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Approximate Solution Method of the Seventh Order KdV Equations by Decomposition Method

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Abstract

In this paper, Adomian decomposition method (ADM) is implemented to approximate the solution of the Korteweg-de Vries (KdV) equations of seventh order, which are Kaup-Kuperschmidt equation and seventh order Kawahara equation. The results obtained by the ADM are compared with the exact solutions. It is found that the ADM is very efficient and convenient and can be applied to a large class of problems. The conservation properties of solution are examined by calculating the first three invariants.

Keywords

Adomian Decomposition Method, Kaup-Kuperschmidt Seventh-Order KdV Equation, Seventh-Order Kawahara Equation, Conservation Laws

1. Introduction

The general seventh-order KdV equation (gsKdV) reads

$$\begin{aligned} &u_t + au^3u_x + bu_x^3 + cuu_xu_{xx} + du^2u_{xxx} + eu_{xx}u_{xxx} \\ &+ fu_xu_{xxxx} + guu_{xxxxx} + u_{xxxxxxx} = 0, \end{aligned} \tag{1}$$

where a, b, c, d, e, f and g are nonzero parameters. One of the well-known particular cases of Equation (1) is called seventh order Kaup Kuperschmidt equation (KK) [1] which can be shown in the form

$$\begin{aligned} &u_t + 2016u^3u_x + 630u_x^3 + 2268uu_xu_{xx} + 504u^2u_{xxx} + 252u_{xx}u_{xxx} \\ &+ 147u_xu_{xxxx} + 42uu_{xxxxx} + u_{xxxxxxx} = 0, \end{aligned} \tag{2}$$

Another form of the seventh-order KdV equation is called seventh order Kawahara equation [2] which can be shown in the form

$$u_t + 6uu_x + u_{xxx} - u_{xxxxx} + \alpha u_{xxxxxxx} = 0, \tag{3}$$

where α is a nonzero constant. These equations were introduced initially by Pomeau *et al.* [3] for discussing the structural stability of KdV equation under a singular perturbation. These equations play an important role in mathematical physics, engineering and applied sciences for investigating travelling solitary wave solutions.

The Adomian decomposition method (ADM) was first proposed by George Adomian in the 1980's [4] [5] [6] [7]. This technique has been shown to solve effectively, easily, and accurately a large class of linear and nonlinear, ordinary or partial, deterministic or stochastic differential equations with approximates which converge rapidly to accurate solutions. This method is well-suited to physical problems since it makes the unnecessary linearization, perturbation problem being solved, sometimes seriously. Conservation laws (CLaws) are of basic importance in the study of evolution equations because they provide physical, conserved quantities for all solutions $u(x,t)$, and they can be used to check the accuracy of numerical solution methods [8] [9] [10] [11] [12]. The paper is arranged in the following manner: in Section 2, we present the ADM; Section 3 presents the CLaws for (KK) and Kawahara seventh-order KdV equations [13] [14]; in Section 4, the ADM is implemented to some problems in addition to studying the properties of CLaws; finally, a brief conclusion is given in Section 5.

2. The Method of Solution

Consider the (gsKdV) equation in an operator form

$$\begin{aligned} &L_t(u) + a(Ku) + b(Mu) + c(Nu) + d(Pu) + e(Qu) \\ &+ f(Ru) + g(Vu) + L_{7x}(u) = 0, \end{aligned} \tag{4}$$

where the notations $Ku = u^3 u_x$, $Mu = u_x^3$, $Nu = uu_x u_{xx}$, $Pu = u^2 u_{xxx}$, $Qu = u_{xx} u_{xxx}$, $Ru = u_x u_{xxxx}$ and $Vu = uu_{xxxxx}$ symbolize the nonlinear terms, respectively. Also, the notation $L_t = \frac{\partial}{\partial t}$ and $L_{7x} = \frac{\partial^7}{\partial x^7}$ symbolize the linear differential operators. Assuming L_t^{-1} the inverse of operator of L_t exists and conveniently by

$$L_t^{-1} = \int_0^t (.)\,\mathrm{d}t \tag{5}$$

Thus, applying the inverse operator L_t^{-1} to (4) yields

$$\begin{aligned} u(x,t) = h(x) - aL_t^{-1}(Ku) - bL_t^{-1}(Mu) - cL_t^{-1}(Nu) - dL_t^{-1}(Pu) \\ - eL_t^{-1}(Qu) - fL_t^{-1}(Ru) - gL_t^{-1}(Vu) - L_t^{-1}(L_{7x}u). \end{aligned} \tag{6}$$

The standard ADM [15] defines the solution $u(x,t)$ by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{7}$$

with u_0 identified as $u(x,0)$. The nonlinear terms *Ku*, *Mu*, *Nu*, *Pu*, *Qu*, *Ru* and Vu can be decomposed into infinite series of polynomial given by

$$Ku = u^2 u_x = \sum_{n=0}^{\infty} A_n, \tag{8}$$

$$Mu = u_x u_{xx} = \sum_{n=0}^{\infty} B_n, \tag{9}$$

$$Nu = uu_{xxx} = \sum_{n=0}^{\infty} C_n, \tag{10}$$

$$Pu = uu_{xxx} = \sum_{n=0}^{\infty} D_n, \tag{11}$$

$$Qu = uu_{xxx} = \sum_{n=0}^{\infty} E_n, \tag{12}$$

$$Ru = uu_{xxx} = \sum_{n=0}^{\infty} F_n, \tag{13}$$

$$Vu = uu_{xxx} = \sum_{n=0}^{\infty} G_n, \tag{14}$$

where A_n, B_n, B_n, D_n, E_n, F_n and G_n are the so-called Adomian polynomials of $u_0, u_1, \cdots, u_n$ defined by equation

$$P_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i(x,t)\right)\right]_{\lambda=0}, n \geq 0. \tag{15}$$

The components $u_n(x,t)$ can be determined sequentially by the standard recursion scheme as:

$$\begin{cases} u_0(x,t) = h(x), \\ u_{n+1} = -aL_t^{-1}(A_n) - bL_t^{-1}(B_n) - cL_t^{-1}(C_n) - dL_t^{-1}(D_n) - eL_t^{-1}(E_n) \\ \qquad - fL_t^{-1}(F_n) - gL_t^{-1}(G_n) - L_t^{-1}(L_{7x}u_n), n \geq 0. \end{cases} \tag{16}$$

3. Conservation Laws

The conservation properties of the solution are examined by calculating the Claws.

1) For KK equation Equation (2), the conservative quantities $I_i\,(i = 1,2,3)$ can be written as

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} u\,\mathrm{d}x, \\ I_2 &= \int_{-\infty}^{\infty}\left(u^3 - \frac{1}{8}u_x^2\right)\mathrm{d}x, \\ I_3 &= \int_{-\infty}^{\infty}\left(u^4 - \frac{3}{4}uu_x^2 + \frac{1}{48}u_{xx}^2\right)\mathrm{d}x, \end{aligned} \tag{17}$$

2) For seventh-order Kawahara equation Equation (3), the conservative quantities $I_i\,(i = 1,2,3)$ can be written as

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} u\,\mathrm{d}x, \\ I_2 &= \int_{-\infty}^{\infty} u^2\,\mathrm{d}x, \\ I_3 &= \int_{-\infty}^{\infty}\left(-u^3 + \frac{1}{2}(u_x)^2 - \frac{1}{2}(u_{xx})^2 + \frac{1}{2}\alpha(u_{xxx})^2\right)\mathrm{d}x, \end{aligned} \tag{18}$$

Since the conservation constants are expected to remain constant during the run of the algorithm to have accurate numerical scheme, conservation constants will be monitored. As various problems of science were modeled by non linear partial differential equations and since therefore the seventh order KdV equation is of high importance, the following examples have been considered.

4. Numerical Examples

Example 1. Consider the seventh-order (KK) equation Equation (2) with initial condition

$$u(x,0)=\frac{1}{3}k^2-\frac{1}{2}k^2\tanh^2(kx),$$

By ADM the recursive relations are

$$\begin{cases} u_0=\frac{1}{3}k^2-\frac{1}{2}k^2\tanh^2(kx), \\ u_{n+1}=-2016L_t^{-1}(A_n)-630L_t^{-1}(B_n)-2268L_t^{-1}(C_n)-504L_t^{-1}(D_n) \\ \qquad -252L_t^{-1}(E_n)-147L_t^{-1}(F_n)-42L_t^{-1}(G_n)-L_t^{-1}(L_{7x}u_n), n\ge 0. \end{cases}$$

The first few components are thus determined as follows:

$$\begin{cases} u_0=\frac{1}{3}k^2-\frac{1}{2}k^2\tanh^2(kx), \\ u_1=\frac{-4k^9t\sinh(kx)}{3\cosh^3(kx)}, \\ u_2=\frac{8k^{16}t^2(2\cosh^2(kx)-3)}{9\cosh^4(kx)}, \end{cases}$$

and so on. Consequently, the solution in a series form is given by

$$u(x,t)=u_0+u_1+u_2+\cdots$$

and in a closed form $u(x,t)=\frac{1}{3}k^2-\frac{1}{2}k^2\tanh^2\left(k\left(x+\frac{4}{3}k^6t\right)\right)$.

The results produced by the proposed method with only few components (n = 5) are compared with the exact solution and listed in Table 1, also the Claws for the seventh-order (KK) equation are given in Table 2. The profile of the solitary wave at $t=0.3$ is displayed in Figure 1.

Table 1. Comparison between exact solution $u(x,t)$ and approximate solution using ADM where $k=0.1$.

x	Exact	ADM	Absolute Error
0.10	0.00333283	0.00333283	9.98865803e^{-19}
0.20	0.00333133	0.00333133	1.99693372e^{-18}
0.30	0.00332884	0.00332884	2.99340524e^{-18}
0.40	0.00332534	0.00332534	3.98748557e^{-18}
0.50	0.00332085	0.00332085	4.97838399e^{-18}

Table 2. Computed quantities I_1, I_2, I_3 for the seventh-order KK equation by ADM.

	I_1			I_2			I_3		
t/x	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
0.1	$3.333e^{-4}$	$9.996e^{-4}$	$1.665e^{-3}$	$3.703e^{-9}$	$1.108e^{-8}$	$1.840e^{-8}$	$3.316e^{-11}$	$9.910e^{-11}$	$1.639e^{-10}$
0.2	$3.333e^{-4}$	$9.996e^{-4}$	$1.665e^{-3}$	$3.703e^{-9}$	$1.108e^{-8}$	$1.840e^{-8}$	$3.316e^{-11}$	$9.910e^{-11}$	$1.639e^{-10}$
0.3	$3.333e^{-4}$	$9.996e^{-4}$	$1.665e^{-3}$	$3.703e^{-9}$	$1.108e^{-8}$	$1.840e^{-8}$	$3.316e^{-11}$	$9.910e^{-11}$	$1.639e^{-10}$
0.4	$3.333e^{-4}$	$9.996e^{-4}$	$1.665e^{-3}$	$3.703e^{-9}$	$1.108e^{-8}$	$1.840e^{-8}$	$3.316e^{-11}$	$9.910e^{-11}$	$1.639e^{-10}$
0.5	$3.333e^{-4}$	$9.996e^{-4}$	$1.665e^{-3}$	$3.703e^{-9}$	$1.108e^{-8}$	$1.840e^{-8}$	$3.316e^{-11}$	$9.910e^{-11}$	$1.639e^{-10}$

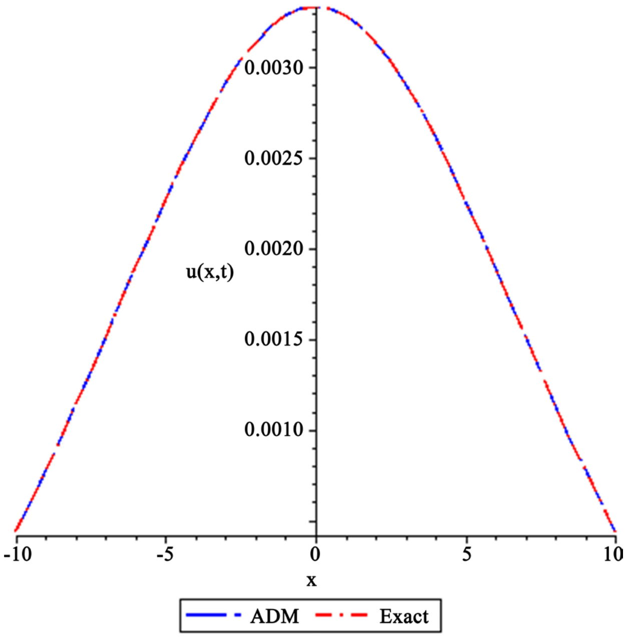

Figure 1. Comparison between exact solution $u(x,t)$ and approximate solution using ADM.

Example 2. Consider the seventh-order Kawahara equation Equation (3) with initial condition

$$u(x,0) = \omega \operatorname{sech}^6(kx),$$

By ADM the recursive relations are

$$\begin{cases} u_0 = \omega \operatorname{sech}^6(kx), \\ u_{n+1} = -6L_t^{-1}(A_n) - L_t^{-1}(L_{3x}u_n) + L_t^{-1}(L_{5x}u_n) - \alpha L_t^{-1}(L_{7x}u_n), n \geq 0. \end{cases}$$

The first few components are thus determined as follows:

$$\begin{cases} u_0 = \omega \operatorname{sech}^6(kx), \\ u_1 = \dfrac{1}{\cosh^{13}(kx)}\Big(12tk\omega \sinh(kx)\big(23328\alpha k^6 \cosh^6(kx) \\ \qquad -215488\alpha k^6 \cosh^4(kx) - 648k^4 \cosh^6(kx) + \cdots\big)\Big), \\ u_2 = \dfrac{1}{\cosh^{20}(kx)}\Big(12t^2k^2\omega\big(108783285811200\alpha^2 k^{12} \cosh^6(kx) \\ \qquad -175649727052800\alpha^2 k^{12} \cosh^4(kx) \\ \qquad +138322888704000\alpha^2 k^{12} \cosh^2(kx) + \cdots\big)\Big), \end{cases}$$

and so on. Consequently, the solution in a series form is given by

$$u(x,t) = u_0 + u_1 + u_2 + \cdots$$

and in a closed form $u(x,t) = \omega \operatorname{sech}^6(k(x - x_0 t))$.

The results produced by the proposed method with only few components (n = 5) are compared with the exact solution and listed in **Table 3**, also the Claws for the seventh-order Kawahara equation are given in **Table 4**. The profile of the solitary wave at $t = 0.3$ is displayed in **Figure 2**.

Table 3. Comparison between exact solution $u(x,t)$ and approximate solution using ADM where $\alpha = \frac{769}{2500}$, $\omega = \frac{86625}{591361}$, $k = \frac{5}{\sqrt{1538}}$ and $x_0 = \frac{180000}{591361}$.

x	Exact	ADM	Absolute Error
0.10	0.14648359	0.14648359	$1.24475423e^{-11}$
0.20	0.14639977	0.14639977	$2.70176889e^{-11}$
0.30	0.14617341	0.14617341	$4.13404041e^{-11}$
0.40	0.14580523	0.14580523	$5.52856164e^{-11}$
0.50	0.14529642	0.14529642	$6.87280812e^{-11}$

Table 4. Computed quantities I_1, I_2, I_3 for the seventh-order Kawahara equation by ADM.

	I_1			I_2			I_3		
t/x	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
0.1	$1.465e^{-2}$	$4.390e^{-2}$	$7.300e^{-2}$	$2.146e^{-3}$	$6.424e^{-3}$	$1.066e^{-2}$	$-3.245e^{-4}$	$-9.697e^{-4}$	$-1.602e^{-3}$
0.2	$1.465e^{-2}$	$4.391e^{-2}$	$7.304e^{-2}$	$2.146e^{-3}$	$6.428e^{-3}$	$1.067e^{-2}$	$-3.245e^{-4}$	$-9.708e^{-4}$	$-1.606e^{-3}$
0.3	$1.465e^{-2}$	$4.392e^{-2}$	$7.308e^{-2}$	$2.145e^{-3}$	$6.430e^{-3}$	$1.068e^{-2}$	$-3.244e^{-4}$	$-9.716e^{-4}$	$-1.609e^{-3}$
0.4	$1.464e^{-2}$	$4.393e^{-2}$	$7.311e^{-2}$	$2.145e^{-3}$	$6.432e^{-3}$	$1.069e^{-2}$	$-3.242e^{-4}$	$-9.721e^{-4}$	$-1.612e^{-3}$
0.5	$1.464e^{-2}$	$4.393e^{-2}$	$7.313e^{-2}$	$2.143e^{-3}$	$6.433e^{-3}$	$1.070e^{-2}$	$-3.239e^{-4}$	$-9.722e^{-4}$	$-1.614e^{-3}$

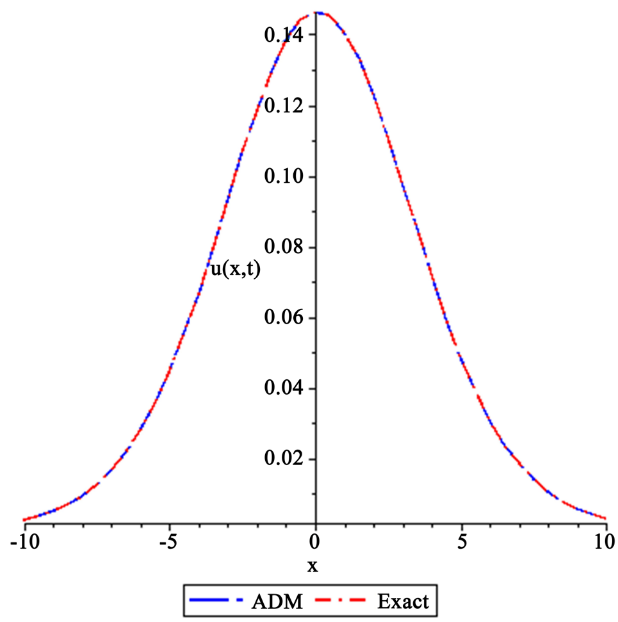

Figure 2. Comparison between exact solution $u(x,t)$ and approximate solution using ADM.

5. Conclusion

In this paper, the ADM was used to solving seventh order KdV equations with initial conditions. We have found out that this method is applicable and efficient technique. All the numerical results obtained by using ADM show very good agreement with the exact solutions for a few terms. The conservation laws are used to assess the accuracy and the efficiency of the method. We have noticed that the method accomplished the aim of preserving conserved quantities, as we saw all invariants were almost constant.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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