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Quasi-Coordinate Search for a Randomly Moving Target

A. A. M. Teamah[1], W. A. Afifi[2,3*]

[1]Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt
[2]Mathematics and Statistics Department, College of Science, Taibah University, Yanbu, KSA
[3]Mathematics and Statistics Department, Canal Higher Institute of Engineering and Technology, Suez, Egypt
Email: *Walaa_affy@yahoo.com

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Abstract

In this paper, we study the quasi-coordinated search technique for a lost target assumed to move randomly on one of two disjoint lines according to a random walk motion, where there are two searchers beginning their search from the origin on the first line and other two searchers begin their search from the origin on the second line. But the motion of the two searchers on the first line is independent from the motion of the other two searchers on the second line. Here we introduce a model of search plan and investigate the expected value of the first meeting time between one of the searchers and the lost target. Also, we prove the existence of a search plan which minimizes the expected value of the first meeting time between one of the searchers and the target.

Keywords

Random Walker, Linear Search, Expected Value, Optimal Search Plane, Stochastic Process

1. Introduction

The searching for a lost target either located or moved is often a time-critical issue, that is, when the target is very important. The primary objective is to find and search for the lost target as soon as possible. The searching for lost targets has recently applications such as the search for a goldmine underground, the search for Landmines and navy mines, the search for the cancer cells in the human body, the search for missing black box of a plane crash in the depth of the sea of ocean, the search for a damaged unit in a large linear system such as telephone lines, and mining system, and so on [1] [2] [3]. Search problem when the lost target is located or moved on the real line has been considered in [4]-[9]. The coordinated search technique discussed on the real line when the located target has symmetric or unsymmetric distribution as in [10] [11] [12]. Also, the coordinated search for a located target in the plane has been examined in [13] [14] [15] [16]. Recently, [17] and [18] proposed and studied a modern search model in the three-dimensional space to find a 3-D randomly located target by one searcher, two searchers and four searchers.

2. Problem Formulations

One of the most complicate problems when a mother loses her son in a way of multiple ways, here the primary objective is finding the lost son, as soon as possible in a minimum time. The survival rate of the son in this region gradually decreases, so the search team must organize itself quickly to begin the mission of the searching for the lost son immediately. Also, when the target is serious as a car, which filled by explosives, and it moves on one road from disjoint roads, and then the search effort must be unrestricted and we can use more than searcher to detect the target at right time.

The search team which consists of 4 searchers will organize itself on 2 straight lines to find the lost target as soon as possible. We clarify a modern technique by collaboration between each two searchers to find the lost person in minimum time. This problem can be characterized as follows.

2.1. The Searching Framework

The space of search: 2 disjoint lines.

The target: The target moves with a random walk motion on one of 2 disjoint straight lines.

The means of search: Looking for the lost target performed by two searches on each line. The searchers start searching for the target from the origins of the two lines with continuous paths and with equal speeds. In addition, the search spaces (2 straight lines) are separated into many distances.

2.2. The Searching Technique

Assume that we have two searchers S_1 and S_2 that start together looking for the lost target from O_1 on L_1. The two searchers coordinate their search about the lost target, where the searcher S_1 searches to the right and goes from the O_1 to H_1, and the searcher S_2 searches to the left and goes from O_1 to $-H_1$, the two searchers S_1 and S_2 reach to H_1 and $-H_1$ in the same time of G_1. Then they come back to O_1 again in the same time of G_2. If one of the two searchers do not find the lost target, then the two searchers S_1 and S_2 begin the new cycle search for the lost target, where they go from O_1 to H_2 and $-H_2$, respectively and they will reach to H_2 and $-H_2$ in the same time of G_3. Then they come back to O_1 again in the same time of G_4 and so on. Also, we have two other searchers S_3 and S_4 start together looking for the lost target from O_2 on the second line L_2, the searcher S_3

searchers to the right and goes from O_2 to $\bar{H}_1$, and the searchers S_4 searches to the left and goes to the left and goes from O_2 to $-\bar{H}_1$, the two searchers S_3 and S_4 reach to $\bar{H}_1$ and $-\bar{H}_1$ in the same time of $\bar{G}_1$. Then they come back to O_2 again in the same time of $\bar{G}_2$. If one of the two searchers not find the lost target, then the two searchers S_3 and S_4 begin the new cycle search for the lost target, where they go from O_2 to $\bar{H}_2$ and $-\bar{H}_2$, respectively and they will reach to $\bar{H}_2$ and $-\bar{H}_2$ in the same time of $\bar{G}_3$, then they come back to O_2 again in the same time of $\bar{G}_4$, and so on. The four searchers return to the O_1 and O_2 after searching successively common distances until the target is found.

2.3. The Movement of the Target and the Searchers

A target is assumed to move randomly on one of two disjoint lines according to a stochastic process $\{S(t), t \in I^+\}, I^+ = \{0,1,2,\cdots\}$. Assume that $\{Z_i\}_{i\geq 0}$ is a sequence of independent identically distributed random variables such as for any $i \geq 1$: $p(Z_i = 1) = p$ and $p(Z_i = -1) = 1 - p = q$, where $p, q > 0$. For $t > 0$, $t \in I^+$,

$$S(t) = \sum_{i=1}^{t} Z_i, S(0) = 0.$$

We assume the searchers S_1 and S_2 begin their search path from O_1 on L_1 with speeds V_1, and the searchers S_3 and S_4 begin their search path from O_2 on L_2 with speeds V_2, following the search paths which are functions $\phi_1 : R^+ \to R$ and $\bar{\phi}_1 : R^+ \to R$ on L_1 and $\phi_2 : R^+ \to R$ and $\bar{\phi}_2 : R^+ \to R$ on L_2, respectively, such that:

$$\left|\phi_1(t_1) - \phi_1(t_2)\right| = \left|\bar{\phi}_1(t_1) - \bar{\phi}_1(t_2)\right| \leq V_1 \left|t_1 - t_2\right|, \tag{1}$$

and

$$\left|\phi_2(t_1) - \phi_2(t_2)\right| = \left|\bar{\phi}_2(t_1) - \bar{\phi}_2(t_2)\right| \leq V_2 \left|t_1 - t_2\right|, \ \forall t_1, t_2 \in I^+, \tag{2}$$

where V_1 and V_2 are constants in R^+ and $\phi_1(0) = \bar{\phi}_1(0) = \phi_2(0) = \bar{\phi}_2(0) = 0$. Let the set of all search paths of the two searchers S_1 and S_2, which satisfy condition (1), be respectively by Φ_{v_1} and $\bar{\Phi}_{v_1}$ respectively and the set of all search paths of the searchers S_3 and S_4 which satisfy condition (2), be represented by Φ_{v_2} and $\bar{\Phi}_{v_2}$, respectively. we represented to the path of S_1 and S_2 by $\phi_0 = (\phi_1, \bar{\phi}_1) \in \Phi_0$ where $\bar{\phi}_0 = (\phi_2, \bar{\phi}_2) \in \bar{\Phi}_0$, where

$$\bar{\Phi}_0 = \left\{(\phi_2, \bar{\phi}_2) : \phi_2 \in \Phi_{v_2}, \bar{\phi}_2 \in \bar{\Phi}_{v_2}\right\}.$$

The search plan of the four searchers be represented by $\hat{\phi} = (\phi_0, \bar{\phi}_0) \in \hat{\Phi}$, where $\hat{\Phi} = \left\{(\phi_0, \bar{\phi}_0) : \phi_0 \in \Phi_0, \bar{\phi}_0 \in \bar{\Phi}_0\right\}$ is the set of all search plan.

We assume that $Z_0 = X$ if the target moves on L_1 and $Z_0 = Y$ if the target moves on L_2 such that $P(Z_0 = X) + P(Z_0 = Y) = 1$. There is a known probability measure $v_1 + v_2 = 1$ on $L_1 \cup L_2$ which describes the location of the target, where v_1 is probability measure induced by the position of the target on L_1, while v_2 on L_2. The first meeting time valued in I^+ defined as

$$\tau_{\hat{\phi}} = \inf\{t : \phi_1(t) = X + S(t) \text{ or } \bar{\phi}_1(t) = X + S(t) \text{ or } \phi_2(t) = Y + S(t) \text{ or } \bar{\phi}_2(t) = Y + S(t)\},$$

where Z_0 is a random variable representing the initial position of the target and valued in $2I$ (or $2I+1$) and independent of $S(t), t > 0$.

At the beginning of the search suppose that the lost target is existing on any integer point on L_1 but more than H_1 or less than $-H_1$ or the lost target is existing on an integer point on L_2 but more than $\bar{H}_1$ or less than $-\bar{H}_1$. Let τ_{ϕ_1} be the first meeting time between S_1 and the target and $\tau_{\hat{\phi}_1}$ be the first meeting time between S_2 and the target and τ_{ϕ_2} be the first meeting time between S_3 and the target and $\tau_{\hat{\phi}_2}$ be the first meeting time between S_4 and the target. The main objective is to find the search plan $\hat{\phi} = (\phi_0, \bar{\phi}_0) \in \hat{\Phi}$ such that $E(\tau_{\hat{\phi}}) < \infty$. In this case $\hat{\phi}$ is said to be a finite search plan, and if $E(\tau^*_{\hat{\phi}}) < E(\tau_{\hat{\phi}}), \forall \hat{\phi} \in \hat{\Phi}$, where E terms to expectation value, then we call $\hat{\phi}^*$ is an optimal search plan.

Given $n > 0$, if z is: $0 \le k_1 \le \frac{n+z}{2} \le n$, where k_1 is integer, then

$$p(S(n) = k_1) = \begin{cases} \binom{n}{k_1} p^{k_1} q^{n-k_1} \\ 0, \text{ if } k_1 \text{ does not exist} \end{cases}$$

2.4. Finite Search Plan

Let $\lambda_1, \lambda_2, \zeta_1, \zeta_2$ be positive integers such that $\zeta_1, \zeta_2 > 1$, $\lambda_1 = k\theta_1$, $\lambda_2 = k\theta_2$, where $k = 1, 2, \cdots$ and θ_1, θ_2 are the least positive integers and $V_1 = V_2 = 1$.

We shall define the sequences $\{G_i\}_{i\ge0}, \{H_i\}_{i\ge0}$ for the searcher S_1 on the first line L_1 and $\{\bar{G}_i\}_{i\ge0}, \{\bar{H}_i\}_{i\ge0}$ for the searcher S_3 on the second line L_2 and the search plans with speeds 1 as follows:

$$G_i = 2^{\frac{1}{2}\left[1-(-1)^{i-1}\right]} \lambda_1 \left(\zeta_1^{\frac{i}{2}+\frac{1}{4}-(-1)^i\frac{1}{4}} - 1 \right), \quad H_i = G_{2i-1},\ i \ge 1 \text{ on } L_1,$$

$$\bar{G}_i = 2^{\frac{1}{2}\left[1-(-1)^{i-1}\right]} \lambda_2 \left(\zeta_2^{\frac{i}{2}+\frac{1}{4}-(-1)^i\frac{1}{4}} - 1 \right), \quad \bar{H}_i = \bar{G}_{2i-1},\ i \ge 1 \text{ on } L_2.$$

We shall define the search path as follows:

for any $t \in I^+$, if $G_i \le t < G_{i+1}$, then

$$\phi_1(t) = \left(\frac{1}{2} H_{\frac{i+1}{2}}\right) + (-1)^{i+1}\left(\frac{1}{2} H_{\frac{i+1}{2}}\right) + (-1)^i (t - G_i),$$

and

$$\bar{\phi}_1(t) = -\phi_1(t).$$

Also, if $\bar{G}_i \le t < \bar{G}_{i+1}$, then

$$\phi_2(t) = \left(\frac{1}{2} \bar{H}_{\frac{i+1}{2}}\right) + (-1)^{i+1}\left(\frac{1}{2} \bar{H}_{\frac{i+1}{2}}\right) + (-1)^i (t - \bar{G}_i),$$

and

$$\bar{\phi}_2(t) = -\phi_2(t).$$

We define the notion

$\varphi_1(t)=S(t)-t,\ \tilde{\varphi}_1(t)=S(t)+t$ on L_1,

$\varphi_2(t)=S(t)-t,\ \tilde{\varphi}_2(t)=S(t)+t$ on L_2,

the searchers S_1 and S_2 return to the origin of L_1 after searching successively common distances $H_1,H_2,H_3,\cdots$, and $-H_1,-H_2,-H_3,\cdots$, respectively and the searchers S_3 and S_4 return to the origin of L_2 after searching successively common distances $\bar{H}_1,\bar{H}_2,\bar{H}_3,\cdots$, and $-\bar{H}_1,-\bar{H}_2,-\bar{H}_3,\cdots$, respectively until the target is found.

Theorem 1: If $\hat{\phi}=(\phi_0,\bar{\phi}_0)\in\hat{\Phi}$ is a search plan defined above, then the expectation $E(\tau_{\hat{\phi}})$ if finite if

$$w_1(x)=\sum_{i=1}^{\infty}(\zeta_1^i-1)p(\tilde{\varphi}_1(G_{2i-1})<-x),$$

$$w_2(x)=\sum_{i=1}^{\infty}(\zeta_1^i-1)p(\varphi_1(G_{2i-1})>-x),$$

$$w_3(x)=\sum_{i=1}^{\infty}(\zeta_1^i(\zeta_1^i-2)+1)p(\tilde{\varphi}_1(G_{2i})<-x),$$

$$w_4(y)=\sum_{i=1}^{\infty}(\zeta_1^i(\zeta_1^i-2)+1)p(\varphi_1(G_{2i})>-x),$$

$$w_5(y)=\sum_{i=1}^{\infty}(\zeta_2^i-1)p(\tilde{\varphi}_2(G_{2i-1})<-y),$$

$$w_6(y)=\sum_{i=1}^{\infty}(\zeta_2^i-1)p(\varphi_2(\tilde{G}_{2i-1})>-y),$$

$$w_7(y)=\sum_{i=1}^{\infty}(\zeta_2^i(\zeta_2^i-2)+1)p(\tilde{\varphi}_2(\tilde{G}_{2i})<-y),$$

and

$$w_8(y)=\sum_{i=1}^{\infty}(\zeta_2^i(\zeta_2^i-2)+1)p(\varphi_2(\tilde{G}_{2i})>-y). \tag{3}$$

are finite.

Proof: Assume that X and Y are independent of $S(t),t>0$, if $X>0$, then $X+S(t)>\phi_1(t)$ until the first meeting between S_1 and the target on L_1, also if $X<0$, then $X+S(t)<\hat{\phi}_1(t)$ until the first meeting between S_2 and the target on L_2. We can apply this assumption on the second line by replacing X by Y and $\phi_1,\bar{\phi}_1$ by $\phi_2,\bar{\phi}_2$ respectively. Hence, for any $i\ge 0$

$$p(\tau_{\hat{\phi}}>t)=p(\tau_{\phi_0}>t \text{ or } \tau_{\hat{\phi}_0}>t),$$

hence

$$\begin{aligned}E(\tau_{\hat{\phi}})&=\int_0^{\infty}p(\tau_{\hat{\phi}}>t)\mathrm{d}t\\ &\le\sum_{i=0}^{\infty}\left[\int_{G_i}^{G_{i+1}}p(\tau_{\phi_0}>G_i)\mathrm{d}t+\int_{\tilde{G}_i}^{\tilde{G}_{i+1}}p(\tau_{\hat{\phi}_0}>\tilde{G}_i)\mathrm{d}t\right]\\ &=\sum_{i=0}^{\infty}\left(2^{\frac{1}{2}\left[1-(-1)^{i+2}\right]}\lambda_1\left(\zeta_1^{\frac{i+1}{2}+\frac{1}{4}-(-1)^{i+1}\frac{1}{4}}-1\right)\right.\end{aligned}$$

$$
\begin{aligned}
&-2^{\frac{1}{2}\left[1-(-1)^{i+1}\right]}\lambda_1\left(\zeta_1^{\frac{i}{2}+\frac{1}{4}-(-1)^i\frac{1}{4}}-1\right)\Bigg)p\left(\tau_{\hat{\phi}_0}>G_i\right)\\
&+\Bigg(2^{\frac{1}{2}\left[1-(-1)^{i+2}\right]}\lambda_2\left(\zeta_2^{\frac{i+1}{2}+\frac{1}{4}-(-1)^{i+1}\frac{1}{4}}-1\right)\\
&-2^{\frac{1}{2}\left[1-(-1)^{i+1}\right]}\lambda_2\left(\zeta_2^{\frac{i}{2}+\frac{1}{4}-(-1)^i\frac{1}{4}}-1\right)\Bigg)p\left(\tau_{\hat{\phi}_0}>\tilde{G}_i\right)\\
&=\lambda_1\Big[\left(\left(\zeta_1-2\right)+1\right)p\left(\tau_{\phi_0}>0\right)+\left(\zeta_1-1\right)p\left(\tau_{\phi_0}>G_1\right)\\
&+\left(\zeta_1\left(\zeta_1-2\right)+1\right)p\left(\tau_{\phi_0}>G_2\right)+\left(\zeta_1^2-1\right)p\left(\tau_{\phi_0}>G_3\right)\\
&+\left(\zeta_1^2\left(\zeta_1-2\right)+1\right)p\left(\tau_{\phi_0}>G_4\right)+\left(\zeta_1^3-1\right)p\left(\tau_{\phi_0}>G_5\right)\\
&+\left(\zeta_1^3\left(\zeta_1-2\right)+1\right)p\left(\tau_{\phi_0}>G_6\right)+\cdots\Big]\\
&+\lambda_2\Big[\left(\left(\zeta_2-2\right)+1\right)p\left(\tau_{\bar{\phi}_0}>0\right)+\left(\zeta_2-1\right)p\left(\tau_{\bar{\phi}_0}>\bar{G}_1\right)\\
&+\left(\zeta_2\left(\zeta_2-2\right)+1\right)p\left(\tau_{\bar{\phi}_0}>\bar{G}_2\right)+\left(\zeta_2^2-1\right)p\left(\tau_{\bar{\phi}_0}>\bar{G}_3\right)\\
&+\left(\zeta_2^2\left(\zeta_2-2\right)+1\right)p\left(\tau_{\bar{\phi}_0}>\bar{G}_4\right)+\left(\zeta_2^3-1\right)p\left(\tau_{\bar{\phi}_0}>\bar{G}_5\right)\\
&+\left(\zeta_2^3\left(\zeta_2-2\right)+1\right)p\left(\tau_{\bar{\phi}_0}>\bar{G}_6\right)+\cdots\Big]
\end{aligned} \tag{4}
$$

to solve Equation (4) we shall find the value of $p\left(\tau_{\phi_0}>G_{2i-1}\right)$, $p\left(\tau_{\bar{\phi}_0}>\bar{G}_{2i-1}\right)$, $p\left(\tau_{\phi_0}>G_{2i}\right)$ and the value of $p\left(\tau_{\bar{\phi}_0}>\bar{G}_{2i}\right)$ as the following

$$
\begin{aligned}
p\left(\tau_{\phi_0}>G_{2i-1}\right)\le&\int_{-\infty}^{0}p\left(x+S\left(G_{2i-1}\right)<-H_i/X=x\right)v_1\left(\mathrm{d}x\right)\\
&+\int_{0}^{\infty}p\left(x+S\left(G_{2i-1}\right)>H_i/X=x\right)v_1\left(\mathrm{d}x\right)
\end{aligned}
$$

We get

$$
\begin{aligned}
p\left(\tau_{\phi_0}>G_{2i-1}\right)\le&\int_{-\infty}^{0}p\left(\tilde{\varphi}_1\left(G_{2i-1}\right)<-x\right)v_1\left(\mathrm{d}x\right)\\
&+\int_{0}^{\infty}p\left(\varphi_1\left(G_{2i-1}\right)>-x\right)v_1\left(\mathrm{d}x\right)
\end{aligned} \tag{5}
$$

also,

$$
\begin{aligned}
p\left(\tau_{\bar{\phi}_0}>\bar{G}_{2i-1}\right)\le&\int_{-\infty}^{0}p\left(\tilde{\varphi}_2\left(\bar{G}_{2i-1}\right)<-y\right)v_2\left(\mathrm{d}y\right)\\
&+\int_{0}^{\infty}p\left(\varphi_2\left(\bar{G}_{2i-1}\right)>-y\right)v_2\left(\mathrm{d}y\right)
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
p\left(\tau_{\phi_0}>G_{2i}\right)\le&\int_{-\infty}^{0}p\left(X+S\left(G_{2i}\right)<-2H_i\right)v_1\left(\mathrm{d}x\right)\\
&+\int_{0}^{\infty}p\left(x+S\left(G_{2i}\right)>2H_i\right)v_1\left(\mathrm{d}x\right)
\end{aligned} \tag{7}
$$

We get

$$
p\left(\tau_{\phi_0}>G_{2i}\right)\le\int_{-\infty}^{0}p\left(\tilde{\varphi}_1\left(G_{2i}\right)<-x\right)v_1\left(\mathrm{d}x\right)+\int_{0}^{\infty}p\left(\varphi_1\left(G_{2i}\right)>-x\right)v_1\left(\mathrm{d}x\right) \tag{8}
$$

$$p\left(\tau_{\phi_0} > \bar{G}_{2i}\right) \le \int_{-\infty}^{0} p\left(\tilde{\varphi}_1\left(\bar{G}_{2i}\right) < -y\right) v_2\left(\mathrm{d}y\right) + \int_{0}^{\infty} p\left(\varphi_1\left(\bar{G}_{2i}\right) > -y\right) v_2\left(\mathrm{d}y\right) \tag{9}$$

substituting by (5), (6), (7) and (8) in (4) we can get

$$\begin{aligned} E\left(\tau_{\hat{\phi}}\right) \le \lambda_1 \Big[& \left(\left(\zeta_1 - 2\right)+1\right) p\left(\tau_{\phi_0} > 0\right) + \left(\zeta_1 - 1\right) p\left(\tau_{\phi_0} > G_1\right) \\ & + \left(\zeta_1\left(\zeta_1 - 2\right)+1\right) p\left(\tau_{\phi_0} > G_2\right) + \left(\zeta_1^2 - 1\right) p\left(\tau_{\phi_0} > G_3\right) \\ & + \left(\zeta_1^2\left(\zeta_1 - 2\right)+1\right) p\left(\tau_{\phi_0} > G_4\right) + \left(\zeta_1^3 - 1\right) p\left(\tau_{\phi_0} > G_5\right) \\ & + \left(\zeta_1^3\left(\zeta_1 - 2\right)+1\right) p\left(\tau_{\phi_0} > G_6\right) + \cdots \Big] \\ + \lambda_2 \Big[& \left(\left(\zeta_2 - 2\right)+1\right) p\left(\tau_{\bar{\phi}_0} > 0\right) + \left(\zeta_2 - 1\right) p\left(\tau_{\bar{\phi}_0} > \bar{G}_1\right) \\ & + \left(\zeta_2\left(\zeta_2 - 2\right)+1\right) p\left(\tau_{\bar{\phi}_0} > \bar{G}_2\right) + \left(\zeta_2^2 - 1\right) p\left(\tau_{\bar{\phi}_0} > \bar{G}_3\right) \\ & + \left(\zeta_2^2\left(\zeta_2 - 2\right)+1\right) p\left(\tau_{\bar{\phi}_0} > \bar{G}_4\right) + \left(\zeta_2^3 - 1\right) p\left(\tau_{\bar{\phi}_0} > \bar{G}_5\right) \\ & + \left(\zeta_2^3\left(\zeta_2 - 2\right)+1\right) p\left(\tau_{\bar{\phi}_0} > \bar{G}_6\right) + \cdots \Big] \end{aligned}$$

hence

$$\begin{aligned} E\left(\tau_{\hat{\phi}}\right) \le \lambda_1 \Bigg[& \left(\left(\zeta_1 - 2\right)+1\right) p\left(\tau_{\phi_0} > 0\right) + \left\{ \int_{-\infty}^{0} w_1\left(x\right) v_1\left(\mathrm{d}x\right) \right. \\ & \left. + \int_{0}^{\infty} w_2\left(x\right) v_1\left(\mathrm{d}x\right) \right\} + \left\{ \int_{-\infty}^{0} w_3\left(y\right) v_2\left(\mathrm{d}y\right) + \int_{0}^{\infty} w_4\left(y\right) v_2\left(\mathrm{d}y\right) \right\} \Bigg] \\ + \lambda_2 \Bigg[& \left(\left(\zeta_2 - 2\right)+1\right) p\left(\tau_{\bar{\phi}_0} > 0\right) + \left\{ \int_{-\infty}^{0} w_5\left(y\right) v_2\left(\mathrm{d}y\right) \right. \\ & \left. + \int_{0}^{\infty} w_6\left(y\right) v_2\left(\mathrm{d}y\right) \right\} + \left\{ \int_{-\infty}^{0} w_7\left(y\right) v_2\left(\mathrm{d}y\right) + \int_{0}^{\infty} w_8\left(y\right) v_2\left(\mathrm{d}y\right) \right\} \Bigg] \end{aligned}$$

where,

$$w_1\left(x\right) = \sum_{i=1}^{\infty} \left(\zeta_1^i - 1\right) p\left(\tilde{\varphi}_1\left(G_{2i-1}\right) < -x\right),$$

$$w_2\left(x\right) = \sum_{i=1}^{\infty} \left(\zeta_1^i - 1\right) p\left(\varphi_1\left(G_{2i-1}\right) > -x\right),$$

$$w_3\left(x\right) = \sum_{i=1}^{\infty} \left(\zeta_1^i\left(\zeta_1 - 2\right)+1\right) p\left(\tilde{\varphi}_1\left(G_{2i}\right) < -x\right),$$

$$w_4\left(x\right) = \sum_{i=1}^{\infty} \left(\zeta_1^i\left(\zeta_1 - 2\right)+1\right) p\left(\varphi_1\left(G_{2i}\right) > -x\right),$$

$$w_5\left(y\right) = \sum_{i=1}^{\infty} \left(\zeta_2^i - 1\right) p\left(\tilde{\varphi}_2\left(\bar{G}_{2i-1}\right) < -y\right),$$

$$w_6\left(y\right) = \sum_{i=1}^{\infty} \left(\zeta_2^i - 1\right) p\left(\varphi_2\left(\bar{G}_{2i-1}\right) > -y\right),$$

$$w_7\left(y\right) = \sum_{i=1}^{\infty} \left(\zeta_2^i\left(\zeta_2 - 2\right)+1\right) p\left(\tilde{\varphi}_2\left(\bar{G}_{2i}\right) < -y\right),$$

and

$$w_8(y)=\sum_{i=1}^{\infty}\left(\zeta_2^i(\zeta_2-2)+1\right)p\left(\varphi_2\left(\bar{G}_{2i}\right)>-y\right).$$

Lemma 1: For any $k\ge 0$, let $a_n\ge 0$ for $n\ge 0$, and $a_{n+1}\le a_n$. Let $\{d_n\}_{n\ge 0}$ be a strictly increasing sequence of integers with $d_0=0$,

$$\sum_{n=k}^{\infty}(d_{n+1}-d_n)a_{d_{n+1}}\le\sum_{k=d_k}^{\infty}a_k\le\sum_{n=k}^{\infty}(d_{n+1}-d_n)a_{d_n},$$

For more details see [1].

Theorem 2: The chosen search plan satisfies

$$w_1(x)\le w_9(|x|),\ w_2(x)\le w_{10}(|x|),$$

$$w_2(x)\le w_{11}(|x|),\ w_4(x)\le w_{12}(|x|),$$

$$w_5(y)\le w_{13}(|y|),\ w_6(y)\le w_{14}(|y|),$$

$$w_7(y)\le w_{15}(|y|),\text{ and }w_8(y)\le w_{16}(|y|),$$

where, $w_9(|x|)$, $w_{10}(|x|)$, $w_{11}(|x|)$, $w_{12}(|x|)$, $w_{13}(|y|)$, $w_{14}(|y|)$, $w_{15}(|y|)$, and $w_{16}(|y|)$ are linear function.

Proof: This theorem will prove for $w_2(x)$ and $w_6(y)$, and by similar way we can prove the other cases

$$w_2(x)=\sum_{i=0}^{\infty}\left(\zeta_1^i-1\right)p\left(\varphi_1(G_{2i-1})>-x\right)$$

and

$$w_6(y)=\sum_{i=0}^{\infty}\left(\zeta_2^i-1\right)p\left(\varphi_2\left(\bar{G}_{2i-1}\right)>-y\right)$$

1) if $x\le 0$, then

$$w_2(x)\le w_2(0)$$

and if $y\le 0$, then

$$w_6(y)\le w_6(0),$$

2) if $x>0$, then

$$w_2(x)=w_2(0)+\sum_{i=0}^{\infty}\left(\zeta_1^i-1\right)p\left(-x<\varphi_1(G_{2i-1})\le 0\right),$$

and if $y>0$, then

$$w_6(y)=w_6(0)+\sum_{i=0}^{\infty}\left(\zeta_2^i-1\right)p\left(-y<\varphi_2\left(\bar{G}_{2i-1}\right)\le 0\right),$$

from Theorem (2), see (Mohamed [1]) we obtain

$$w_2(0)=\sum_{i=0}^{\infty}\left(\zeta_1^i-1\right)p\left(\varphi_1(G_{2i-1})>0\right)\le\sum_{i=1}^{\infty}\left(\zeta_1^i-1\right)\varepsilon^{G_{2i-1}},\ 0<\varepsilon<1$$

and

$$w_6(0)=\sum_{i=0}^{\infty}\left(\zeta_2^i-1\right)p\left(\varphi_2\left(\bar{G}_{2i-1}\right)>0\right)\le\sum_{i=1}^{\infty}\left(\zeta_2^i-1\right)\varepsilon^{\bar{G}_{2i-1}},\ 0<\varepsilon<1$$

Let us define the following

1) $V(n)=\varphi_1(n\theta_1)/2=\sum_{i=1}^{n}W_i$, where $\{W_i\}$ is a sequence of (i. i. d. r. v.) $\bar{V}(n)=\varphi_2(n\theta_2)/2=\sum_{i=1}^{n}\bar{W}_i$, where $\{\bar{W}_i\}$ is a sequence of (i. i. d. r. v.).

2) $d_n=G_{2n-1}/\theta_1=k(\zeta_1^n-1)$, $\bar{d}_n=\bar{G}_{2n-1}/\theta_2=k(\zeta_2^n-1)$.

3) $a(n)=\frac{n}{n+k}p(-x/2<V(n)\le 0)=\sum_{i=0}^{|x|/2}p[-(j+1)<V(n)\le(-j)]$,

$\bar{a}(n)=\frac{n}{n+k}p(-y/2<\bar{V}(n)\le 0)=\sum_{i=0}^{|y|/2}p[-(j+1)<\bar{V}(n)\le(-j)]$,

4) m_1 is an integer such that $dm_1=b_1|x|+b_2$, and m_2 is an integer such that $dm_2=\bar{b}_1|y|+\bar{b}_2$,

5) $\alpha_1=\frac{\zeta_1}{(\zeta_1-1)k}$, and $\alpha_2=\frac{\zeta_2}{(\zeta_2-1)k}$,

and

6) $U_1(j,j+1)=\sum_{n=0}^{\infty}p[-(j+1)<V(n)<(-j)]$,

$\bar{U}_1(j,j+1)=\sum_{n=0}^{\infty}p[-(j+1)<\bar{V}(n)\le(-j)]$,

then $U_1(j,j+1)$ and $\bar{U}_1(j,j+1)$ satisfies the condition of the renewal equation, for more details see [19].

If $n>d_{m_1}$ and $n>d_{m_2}$ then by Theorem (2) see (Mohamed [1]) $a(n)$ and $\bar{a}(n)$ are non increasing and we can apply Lemma (2) to obtain

$$\begin{aligned}
w_2(x)-w_2(0)&=\sum_{i=1}^{\infty}(\zeta_1^i-1)p(-x<\varphi_1(G_{2i-1})\le 0)\\
&=\sum_{n=1}^{n_1}\zeta_1^n a(d_n)+\sum_{n=n_1+1}^{\infty}\zeta_1^n a(d_n)\\
&\le\sum_{n=1}^{n_1}\zeta_1^n+\alpha_1\sum_{n=n_1+1}^{\infty}(d_n-d_{n-1})a(d_n)\\
&\le\sum_{n=1}^{n_1}\zeta_1^n+\alpha_1\sum_{n=d_{m_1}}^{\infty}a(n)\\
&\le\sum_{n=1}^{n_1}\zeta_1^n+\alpha_1\sum_{n=d_{m_1}}^{\infty}\sum_{i=0}^{|x|/2}p[-(j+1)<V(n)\le(-j)]\\
&\le\sum_{n=1}^{n_1}\zeta_1^n+\alpha_1\sum_{j=0}^{|x|/2}U_1(j,j+1)
\end{aligned}$$

and

$$\begin{aligned}
w_6(x)-w_6(0)&=\sum_{i=0}^{\infty}(\zeta_2^i-1)p(-y<\varphi_2(G_{2i-1})\le 0)\\
&=\sum_{n=1}^{n_2}\zeta_2^n a(\bar{d}_n)+\sum_{n=n_2+1}^{\infty}\zeta_2^n a(\bar{d}_n)\\
&\le\sum_{n=1}^{n_2}\zeta_2^n+\alpha_2\sum_{n=n_2+1}^{\infty}(\bar{d}_n-\bar{d}_{n-1})a(\bar{d}_n)
\end{aligned}$$

$$\leq \sum_{n=1}^{n_2} \zeta_2^n + \alpha_2 \sum_{n=d_{m_2}}^{\infty} \bar{a}(n)$$

$$\leq \sum_{n=1}^{n_2} \zeta_2^n + \alpha_2 \sum_{j=0}^{|y|/2} \bar{U}_1(j, j+1)$$

Since $U_1(j, j+1)$ and $\bar{U}_1(j, j+1)$ satisfied the condition of the renewal equation, hence $U_1(j, j+1)$ and $\bar{U}_1(j, j+1)$ is bounded for all j by a constant, so

$$w_2(x) \leq w_2(0) + N_1 + N_2|x| = w_{10}(|x|),$$

and

$$w_6(x) \leq w_6(0) + \bar{N}_1 + \bar{N}_2|x| = w_{14}(|y|).$$

Theorem 3: If $\hat{\phi} = (\phi_0, \bar{\phi}_0) \in \hat{\Phi}$ is a finite search plan, then $E|Z_0|$ is finite.

Proof: If $E(\tau_{\hat{\phi}}) < \infty$, then $p(\tau_{\hat{\phi}} \text{ is finite}) = 1$ and so

$$p(\tau_{\phi_0} \text{ is finite}) + p(\tau_{\hat{\phi}_0} \text{ is finite}) = 1,$$

then, we conclude that

$$p(\tau_{\phi_0} \text{ is finite}) = 1 \text{ and } p(\tau_{\bar{\phi}_0} \text{ is finite}) = 0,$$

or

$$p(\tau_{\phi_0} \text{ is finite}) = 0 \text{ and } p(\tau_{\bar{\phi}_0} \text{ is finite}) = 1.$$

On the first line L_1 if $p(\tau_{\phi_0} \text{ is finite}) = 1$, then $X_0 = \phi(\tau_{\phi_0}) - S(\tau_{\phi_0})$ with probability one and hence

$$E|X_0| \leq E(\tau_{\phi_0}) + E|S(\tau_{\phi_0})|.$$

If $E(\tau_{\phi_0}) < \infty$, but $|S(\tau_{\phi_0})| \leq \tau_{\phi_0}$, then $E|S(\tau_{\phi_0})| \leq E(\tau_{\phi_0})$ and $E|X_0| < \infty$.

On the second line L_2 if $p(\tau_{\bar{\phi}_0} \text{ is finite}) = 1$, then $Y_0 = \phi(\tau_{\bar{\phi}_0}) - S(\tau_{\bar{\phi}_0})$ with probability one, by the same way we can get $E|Y_0|$ is finite on the second line L_2.

3. Existence of an Optimal Search Plan

Theorem 4: Let for any $t \in I^+$, let $S(t)$ be a process. The mapping $\hat{\phi} \to E(\tau_{\hat{\phi}}) \in R^+$ is lower semi-continuous on $\hat{\Phi}(t)$.

Proof: Let $I(\hat{\phi}, t)$ be the indicator function of the set $\{\tau_{\hat{\phi}} \geq t\}$ by the Fatou Lebesque theorem see (Stone [16]) we get

$$E(\tau_{\hat{\phi}}) = E\left[\sum_{t=1}^{\infty} I(\hat{\phi}, t)\right]$$
$$= E\left[\sum_{t=1}^{\infty} \liminf_{i\to\infty} I(\hat{\phi}_n, t)\right] \leq \liminf_{i\to\infty} E(\tau_{\hat{\phi}}),$$

for any sequence $\hat{\phi}_n \to \hat{\phi}$ in $\hat{\Phi}(t)$ is sequentially compact [20], thus the mapping $\hat{\phi} \to E(\tau_{\hat{\phi}})$ is lower semi continuous on $\hat{\Phi}(t)$, then this mapping attains its minimum.

4. Conclusions

We have described a new kind of search technique to find a lost moving target on one of two disjoint lines. The motion of the four searchers on the two lines in the quasi-coordinated search technique is independent, and this helps us to find the lost target without waste of time and cost, especially if this target is valuable as the search for lost children. Actually we calculated the finite search plan. Also; we proved the existence of an optimal search plan which minimizes the expected value of the first meeting time between one of the searchers and the target.

In the future work, we will introduce an important search problem, looking for a randomly moving target as a general case and the searchers will begin their mission from any point on the line.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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