompressible flows. Here, $t \geq 0$ is time, $x \in R^2$ is the spatial coordinate, and $\rho = \rho(x,t)$, $u = (u^1, u^2)(x,t)$, $\theta = (\theta^1, \theta^2)(x,t)$ and $P = P(x,t)$, are the fluid density, velocity, temperature and pressure, respectively. The constant $\mu > 0$ and $\kappa > 0$ are the viscosity coefficient and the thermal expansion coefficient of the flow respectively.

The initial data ρ_0, u_0 and θ_0 are given by

$$\rho(x,0)=\rho_0(x),\quad \rho u(x,0)=\rho_0 u_0(x),\quad \theta(x,0)=\theta_0(x). \tag{2}$$

There has been a long history, studying the existence of solutions to Boussinesq equations. In recent years, much attention has attracted by Boussinesq equations with $\rho>0$. For example, when $\mu>0$, $\kappa>0$, Ishimura-Morimoto [1] gave blow-up criterion in the 3D. Next, for the cases of "partial viscosity", in [2], Fan-Zhou proved blow-up criterion of Equations (1) with $\mu=0$, $\kappa>0$. For general initial data in H^m and $m\geq 3$ cases, Hou and Li [3] come up with the global well-posed solution of the proof for the incompressible Boussinesq equations in two-dimensions. When the equation was not viscous, such as $\mu=\kappa=0$, Dongho Chae and Hee-Seok Nam [4] studied the local existence of solution of the Boussinesq equations and provided a blow-up criterion for the smooth solutions in the Sobolev spaces $H^m(R^2)$ and $m>2$. In $\rho\geq 0$ case, Hou and Jiu [5] considered the local existence and uniqueness of the strong solutions of the density-dependent viscous Boussinesq equations for incompressible fluid in R^3 with $\mu>0$, $\kappa>0$. But the case of the 3D case [5] cannot be used in 2D case. However, the two-dimensional case is an open problem. Recently, we mention that Liang [6] has come up with energy estimation of the Navier-Stokes equation with vacuum as far-field density in a bounded sphere, then extends to the entire two-dimensional space to obtain the existence of a local strong solution of the incompressible Navier-Stokes equations. In fact, if the temperature function is zero (*i.e.*, $\theta=0$), then (1) reduces to the Navier-Stokes equations [7]. Comparing with the Navier-Stokes equation and Euler equation, Boussinesq equations exist a complicated nonlinear relationship between velocity and pressure [8]. As a result, the study of Boussinesq equations is more complicated. Based on [6], we will show the existence and uniqueness of strong solution to the Cauchy problem (1) and (2).

This article has two difficulties. Firstly, it is difficult to control the L^p-norm ($p>2$) of the velocity u with the L^2-norm of its gradient. To overcome this difficulty, in light of [6] [9], we introduce $\bar{x}\triangleq\left(e+|x|^2\right)^{\frac{1}{2}}\log^{1+\gamma_0}\left(e+|x|^2\right)$ ($\gamma_0>0$), and set up a Hardy-type inequality (such as (14)) to bound the L^p-norm of $u\bar{x}^{-\gamma}$ taking the place of the velocity u [10]. We acquire a pivotal inequality (such as (22)), which can control the L^p-norm of ρu. Moreover, in incompressible Boussinesq equations, there are strong coupled terms that bring us some new difficulties, such as $\||u||\theta|\|$ and $\||u||\nabla\theta|\|$. For the purpose of controling $\||u||\theta|\|$ and $\||u||\nabla\theta|\|$, which are infered from the coupled term $u\cdot\nabla\theta$ and integration, we make use of a spatial weighted mean estimate of θ and $\nabla\theta$ (*i.e.*, $\bar{x}^{a/2}\theta$ and $\bar{x}^{a/2}\nabla\theta$, such as (18), (42)). Particularly, the focus of this article is to do a priori estimate in a bounded ball B_{R_0}. Through the above key steps, we can easily get the existence and uniqueness of strong solution to the Cauchy problem (1) and (2) by a standard limit procedure.

Theorem 1.1. *For each positive constant* $q>2$ *and* $a>1$, *Let the initial data* (ρ_0,u_0,θ_0) *satisfy*

$$\begin{cases}\rho_0 \ge 0,\ \rho_0 \bar{x}^a \in L^1 \cap H^1 \cap W^{1,q},\ \nabla u_0 \in L^2,\ \sqrt{\rho_0} u_0 \in L^2, \\ \theta_0 \ge 0,\ \theta_0 \bar{x}^{\frac{a}{2}} \in L^2,\ \nabla \theta \in L^2,\ \operatorname{div} u_0 = 0.\end{cases} \tag{3}$$

Then set $T_1 > 0$ is a small time, for the problem (1)-(2) make a unique strong solution (ρ, u, P, θ) on $R^2 \times [0, T_1]$ satisfies the following properties:

$$\begin{cases}0 \le \rho \in C\left([0,T_1]; L^1 \cap H^1 \cap W^{1,q}\right), \rho \bar{x}^a \in L^\infty\left(0,T; L^1 \cap H^1 \cap W^{1,q}\right), \\ \sqrt{\rho} u, \nabla u, \bar{x}^{-1} u, \sqrt{t}\sqrt{\rho} u_t, \sqrt{t}\nabla P, \sqrt{t}\nabla^2 u \in L^\infty\left(0,T_1; L^2\right), \\ \nabla \theta \in L^2\left(0,T_1; H^1\right), \sqrt{t}\nabla u \in L^2\left(0,T_1; W^{1,q}\right), \\ \theta, \theta \bar{x}^{a/2}, \nabla\theta, \sqrt{t}\theta_t, \sqrt{t}\nabla^2\theta, \sqrt{t}\nabla\theta\bar{x}^{a/2} \in L^\infty\left(0,T_1; L^2\right), \\ \nabla u \in L^2\left(0,T_1; H^1\right) \cap L^{(q+1)/q}\left(0,T_1; W^{1,q}\right), \\ \nabla P \in L^2\left(0,T_1; L^2\right) \cap L^{(q+1)/q}\left(0,T_1; L^q\right), \\ \theta_t, \nabla\theta\bar{x}^{a/2} \in L^2\left(0,T_1; L^2\right), \\ \sqrt{\rho} u_t, \sqrt{t}\nabla u_t, \sqrt{t}\nabla\theta_t, \sqrt{t}\bar{x}^{-1} u_t \in L^2\left(R^2 \times (0,T_1)\right),\end{cases} \tag{4}$$

and

$$\inf_{0 \le t \le T_1} \int_{B_N} \rho(x,t)\,\mathrm{d}x \ge \frac{1}{4}\int \rho_0(x)\,\mathrm{d}x, \tag{5}$$

for the constant $N > 0$ and $B_N \triangleq \left\{x \in R^2 \,|\, |x| < N\right\}$.

2. A Priori Estimates

The main duty in the present paper is to establish crucial energy estimates in the bounded domain. Next, we are going to establish the a priori estimates of ψ, which will be the main effort of this section. We define

$$\psi(t) \triangleq 1 + \left\|\rho^{1/2} u\right\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla\theta\|_{L^2} + \left\|\bar{x}^{a/2}\theta\right\|_{L^2} + \left\|\bar{x}^{a/2}\rho\right\|_{L^1 \cap H^1 \cap W^{1,q}}.$$

Proposition 2.1 *Suppose* (ρ_0, u_0, θ_0) *satisfies* (3). *Let* (ρ, u, P, θ) *be the solution to the initial-boundary-value problem* (1) *on* $B_N \triangleq \left\{x \in R^2 \,|\, |x| < N\right\}$. *Then there exists a small positive time* $T_1 > 0$ *and* C *which depends on* $\mu, \kappa, q, a, \gamma_0, N_1$, *and* ψ, *such that*

$$\begin{aligned}&\sup_{t \in [0,T_1]}\left(\psi(t) + \sqrt{t}\left\|\sqrt{\rho} u_t\right\|_{L^2} + \sqrt{t}\|\theta_t\|_{L^2} + \sqrt{t}\left\|\nabla^2 u_t\right\|_{L^2} + \sqrt{t}\|\nabla P\|_{L^2} + \sqrt{t}\left\|\nabla^2\theta\right\|_{L^2}\right) \\ &+ \int_0^T\left(\left\|\sqrt{\rho} u_t\right\|_{L^2}^2 + \left\|\nabla^2 u\right\|_{L^2}^2 + \left\|\nabla^2\theta\right\|_{L^2}^2 + \|\theta_t\|_{L^2}^2 + \left\|\nabla\theta\bar{x}^{a/2}\right\|_{L^2}^2\right)\mathrm{d}t \\ &+ \int_0^{T_1}\left(\left\|\nabla^2 u\right\|_{L^q}^{(q+1)/q} + \|\nabla P\|_{L^q}^{(q+1)/q} + t\left\|\nabla^2 u\right\|_{L^q}^2 + t\|\nabla P\|_{L^q}^2\right) \\ &+ \int_0^{T_1}\left(t\|\nabla u_t\|_{L^2}^2 + t\|\nabla H_t\|_{L^2}^2\right)\mathrm{d}t \le C.\end{aligned} \tag{6}$$

In addition

$$E_1 \triangleq \left\|\rho_0^{1/2} u_0\right\|_{L^2} + \|\nabla u_0\|_{L^2} + \|\nabla\theta_0\|_{L^2} + \left\|\bar{x}^{a/2}\theta_0\right\|_{L^2} + \left\|\bar{x}^{a/2}\rho_0\right\|_{L^1 \cap H^1 \cap W^{1,q}}.$$

The validity of Proposition 2.1 is at the end of this section. Next, we will start the standard energy estimation for (ρ, u, P, θ) and the L^p-norm of the density.

Next, we start with the standard energy estimates.

Lemma 2.1 *Assume that problem* (1) *have a smooth solution* (ρ,u,P,θ) *to the initial-boundary-value, in the* $B_{R_0}=\{x\in R^2 \mid |x|<R_0\}$ *and* $R_0>0$. *When for arbitrary* $t>0$

$$\sup_{s\in[0,t]}\left(\|\rho\|_{L^1\cap L^\infty}+\left\|\rho^{1/2}u\right\|_{L^2}^2+\|\theta\|_{L^2}^2\right)+\int_0^t\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\mathrm{d}s\le C, \tag{7}$$

moreover, C relies on $\mu,\kappa,q,a,\gamma_0,N_0$ and $\psi(t)$.

Proof: From the mass Equation $(1)_1$, we can deduce

$$\sup_{s\in[0,t]}\left(\left\|\rho^{1/2}u\right\|_{L^2}^2+\|\theta\|_{L^2}^2\right)+\int_0^t\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\mathrm{d}s\le C, \tag{8}$$

owing to $\mathrm{div}u=0$ and the continuity Equation $(1)_1$ [11], we obtain

$$\sup_{s\in[0,t]}\|\rho\|_{L^1\cap L^\infty}\le C. \tag{9}$$

Inequalities (8) and (9) complete the proof.

Next, spatial weighted estimates of density and temperature have yet to be proven.

Lemma 2.2 *Let the assumptions in Lemma* 2.1 *be satisfied. Where* $T_2>0$ *is a small time and relies only on* $\mu,\kappa,q,a,\gamma_0,N$, *and* ψ, *then for arbitrary* $t\in(0,T_2]$

$$\sup_{s\in[0,t]}\left(\left\|\rho\bar{x}^a\right\|_{L^1}+\left\|\theta\bar{x}^{a/2}\right\|_{L^2}^2\right)+\int_0^t\left\|\nabla\theta\bar{x}^{a/2}\right\|_{L^2}^2\mathrm{d}s\le C. \tag{10}$$

Proof: First, for R_0, let $\varphi_{R_0}\in C_0^\infty\left(B_{R_0}\right)$ satisfy

$$0\le\varphi_{R_0}\le 1,\ \varphi_{R_0}(x)=1,\ \text{if } |x|\le N/2,\ \left|\nabla\varphi_{R_0}\right|\le CN^{-1}. \tag{11}$$

From Equations $(1)_1$ and (14) we can deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\varphi_{R_0}\mathrm{d}x=\int\rho u\cdot\nabla\varphi_{R_0}\mathrm{d}x\ge -CN^{-1}\left(\int\rho\mathrm{d}x\right)^{1/2}\left(\int\rho|u|^2\right)^{1/2}\ge-\hat{C}N^{-1}, \tag{12}$$

integrating (12) and using (5) give

$$\inf_{t\in[0,T_2]}\int_{B_{R_0}}\rho\mathrm{d}x\ge\inf_{t\in[0,T_2]}\int\rho\varphi_{R_0}\mathrm{d}x\ge\int\rho_0\varphi_{R_0}-\hat{C}N^{-1}T_2\ge 1/4, \tag{13}$$

where, $T_2\triangleq\min\left\{1,\left(N/4\hat{C}\right)\right\}$. It follows from (13), (9) and ([2] Lemma 2.3) that for arbitrary $v\in\tilde{D}^{1,2}$ we can obtain

$$\left\|v\bar{x}^{-\gamma}\right\|_{(2+\epsilon)/\bar{\gamma}}^2\le C(\epsilon,\gamma)\left\|\rho^{1/2}v\right\|_{L^2}^2+C(\epsilon,\gamma)\|\nabla v\|_{L^2}^2, \tag{14}$$

where $\bar{\gamma}=\min\{1,\gamma\}$. From now on, using multiplying Equations $(1)_1$ by $\bar{x}^a$ and integrating, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\bar{x}^a\mathrm{d}x&\le C\int\rho|u|\bar{x}^{a-1}\log^{1+\gamma_0}\left(e+|x|^2\right)\mathrm{d}x\\&\le C\left\|\rho\bar{x}^{a-1+\frac{8}{8+a}}\right\|_{L^{\frac{8+a}{7+a}}}\left\|u\bar{x}^{-\frac{4}{8+a}}\right\|_{L^{8+a}}\\&\le C\|\rho\|_{L^\infty}^{\frac{1}{8+a}}\left\|\rho\bar{x}^a\right\|_{L^1}^{\frac{7+a}{8+a}}\left(\left\|\rho^{1/2}u\right\|_{L^2}+\|\nabla u\|_{L^2}\right)\\&\le C\left(1+\left\|\rho\bar{x}^a\right\|_{L^1}\right)\left(1+\|\nabla u\|_{L^2}^2\right),\end{aligned} \tag{15}$$

using Gronwall's inequality and (7), we find

$$\sup_{s\in[0,t]}\left\|\rho\bar{x}^{a}\right\|_{L^1}\le C\exp\left\{C\int_0^t\left(1+\|\nabla u\|_{L^2}^2\right)\mathrm{d}s\right\}\le C. \tag{16}$$

Next, multiplying Equations $(1)_3$ by $\theta\bar{x}^{a}$ and integrating, we infer

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\theta\bar{x}^{\frac{a}{2}}\right\|_{L^2}^2+k\left\|\nabla\theta\bar{x}^{\frac{a}{2}}\right\|_{L^2}^2\\
&=\frac{k}{2}\int|\theta|^2\Delta\bar{x}^{a}\mathrm{d}x+\frac{1}{2}\int|\theta|^2u\cdot\nabla\bar{x}^{a}\mathrm{d}x\\
&\le C\int|\theta|^2\bar{x}^{a}\bar{x}^{-2}\log^{2(1+\gamma_0)}\left(e+|x|^2\right)\mathrm{d}x+\left\|\theta\bar{x}^{\frac{a}{2}}\right\|_{L^4}\left\|\theta\bar{x}^{\frac{a}{2}}\right\|_{L^2}\left\|u\bar{x}^{-\frac{3}{4}}\right\|_{L^4}\\
&\le C\left\|\theta\bar{x}^{\frac{a}{2}}\right\|_{L^2}^2+C\left\|\theta\bar{x}^{\frac{a}{2}}\right\|_{L^4}^2+C\left\|\theta\bar{x}^{\frac{a}{2}}\right\|_{L^4}^2\left(\left\|\rho^{1/2}u\right\|_{L^2}^2+\|\nabla u\|_{L^2}^2\right)\\
&\le C\left(1+\|\nabla u\|_{L^2}^2\right)\left\|\theta\bar{x}^{\frac{a}{2}}\right\|_{L^2}^2+\frac{\kappa}{2}\left\|\nabla\theta\bar{x}^{\frac{a}{2}}\right\|_{L^2}^2,
\end{aligned} \tag{17}$$

due to Gagliardo-Nirenberg inequality [12], (7), (14). Then using Gronwall's inequality and (7), we find

$$\sup_{s\in[0,t]}\left\|\theta\bar{x}^{a/2}\right\|_{L^2}^2+\int_0^t\left\|\nabla\theta\bar{x}^{a/2}\right\|_{L^2}^2\mathrm{d}s\le C, \tag{18}$$

which together with (16) gives (10). We complete the proof.

Lemma 2.3 *Suppose that* (ρ,u,P,θ) *and* T_2 *of Lemma* 2.1 *and Lemma* 2.2 *hold. There is a positive constant* $\zeta>1$, *for all* $t\in(0,T_2]$

$$\begin{aligned}
&\sup_{t\in[0,T]}\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)+\int_0^T\left(\left\|\rho^{\frac{1}{2}}u_s\right\|_{L^2}^2+\left\|\nabla^2u\right\|_{L^2}^2+\|\theta_s\|_{L^2}^2+\left\|\nabla^2\theta\right\|_{L^2}^2\right)\mathrm{d}s\\
&\le C+C\int_0^t\psi^{\zeta}(s)\mathrm{d}s.
\end{aligned} \tag{19}$$

Proof: In Equations $(1)_2$, multiplying both sides by u_t, and integrating, we get

$$\mu\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla u|^2\mathrm{d}x+\int\rho|u_t|^2\mathrm{d}x\le C\int\rho|u|^2|\nabla u|^2\mathrm{d}x. \tag{20}$$

Now, it follows from (7), (10), and (14) that for arbitrary $\epsilon>0$, $\gamma>0$,

$$\begin{aligned}
\left\|\rho^{\gamma}v\right\|_{L^{(2+\epsilon)/\tilde{\gamma}}}&\le C\left\|\rho^{\gamma}\bar{x}^{\frac{3\tilde{\gamma}a}{4(2+\epsilon)}}\right\|_{L^{\frac{4(2+\epsilon)}{3\tilde{\gamma}}}}\left\|v\bar{x}^{-\frac{3\tilde{\gamma}a}{4(2+\epsilon)}}\right\|_{L^{\frac{4(2+\epsilon)}{\tilde{\gamma}}}}\\
&\le C\|\rho\|_{L^\infty}^{\frac{4(2+\epsilon)\gamma-3\tilde{\gamma}}{4(2+\epsilon)}}\left\|\rho\bar{x}^{a}\right\|_{L^1}^{\frac{3\tilde{\gamma}}{4(2+\epsilon)}}\left(\left\|\rho^{1/2}v\right\|_{L^2}+\|\nabla v\|_{L^2}\right)\\
&\le C\left(\left\|\rho^{1/2}v\right\|_{L^2}+\|\nabla v\|_{L^2}\right),
\end{aligned} \tag{21}$$

where $\tilde{\gamma}=\min\{1,\gamma\}$ and $v\in\tilde{D}^{1,2}\left(B_{R_0}\right)$. Particularly, this together with (7) and (14) derives

$$\left\|\rho^{\gamma}u\right\|_{L^{(2+\epsilon)/\tilde{\gamma}}}+\left\|u\bar{x}^{-\gamma}\right\|_{L^{(2+\epsilon)/\tilde{\gamma}}}\le C\left(1+\|\nabla v\|_{L^2}\right). \tag{22}$$

Using Hölder's and Gagliardo-Nirenberg inequalities, we deduce that

$$\begin{aligned}\int \rho|u|^2|\nabla u|^2\,\mathrm{d}x &\le C\left\|\rho^{1/2}u\right\|_{L^8}^2\|\nabla u\|_{L^{8/3}}^2\\ &\le C\left\|\rho^{1/2}u\right\|_{L^8}^2\|\nabla u\|_{L^2}^{3/2}\|\nabla u\|_{H^1}^{1/2}\\ &\le C\psi^{\zeta}+\varepsilon\left\|\nabla^2 u\right\|_{L^2}^2,\end{aligned} \tag{23}$$

where $\zeta > 1$.

Substituting (23) into (20) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla u|^2\,\mathrm{d}x+\int\rho|u_t|^2\,\mathrm{d}t\le\epsilon\left\|\nabla^2u\right\|_{L^2}^2+C\psi^{\zeta}. \tag{24}$$

Now, it follows from Equations $(1)_3$ that

$$\begin{aligned}&\kappa\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta\|_{L^2}^2+\|\theta_t\|_{L^2}^2+\kappa^2\|\Delta\theta\|_{L^2}^2\\ &\le C\left\||u||\nabla\theta|\right\|_{L^2}^2\\ &\le C\left\|\bar{x}^{-a/4}u\right\|_{L^8}^2\left\|\bar{x}^{-a/2}\nabla\theta\right\|_{L^2}\|\nabla\theta\|_{L^4}\\ &\le C\left\|\bar{x}^{-a/2}\nabla\theta\right\|_{L^2}^2+C\psi^{\zeta},\end{aligned} \tag{25}$$

owing to (22) and Gagliardo-Nirenberg inequality, multiplying (25) by $\kappa^{-1}(C_0+1)$ and the resulting inequality to (24) imply

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2+(C_0+1)\|\nabla\theta\|_{L^2}^2\right)+\left\|\rho^{1/2}u_t\right\|_{L^2}^2+\frac{\kappa}{2}\|\Delta\theta\|_{L^2}^2\\ &\le C\left\|\bar{x}^{-a/2}\nabla\theta\right\|_{L^2}^2+\epsilon\left\|\nabla^2u\right\|_{L^2}^2+C\psi^{\zeta},\end{aligned} \tag{26}$$

where (ρ,u,P,θ) satisfies Stokes system, so the regularity estimates [13] on the weak solutions show for all $p\in(1,\infty)$

$$\left\|\nabla^2u\right\|_{L^p}+\|\nabla p\|_{L^p}\le C\left(\|\rho u_t\|_{L^p}+\left\|\rho|u||\nabla u|\right\|_{L^p}\right). \tag{27}$$

Making use of (27), (6), (22) and Gagliardo-Nirenberg inequality, one has

$$\begin{aligned}\left\|\nabla^2u\right\|_{L^2}^2+\|\nabla P\|_{L^2}^2 &\le C\|\rho u_t\|_{L^2}^2+\|\rho u\cdot\nabla u_t\|_{L^2}^2\\ &\le C\|\rho\|_{L^\infty}\left\|\sqrt{\rho}u_t\right\|_{L^2}^2+C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^4}^2\\ &\le C\left\|\sqrt{\rho}u_t\right\|_{L^2}^2+C\|\rho u\|_{L^4}^2\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}\\ &\le C\left\|\sqrt{\rho}u_t\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla^2u\right\|_{L^2}^2+C\psi^{\zeta}.\end{aligned} \tag{28}$$

Finally, inserting (28) into (26) and choosing ϵ small enough to hold

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2+(C_0+1)\|\nabla\theta\|_{L^2}^2\right)+\frac{1}{2}\left\|\rho^{1/2}u_t\right\|_{L^2}^2+\frac{\kappa}{2}\|\Delta\theta\|_{L^2}^2\\ &\le C\left\|\bar{x}^{-a/2}\nabla\theta\right\|_{L^2}^2+C\psi^{\zeta}.\end{aligned} \tag{29}$$

Integrating (29), using ([2] Lemma 2.4), (9), and (28), we obtain (18). Hence Lemma 2.3 is proved.

Lemma 2.4 *Suppose that* (ρ,u,P,θ) *and* T_2 *of Lemma* 2.1 *and Lemma* 2.2 *satisfy*

$$\sup_{s\in[0,t]}\left(s\left\|\rho^{1/2}u_s\right\|_{L^2}^2+s\left\|\theta_s\right\|_{L^2}^2\right)+\int_0^t\left(s\left\|\nabla u_s\right\|_{L^2}^2+s\left\|\nabla\theta_s\right\|_{L^2}^2\right)\mathrm{d}x \\ \le C\exp\left\{C\int_0^t\psi^{\zeta}\mathrm{d}s\right\}. \tag{30}$$

Proof: Differentiating both side of Equations $(1)_2$ with respect to t, then multiplying both sides by u_t and integrating gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\left|u_t\right|^2\mathrm{d}x+\mu\int\left|\nabla u_t\right|^2\mathrm{d}x \\ \le C\int\rho|u|\left|u_t\right|\left(\nabla u_t+|\nabla u|^2+|u|\left|\nabla^2u\right|\right)\mathrm{d}x+C\int\rho|u|^2|\nabla u|\left|\nabla u_t\right|\mathrm{d}x \\ +C\int\rho\left|u_t\right|^2|\nabla u|\mathrm{d}x \\ \triangleq\sum_{i=1}^{3}\bar{M}_j. \tag{31}$$

Making use of (21), (22) combined with Gagliardo-Nirenberg inequality and Hölder's inequality combined with (21) and (28) leads to

$$\bar{M}_1\le C\left\|\rho^{1/2}u\right\|_{L^6}\left\|\rho^{1/2}u_t\right\|_{L^2}^{1/2}\left\|\rho^{1/2}u_t\right\|_{L^6}^{1/2}\left(\left\|\nabla u_t\right\|_{L^2}+\left\|\nabla u\right\|_{L^4}^2\right) \\ +C\left\|\rho^{1/4}u\right\|_{L^{12}}^2\left\|\rho^{1/2}u_t\right\|_{L^2}^{1/2}\left\|\rho^{1/2}u_t\right\|_{L^6}^{1/2}\left\|\nabla^2u\right\|_{L^2} \\ \le C\left(1+\left\|\nabla u\right\|_{L^2}^2\right)\left\|\rho^{1/2}u_t\right\|_{L^2}^{1/2}\left(\left\|\rho^{1/2}u_t\right\|_{L^2}+\left\|\nabla u_t\right\|_{L^2}\right)^{1/2} \\ \cdot\left(\left\|\nabla u_t\right\|_{L^2}+\left\|\nabla u\right\|_{L^2}^2+\left\|\nabla u\right\|_{L^2}\left\|\nabla^2u\right\|_{L^2}+\left\|\nabla^2u\right\|_{L^2}\right) \\ \le\frac{\mu}{4}\left\|\nabla u_t\right\|_{L^2}^2+C\psi^{\zeta}\left\|\rho^{1/2}u_t\right\|_{L^2}^2+C\psi^{\zeta}+C\left(1+\left\|\nabla u\right\|_{L^2}^2\right)\left\|\nabla^2u\right\|_{L^2}^2,$$

$$\bar{M}_2+\bar{M}_3 \\ \le C\left\|\rho^{1/2}u\right\|_{L^8}^2\left\|\nabla u\right\|_{L^4}\left\|\nabla u_t\right\|_{L^2}+C\left\|\rho^{1/2}u_t\right\|_{L^6}^{3/2}\left\|\rho^{1/2}u_t\right\|_{L^2}^{1/2}\left\|\nabla u\right\|_{L^2} \\ \le\frac{\mu}{4}\left\|\nabla u_t\right\|_{L^2}^2+C\psi^{\zeta}\left\|\rho^{1/2}u_t\right\|_{L^2}^2+C\left(\psi^{\zeta}+\left\|\nabla^2u\right\|_{L^2}^2\right).$$

In summary, we conclude from (31) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\rho^{1/2}u_t\right\|_{L^2}^2+\mu\left\|\nabla u_t\right\|_{L^2}^2\le C\psi^{\zeta}\left(1+\left\|\rho^{1/2}u_t\right\|_{L^2}^2\right). \tag{32}$$

Differentiating both side of Equations $(1)_3$ with respect to t, then multiplying both sides by θ^t and integrating hold

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left|\theta_t\right|^2\mathrm{d}x+\kappa\int\left|\nabla\theta_t\right|^2\mathrm{d}x \\ =\int\nabla\theta_t\cdot u_t\cdot\theta\mathrm{d}x+\int\theta_t\cdot\nabla u\cdot\theta_t\mathrm{d}x \\ \le C\psi^{\zeta}\left(\left\|\theta_t\right\|_{L^2}^2+\left\|\rho^{1/2}u_t\right\|_{L^2}^2\right)+C\left\|\nabla u_t\right\|_{L^2}^2. \tag{33}$$

Next, multiplying (32) by $u^{-1}\left(C_1+1\right)$ and using (33) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\left(C_1+1\right)\left\|\rho^{1/2}u_t\right\|_{L^2}^2+C\left\|\nabla u_t\right\|_{L^2}^2\right)\le C\psi^{\zeta}\left(1+\left\|\rho^{1/2}u_t\right\|_{L^2}^2\right). \tag{34}$$

Multiplying in by t, then by means of the Gronwall's inequality and (18) we arrive at (30). We complete the proof of Lemma 2.4.

Lemma 2.5 *Suppose that* (ρ, u, P, θ) *and* T_2 *of Lemma* 2.1 *and Lemma* 2.2 *hold, there exists a constant* $\zeta > 0$ *for each* $t \in (0, T]$ *satisfies*

$$\begin{aligned}&\sup_{s\in[0,t]}\left(s\|\nabla^2 u\|_{L^2}^2 + s\|\nabla^2\theta\|_{L^2}^2 + s\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2\right) + \int_0^t s\|\Delta\theta\bar{x}^{a/2}\|_{L^2}^2\,\mathrm{d}s\\&\le C\exp\left\{C\exp\left\{C\int_0^t \psi^\zeta\,\mathrm{d}s\right\}\right\}.\end{aligned} \tag{35}$$

Proof: Multiplying Equations (1)₃ by $\Delta\theta\bar{x}^a$ and integrating by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\theta|^2\bar{x}^a\mathrm{d}x + \kappa\int|\Delta\theta|^2\bar{x}^a\mathrm{d}x\\&\le C\int|\nabla u||\nabla\theta|^2\bar{x}^a dx + C\int|u||\nabla\theta|^2|\nabla\bar{x}^a|\mathrm{d}x + C\int|\nabla\theta||\Delta\theta||\nabla\bar{x}^a|\mathrm{d}x\\&\triangleq\sum_{i=1}^3\hat{M}_i.\end{aligned} \tag{36}$$

We then deduce:

$$\hat{M}_1 \le C\|\nabla u\|_{L^\infty}\left\|\nabla\theta\bar{x}^{\frac{a}{2}}\right\|_{L^2}^2 \le C\left(\psi^\zeta + \|\nabla^2 u\|_{L^q}^{(q+1)/q}\right)\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2,$$

$$\begin{aligned}\hat{M}_2 + \hat{M}_3 &\le C\left\||\nabla\theta|^{2-\frac{3}{3a}}\bar{x}^{a-\frac{1}{3}}\right\|_{L^{\frac{6a}{6a-2}}}\left\|u\bar{x}^{-\frac{1}{3}}\right\|_{L^{6a}}\left\||\nabla\theta|^{\frac{2}{3a}}\right\|_{L^{6a}}\\&\quad + C\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2 + \frac{\kappa}{4}\|\Delta\theta\bar{x}^{a/2}\|_{L^2}^2\\&\le C\psi^\zeta\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^{(6a-2)/3a}\|\nabla\theta\|_{L^4}^{2/3a} + C\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2 + \frac{\kappa}{4}\|\Delta\theta\bar{x}^{a/2}\|_{L^2}^2\\&\le C\left(\psi^\zeta + 1\right)\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2 + \|\nabla\theta\|_{L^4}^2 + \frac{\kappa}{4}\|\Delta\theta\bar{x}^{a/2}\|_{L^2}^2\\&\le C\left(\psi^\zeta + 1\right)\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2 + \frac{\kappa}{2}\|\Delta\theta\bar{x}^{a/2}\|_{L^2}^2.\end{aligned}$$

Substituting the above estimates into (36) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\theta|^2\bar{x}^a\mathrm{d}x + \kappa\int|\Delta\theta|^2\bar{x}^a\mathrm{d}x \le C\left(\psi^\zeta + \|\nabla^2 u\|_{L^q}^{(q+1)/q} + 1\right)\|\nabla\theta\bar{x}^{a/2}\|_{L^2}^2. \tag{37}$$

where, we claim that

$$\int_0^t\left(\|\nabla^2 u\|_{L^q}^{(q+1)/q} + \|\nabla P\|_{L^q}^{(q+1)/q} + s\|\nabla^2 u\|_{L^q}^2 + s\|\nabla P\|_{L^q}^2\right) \le C\exp\left\{C\int_0^t\psi^\zeta(s)\,\mathrm{d}s\right\}. \tag{38}$$

However, choosing $p = q$ in ([2] Lemma 2.4), we deduce from (7), (21) and Gagliardo-Nirenberg inequality that

$$\begin{aligned}&\|\nabla^2 u\|_{L^q} + \|\nabla P\|_{L^q}\\&\le C\left(\|\rho u_t\|_{L^q} + \|\rho u\cdot\nabla u\|_{L^q}\right) \le C\left(\|\rho u_t\|_{L^q} + \|\rho u\|_{L^{2q}}\|\nabla u\|_{L^{2q}}\right)\\&\le C\|\rho u_t\|_{L^2}^{2(q-1)/(q^2-2)}\|\rho u_t\|_{L^{q^2}}^{(q^2-2q)/(q^2-2)} + C\psi^\zeta\|\nabla^2 u\|_{L^2}^{1-1/q}\\&\le C\left(\|\rho u_t\|_{L^2}^{2(q-1)/(q^2-2)}\|\nabla u_t\|_{L^2}^{(q^2-2q)/(q^2-2)} + \|\rho u_t\|_{L^2}\right) + C\psi^\zeta\|\nabla^2 u\|_{L^2}^{1-1/q},\end{aligned} \tag{39}$$

using (18) and (30), we conclude

$$
\begin{aligned}
&\int_0^t\left(\left\|\nabla^2 u\right\|_{L^q}^{(q+1)/q}+\|\nabla P\|_{L^q}^{(q+1)/q}\right)\mathrm{d}s \\
&\le C\int_0^t s^{-(q+1)/2q}\left(s\left\|\rho^{1/2}u_t\right\|_{L^2}^2\right)^{\frac{q^2-1}{q\left(q^2-2\right)}}\left(s\left\|\nabla u_t\right\|_{L^2}^2\right)^{\frac{(q-2)(q+1)}{2\left(q^2-2\right)}}\mathrm{d}s \\
&\quad+C\int_0^t\left\|\rho^{1/2}u_t\right\|_{L^2}^{\frac{q+1}{q}}\mathrm{d}s+C\int_0^t\psi^{\zeta}\left\|\nabla^2 u\right\|_{L^2}^{\frac{q^2-1}{q^2}}\mathrm{d}s \\
&\le C\sup_{t\in[0,T]}\left(s\left\|\rho^{1/2}u_t\right\|_{L^2}^2\right)^{\frac{q^2-1}{q\left(q^2-2\right)}}\int_0^t s^{-(q+1)/2q}\left(s\left\|\nabla u_t\right\|_{L^2}^2\right)^{\frac{(q-2)(q+1)}{2\left(q^2-2\right)}}\mathrm{d}s \\
&\quad+C\int_0^t\left(\left\|\rho^{1/2}u_t\right\|_{L^2}^2+\left\|\nabla^2 u\right\|_{L^2}^2\right)\mathrm{d}s \\
&\le C\exp\left\{C\int_0^t\varphi^{\zeta}\mathrm{d}s\right\}\left(1+\int_0^t\left(s^{-\frac{q^3+q^2-2q-2}{q^3+q^2-2q}}+s\left\|\nabla u_t\right\|_{L^2}^2\right)\mathrm{d}s\right) \\
&\le C\exp\left\{C\int_0^t\varphi^{\zeta}\mathrm{d}s\right\},
\end{aligned} \tag{40}
$$

and

$$
\begin{aligned}
&\int_0^t\left(s\left\|\nabla^2 u\right\|_{L^q}^2+s\|\nabla P\|_{L^q}^2\right)\mathrm{d}s \\
&\le C\int_0^t\left(s\left\|\rho u_t\right\|_{L^2}^2\right)^{2(q-1)/\left(q^2-2\right)}\left(s\left\|\nabla u_t\right\|_{L^2}^2\right)^{\left(q^2-2q\right)/\left(q^2-2\right)}\mathrm{d}s \\
&\quad+\int_0^t\left\|\rho^{1/2}u_t\right\|_{L^2}^2\mathrm{d}s+C\int_0^t s\left(\left\|\nabla^2 u\right\|_{L^2}^2\right)^{1-1/q}\mathrm{d}s \\
&\le C\int_0^t\left\|\rho^{1/2}u_t\right\|_{L^2}^2\mathrm{d}s+C\int_0^t s\left\|\nabla u_t\right\|_{L^2}^2\mathrm{d}s+C\int_0^t s\left\|\nabla^2 u\right\|_{L^2}^2\mathrm{d}s \\
&\le C\exp\left\{C\int_0^t\psi^{\zeta}\mathrm{d}s\right\}.
\end{aligned} \tag{41}
$$

The desired (38) comes from equalities (39)-(40). Thus, multiplying (37) by t and using Gronwall's inequality, (19), (20), and (38) to deduce

$$
\sup_{s\in[0,T]}\left(s\left\|\nabla\theta\overline{x}^{a/2}\right\|_{L^2}^2\right)+\int_0^t s\left\|\Delta\theta\overline{x}^{a/2}\right\|_{L^2}^2\mathrm{d}s\le C\exp\left\{C\exp\left\{C\int_0^t\varphi^{\zeta}\mathrm{d}s\right\}\right\}. \tag{42}
$$

Now, combining Equations $(1)_3$, Hölder and Gagliardo-Nirenberg inequality that, we acquire

$$
\begin{aligned}
\left\|\nabla^2\theta\right\|_{L^2}^2&\le C\left\|\theta_t\right\|_{L^2}^2+C\left\||u|\nabla\theta\right\|_{L^2}^2 \\
&\le C\left\|\theta_t\right\|_{L^2}^2+C\left\|u\overline{x}^{-a/4}\right\|_{L^8}^2\left\|\nabla\theta\overline{x}^{a/2}\right\|_{L^2}\left\|\nabla\theta\right\|_{L^4} \\
&\le C\left\|\theta_t\right\|_{L^2}^2+\left\|\nabla\theta\overline{x}^{a/2}\right\|_{L^2}^2+C\left\|u\overline{x}^{-a/4}\right\|_{L^8}^4\left\|\nabla\theta\right\|_{L^4}^2 \\
&\le C\left\|\theta_t\right\|_{L^2}^2+C\left\|\nabla\theta\overline{x}^{a/2}\right\|_{L^2}^2+\frac{1}{4}\left\|\nabla^2\theta\right\|_{L^2}^2+C\left(1+\left\|\nabla u\right\|_{L^2}^8\right),
\end{aligned} \tag{43}
$$

which together with (28) gives that

$$
\begin{aligned}
&\left\|\nabla^2 u\right\|_{L^2}^2+\|\nabla P\|_{L^2}^2+\left\|\nabla^2\theta\right\|_{L^2}^2 \\
&\le C\left(\left\|\sqrt{\rho}u_t\right\|_{L^2}^2+\left\|\theta_t\right\|_{L^2}^2+C\left\|\nabla\theta\overline{x}^{a/2}\right\|_{L^2}^2\right)+C\left(1+\left\|\nabla u\right\|_{L^2}^8\right).
\end{aligned} \tag{44}
$$

Finally, multiplying (44) by $\overline{t}$, from (20), (30), and (42) we get

$$\sup_{\bar{t}\in[0,T]}\left(\bar{t}\left\|\nabla^2 u\right\|_{L^2}^2+\bar{t}\left\|\nabla P\right\|_{L^2}^2+\bar{t}\left\|\nabla^2\theta\right\|_{L^2}^2\right)\le C\exp\left\{C\exp\left\{C\int_0^t\varphi^\zeta\,\mathrm{d}s\right\}\right\}, \tag{45}$$

which combined with (42) implies (30) and thus finishes the proof Lemma 2.6.

Lemma 2.6 *Suppose that* (ρ,u,P,θ) *and* T_2 *of Lemma* 2.1 *and Lemma* 2.2 *hold. For a constant* $C>0$ *dependent on T hold*

$$\sup_{t\in[0,T]}\left\|\rho\bar{x}^a\right\|_{L^1\cap H^1\cap W^{1,q}}\le\exp\left\{C\exp\left\{C\int_0^t\varphi^\zeta\,\mathrm{d}s\right\}\right\}. \tag{46}$$

Proof. The lemma is analogous to that in ([2] Lemma 3.7) and is left to the reader. No proof will be given for Lemma 2.7.

Using the priori estimates given in Lemma 2.1-Lemma 2.6, gives Proposition 3.1 immediately.

3. Proof of Theorem 1.1

Now, combining Lemma 2.1-Lemma 2.6 and using a standard method, we obtain Proof of Theorem 1.1. In this paper, we mainly make prior estimates. The other steps are omitted here.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Ishimura, N. and Morimoto, H. (1999) Remarks on the Blow-Up Criterion for the 3D Boussinesq Equations. *Mathematical Models and Methods in Applied Sciences*, **9**, 1323-1332. https://doi.org/10.1142/S0218202599000580

[2] Fan, J. and Zhou, Y. (2009) A Note on Regularity Criterion for the 3D Boussinesq System with Partial Viscosity. *Applied Mathematics Letters*, **22**, 802-805. https://doi.org/10.1016/j.aml.2008.06.041

[3] Hou, T.Y. and Li, C. (2005) Global Well-Posedness of the Viscous Boussinesq Equations. *Discrete and Continuous Dynamical Systems*, **12**, 1-12. https://doi.org/10.3934/dcds.2005.12.1

[4] Hou, W. and Jiu, Q.S. (2008) On Local Strong Solutions of Density-Dependent Boussinesq Equations.

[5] Liang, Z.L. (2015) Local Existence Strong Solution and Blow-Up Criterion for the 2D Nonhomogeneous Incompressible Fluids. *Journal of Differential Equations*, **7**, 2633-2654. https://doi.org/10.1016/j.jde.2014.12.015

[6] Lü, B.Q., Xu, Z.H. and Zhong, X. (2015) On Local Strong Solutions to the Cauchy Problem of Two-Dimensional Density-Dependent Magnetohydrodynamic Equations with Vacuum. arXiV:1506.02156V1

[7] Choe, H.J. and Kim, H. (2003) Strong Solutions of the Navier-Stokes Equations for Nonhomogeneous Incompressible Fluids. *Communications in Partial Differential Equations*, **28**, 1183-1201. https://doi.org/10.1081/PDE-120021191

[8] Simon, J. (1990) Nonhomogeneous Viscous Incompressible Fluids: Existence of Velocity, Density, and Pressure. *SIAM Journal on Mathematical Analysis*, **21**,

1093-1117. https://doi.org/10.1137/0521061

[9] Chae, D. and Nam, H.S. (1999) Local Existence and Blow-Up Criterion of Hölder Continuous Solutions of the Bonssinesq Equations. *Nagoya Mathematical Journal*, **155**, 55-80. https://doi.org/10.1017/S0027763000006991

[10] Lions, P.L. (1996) Mathematical Topicas in Fluid Mechanics, Vol. I: Incompressible Models. Oxford University Press, Oxford.

[11] Coifman, R., Lions, P.L., Meyer, Y. and Semmes, S. (1993) Compensated Compactness and Hardy Spaces. *Journal de Mathématiques Pures et Appliquées*, **72**, 247-286.

[12] Galdi, G. (1994) An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. I: Linerized Steady Promblems, Springer Tracts in Natural Philosophy 38. Vol. 1, Springer-Verlag, New York.

[13] Nirenberg, L. (1959) On Elliptic Partial Differential Equations. *Annali della Scuola Normale Superiore di Pisa*, **13**, 115-162.