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The Aleksandrov Problem in Non-Archimedean 2-Fuzzy 2-Normed Spaces

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Abstract

We introduce the definition of non-Archimedean 2-fuzzy 2-normed spaces and the concept of isometry which is appropriate to represent the notion of area preserving mapping in the spaces above. And then we can get isometry when a mapping satisfies AOPP and (*) (in article) by applying the Benz's theorem about the Aleksandrov problem in non-Archimedean 2-fuzzy 2-normed spaces.

Keywords

Non-Archimedean 2-Fuzzy 2-Normed Space, Isometry, Benz's Theorem

1. Introduction

Let X, Y be two metric spaces. For a mapping $f: X \to Y$, for all $x_1, x_2 \in X$, if f satisfies,

$$d_Y\left(f\left(x_1\right), f\left(x_2\right)\right) = d_X\left(x_1, x_2\right)$$

where $d_X(\cdot,\cdot), d_Y(\cdot,\cdot)$ denote the metrics in the spaces X, Y, then f is called an isometry. It means that for some fixed number $p > 0$, assume that f preserves distance p; *i.e.*, for all x_1, x_2 in X, if $d_X\left(x_1, x_2\right) = p$, we can get $d_Y\left(f\left(x_1\right), f\left(x_2\right)\right) = p$. Then we say p is a conservative distance for the mapping f. Whether there exists a single conservative distance for some f such that f is an isometry from X to Y, is the basic issue of conservative distances. It is called the Aleksandrov problem.

Theorem 1.1. ([1]) *Let* X, Y *be two real normed linear spaces* (*or NLS*) *with* $\dim X > 1$, $\dim Y > 1$ *and Y is strictly convex, assume that a fixed real number* $p > 0$ *and that a fixed integer* $N > 1$. *Finally, if* $f: X \to Y$ *is a mapping satisfies*

1) $\|x_1 - x_2\| = p \Rightarrow \|f(x_1) - f(x_2)\| \le p$

2) $\|x_1 - x_2\| = N \cdot p \Rightarrow \|f(x_1) - f(x_2)\| \ge N \cdot p$

for all $x_1, x_2 \in X$. Then *f* is an affine isometry. we can call Benz's theorem.

We can see some results about the Aleksandrov problem in different spaces in [2]-[10]. A natural question is that: Whether the Aleksandrov problem can be proved in non-Archimedean 2-fuzzy 2-normed spaces under some conditions. So in this article, we will give the definition of non-Archimedean 2-fuzzy 2-normed spaces according to [11] [12] [13] [14], then by applying the Benz's theorem to fix the value of p and N to solve problems.

If a function from a field K to $[0,\infty)$ satisfies

(T_1) $|a| \ge 0, |a| = 0 \Leftrightarrow a = 0$;

(T_2) $|ab| = |a||b|$;

(T_3) $|a+b| \le \max\{|a|,|b|\}$.

for all $a, b \in K$, then the field K is called a non-Archimedean field.

We can know $|-1| = |1| = 1$, $|a| \le 1$ for all $a \in N$ from the above definition. An example of a non-Archimedean valuation (or NAV) is the function $|\cdot|$ taking $|0| = 0$ and others into 1.

In 1897, Hence in [15] found that *p*-adic numbers play a vital role in the complex analysis, the norm derived from *p*-adic numbers is the non-Archimedean norm, the analysis of the non-Archimedean has important applications in physics.

Definition 1.2. *Let X be a vector space and dim* $X \ge 2$. *A function* $\|\cdot,\cdot\| : X \to [0,\infty)$ *is called non-Archimedean* 2*-norm, if and only if it satisfies*

(T_1) $\|x_1, x_2\| \ge 0$, $\|x_1, x_2\| = 0$ iff x_1, x_2 are linearly dependent;

(T_2) $\|x_1, x_2\| = \|x_2, x_1\|$;

(T_3) $\|rx_1, x_2\| = |r|\|x_1, x_2\|$;

(T_4) $\|x_1 + x_2, y\| \le \max\{\|x_1, y\|, \|x_2, y\|\}$

for all $x_1, x_2, y \in X, r \in K$. Then $(X, \|\cdot,\cdot\|)$ is called non-Archimedean 2-normed space over the field *K*.

Definition 1.3. *An NAV* $|\cdot|$ *in a linear space X over a field K. A function* $F : X \times \mathbb{R} \to [0,1]$ *is said to be a non-Archimedean fuzzy norm on X, if and only if for all* $x, x_1, x_2 \in X$ *and* $s, t \in \mathbb{R}$,

(F1) $F(x,s) = 0$ with $s \le 0$,

(F2) $F(x,s) = 1$ iff $x = 0$ for all $s > 0$,

(F3) $F(cx,s) = F\left(x, \frac{s}{|c|}\right)$, for $c \ne 0$ and $c \in K$,

(F4) $F(x_1 + x_2, s + t) \ge \min\{F(x_1, s), F(x_2, t)\}$,

(F5) $F(x,*)$ is a nondecreasing function of $s \in R$ and $\lim_{s\to\infty} F(x,s) = 1$.

Then (X, F) is known as a non-Archimedean fuzzy normed space (or F-NANS).

Theorem 1.4. *Let* (X, F) *be an F-NANS. Assume the condition that:*

(F6) $F(x,s) > 0$ for all $s > 0 \Rightarrow x = 0$.

Define $\|x\|_\alpha = \inf\{s : F(x,s) \ge \alpha\}, \alpha \in (0,1)$. We call these α-norms on X or the fuzzy norm on X.

Proof: 1) Let $\|x\|_\alpha = 0$, it implies that $\inf\{s : F(x,s) \ge \alpha\} = 0$, then for all $s \in R$, $s > 0$, $F(x,s) \ge \alpha > 0$, so $x = 0$;

Conversely, assume that $x = 0$, by (F2), $F(x,s) = 1$ for all $s > 0$, then $\inf\{s : F(x,s) \ge \alpha\} = 0$ for all $\alpha \in (0,1)$, so $\|x\|_\alpha = 0$.

2) By (F3), if $c \ne 0$, then

$$\|cx\|_\alpha = \inf\{s : F(cx,s) \ge \alpha\} = \inf\left\{s : F\left(x, \frac{s}{|c|}\right) \ge \alpha\right\}$$

Let $t = \frac{s}{|c|}$, then

$$\|cx\|_\alpha = \inf\{|c|t : F(x,t) \ge \alpha\} = |c| \inf\{t : F(x,t) \ge \alpha\} = |c| \cdot \|x\|_\alpha$$

If $c = 0$, then

$$\|cx\|_\alpha = 0 = c\|x\|_\alpha$$

3) We have

$$\begin{aligned}
&\max\{\|x\|_\alpha, \|y\|_\alpha\} \\
&= \max\{\inf\{s, F(x,s) \ge \alpha\}, \inf\{t, F(x,t) \ge \alpha\}\} \\
&= \inf\{\max\{s,t\}, F(x,s) \ge \alpha, F(x,t) \ge \alpha\} \\
&\ge \inf\{s+t, F(x+y, s+t) \ge F(x+y, \max\{s,t\}) \\
&\qquad \ge \min\{F(x,s) \ge \alpha, F(x,t) \ge \alpha\} \ge \alpha\} \\
&\ge \inf\{r, F(x+y,r) \ge \alpha\} = \|x+y\|_\alpha
\end{aligned}$$

□

Example 1.5. *Let $(X, \|\cdot\|)$ be a non-Archimedean normed space. Define*

$$F(x,s) = \begin{cases} \frac{s}{s+\|x\|}, & s > 0, \\ 0, & s \le 0. \end{cases}$$

for all $x \in X$, Then (X, F) is a F-NANS.

Definition 1.6. *Let Z be any non-empty set and $\Im(Z)$ be the set of all fuzzy sets on Z. For $Z_1, Z_2 \in \Im(Z)$ and $\lambda \in K$, define*

$$Z_1 + Z_2 = \{(z_1 + z_2, \mu_1 \wedge \mu_2) \mid (z_1, \mu_1) \in Z_1, (z_2, \mu_2) \in Z_2\}$$

and

$$\lambda Z_1 = \{(\lambda z_1, \mu_1) \mid (z_1, \mu_1) \in Z_1\}$$

Definition 1.7. *A non-Archimedean fuzzy linear space $\hat{X} = X \times (0,1]$ over the field K, we define the addition and scalar multiplication operation of X as following: $(x_1, \mu_1) + (x_2, \mu_2) = (x_1 + x_2, \mu_1 \wedge \mu_2)$, $\lambda(x_1, \mu_1) = (\lambda x_1, \mu_1)$, if for every $(x_1, \mu_1) \in X$, we have a related non-negative real numebr, $\|(x_1, \mu_1)\|$ is the fuzzy norm of (x_1, μ_1) in such that*

(T_1) $\|(x_1, \mu_1)\| = 0 \Leftrightarrow x_1 = 0, \mu_1 \in (0,1]$;

(T$_2$) $\left\|\lambda\left(x_1,\mu_1\right)\right\|=|\lambda|\left\|\left(x_1,\mu_1\right)\right\|$;

(T$_3$) $\left\|\left(x_1,\mu_1\right)+\left(x_2,\mu_2\right)\right\|\le\max\left\{\left\|\left(x_1,\mu_1\wedge\mu_2\right),\left(x_2,\mu_1\wedge\mu_2\right)\right\|\right\}$;

(T$_4$) $\left\|\left(x_1,\vee_t\mu_t\right)\right\|=\wedge_t\left\|\left(x_1,\mu_t\right)\right\|$ for all $\mu_t\in(0,1]$.

for every $\left(x_1,\mu_1\right),\left(x_2,\mu_2\right)\in X,\lambda\in K$, then we say that X is an F-NANS.

Definition 1.8. *Let X be a non-empty non-Archimedean field set,* $\Im(X)$ *be the set of all fuzzy sets on X. If* $f_1\in\Im(X)$, *then* $f_1=\left\{\left(x_1,\mu_1\right):x_1\in X,\mu_1\in(0,1]\right\}$. *Clearly,* $\left|f_1\left(x_1\right)\right|\le 1$, *so* f_1 *is a bounded function. Let* $K\in\mathbb{Q}$, *then* $\Im(X)$ *is a non-Archimedean linear space over the field K and the addition, scalar multiplication are defined as follows*

$$f_1+f_2=\left\{\left(x_1,\mu_1\right)+\left(x_2,\mu_2\right)\right\}=\left\{\left(x_1+x_2,\mu_1\wedge\mu_2\right)\mid\left(x_1,\mu_1\right)\in f_1,\left(x_2,\mu_2\right)\in f_2\right\}$$

and

$$\lambda f_1=\left\{\left(\lambda x_1,\mu_1\right)\mid\left(x_1,\mu_1\right)\in f_1\right\}$$

If for every $f\in\Im(X)$, there is a related non-negative real number $\|f\|$ called the norm of f in such that for all $f_1=\left(x_1,\mu_1\right),f_2=\left(x_2,\mu_2\right)\in\Im(X)$

(T$_1$) $\|f\|=0$ iff $f=0$. For

$$\begin{aligned}\|f\|&=\left\{\left\|\left(x_1,\mu_1\right)\right\|\right\}=0\\&\Leftrightarrow x_1=0,\mu_1\in(0,1]\\&\Leftrightarrow f=0.\end{aligned}$$

(T$_2$) $\|\lambda f\|=|\lambda|\|f\|,\lambda\in K$. For

$$\|\lambda f\|=\left\{\left\|\lambda\left(x_1,\mu_1\right)\right\|\right\}=\left\{|\lambda|\left\|\left(x_1,\mu_1\right)\right\|\right\}=|\lambda|\|f\|$$

(T$_3$) $\left\|f_1+f_2\right\|\le\max\left\{\left\|f_1\right\|,\left\|f_2\right\|\right\}$. For

$$\begin{aligned}\left\|f_1+f_2\right\|&=\left\{\left\|\left(x_1,\mu_1\right)+\left(x_2,\mu_2\right)\right\|\right\}\\&=\left\{\left\|\left(x_1+x_2\right),\left(\mu_1\wedge\mu_2\right)\right\|\right\}\\&\le\max\left\{\left\|\left(x_1,\mu_1\wedge\mu_2\right)\right\|,\left\|\left(x_2,\mu_1\wedge\mu_2\right)\right\|\right\}\\&\le\max\left\{\left\|f_1\right\|,\left\|f_2\right\|\right\}\end{aligned}$$

Then the linear space $\Im(X)$ is a non-Archimedean normed space.

Definition 1.9. ([4]) *A 2-fuzzy set on X is a fuzzy set on* $\Im(X)$.

Definition 1.10. *A NAV* $|\cdot,\cdot|$ *in a linear space* $\Im(X)$ *over a field K. If a function* $F:\Im(X)^2\times\mathbb{R}\to[0,1]$ *is a non-Archimedean 2-fuzzy 2-norm on X (or a fuzzy 2-norm on* $\Im(X)$*), iff for all* $f_1,f_2,f_3\in\Im(X)$, $s,t\in\mathbb{R}$,

(F1) $F\left(f_1,f_2,s\right)=0$ for $s\le 0$;

(F2) $F\left(f_1,f_2,s\right)=1$ iff f_1,f_2 are linearly dependent for all $s>0$;

(F3) $F\left(f_1,f_2,s\right)=N\left(f_2,f_1,s\right)$;

(F4) $F\left(cf_1,f_2,s\right)=N\left(f_1,f_2,\frac{s}{|c|}\right)$, for $c\ne 0$ and $c\in K$;

(F5) $F\left(f_1,f_2+f_3,s+t\right)\ge\min\left\{F\left(f_1,f_2,s\right),F\left(f_1,f_3,t\right)\right\}$;

(F6) $F(f_1, f_2, *)$ is a nondecreasing function of R and $\lim_{s\to\infty} F(f_1, f_2, s) = 1$;

Then $(\Im(X), F)$ is called a non-Archimedean fuzzy 2-normed space (or FNA-2) or (X, F) is a non-Archimedean 2-fuzzy 2-normed space.

Theorem 1.11. *Let* $(\Im(X), F)$ *be an FNA-2. Suppose the condition that:*

(F7) $N(f_1, f_2, s) > 0$ for all $s > 0$ $\Rightarrow f_1$ and f_2 are linearly dependent.

Define $\|f_1, f_2\|_\alpha = \inf\{t : N(f_1, f_2, s) \ge \alpha, \alpha \in (0,1)\}$. We call these α-2-norms on $\Im(X)$ or the 2-fuzzy 2-norm on *X*.

Proof: It is similar to the proof of Theorem 1.4. □

2. Main Result

From now on, if we have no other explanation, let $\dim \Im(X) \ge 2$, $\dim \Im(Y) \ge 2$. $\blacktriangle = \|f-h, g-h\|_\alpha$, $\blacktriangledown = \|\psi(f)-\psi(h), \psi(g)-\psi(h)\|_\beta$

Definition 2.1. *Let* $\Im(X), \Im(Y)$ *be two FNA-2 and a mapping* $\psi : \Im(X) \to \Im(Y)$. *If for all* $f, g, h \in \Im(X)$ *and* $\alpha, \beta \in (0,1)$, *we have*

$$\|\psi(f)-\psi(h), \psi(g)-\psi(h)\|_\beta = \|f-h, g-h\|_\alpha \quad (\nabla)$$

then ψ is called 2-isometry.

Definition 2.2. *For a mapping* $\psi : \Im(X) \to \Im(Y)$ *and* $f, g, h \in \Im(X)$

1) If $\blacktriangle = 1$, then $\blacktriangledown = 1$, we say ψ satisfies the area one preserving property (AOPP).

2) If $\blacktriangle = n$, then $\blacktriangledown = n$, we say ψ satisfies the area *n* for each *n* (AnPP).

Definition 2.3. *We say a mapping* $\psi : \Im(X) \to \Im(Y)$ *preserves collinear, if* f, g, h *mutually disjoint elements of* $\Im(X)$, *then exist some real number t we have*

$$\psi(g)-\psi(h) = t(\psi(f)-\psi(h))$$

Next, we denote $\|\psi(f)-\psi(h), \psi(g)-\psi(h)\|_\beta \le \|f-h, g-h\|_\alpha \quad (*)$.

Lemma 2.4. *Let* $\Im(X)$ *and* $\Im(Y)$ *be two FNA-2. If* $\blacktriangle \le 1$, *a mapping* $\psi : \Im(X) \to \Im(Y)$ *satisfies* $(*)$ *and AOPP, then we can get* (∇) *where* $\blacktriangle \le 1$.

Proof: 1) Firstly, we prove that *f* preserves collinear. We assume that $\blacktriangle = 0$, according to $(*)$, we get

$$\|\psi(f)-\psi(h), \psi(g)-\psi(h)\|_\beta = 0$$

then $\psi(f)-\psi(h)$ and $\psi(g)-\psi(h)$ are linearly dependent. So we obtain that ψ preserves collinear.

2) Secondly, we prove that when $\blacktriangle \le 1$, we can get (∇).

If

$$\blacktriangledown < \blacktriangle$$

Let $\omega = h + \dfrac{f-h}{\|f-h, g-h\|_\alpha}$, then $\|\omega - h, g-h\|_\alpha = 1$, so

$$\left\|\psi(\omega)-\psi(h),\psi(g)-\psi(h)\right\|_{\beta}=1 \quad (\Delta)$$

Since

$$\|\omega-f,g-h\|_{\alpha}=\left\|\frac{f-h}{\|f-h,g-h\|}-(f-h),g-h\right\|=1-\blacktriangle$$

according to $(*)$, we have

$$\left\|\psi(\omega)-\psi(f),\psi(g)-\psi(h)\right\|_{\beta}\le\|\omega-f,g-h\|_{\alpha}=1-\blacktriangle$$

Since f preserves collinear, so there exists a real number s such that

$$\psi(\omega)-\psi(h)=s\left(\psi(f)-\psi(h)\right)$$

and

$$\psi(\omega)-\psi(f)=(s-1)\left(\psi(f)-\psi(h)\right)$$

So, we get

$$\begin{aligned}&\left\|\psi(\omega)-\psi(h),\psi(g)-\psi(h)\right\|_{\beta}\\&=|s|\blacktriangledown\\&\le|s-1|\blacktriangledown+\blacktriangledown\\&=\left\|\psi(\omega)-\psi(f),\psi(g)-\psi(h)\right\|_{\beta}+\blacktriangledown\\&<1-\blacktriangle+\blacktriangle=1\end{aligned}$$

This contradicts with Δ. □

Lemma 2.5. *Let* $\Im(X)$ *and* $\Im(Y)$ *be two FNA-2. If a mapping* $\psi:\Im(X)\to\Im(Y)$ *satisfies AOPP and preserves collinear, then*

1) ψ is an injective;

2) if $\phi(f)=\psi(f)-\psi(0)$, then $\phi(f+g)=\phi(f)+\phi(g)$ and $\phi(\lambda f)=\lambda\phi(f)$ with $0<\lambda<1$.

Proof: 1) We prove ψ is injective. Let $f,g\in\Im(X)$, since dim $\Im(X)\ge 2$, there exists an element $h\in\Im(X)$ such that $f-h,g-h$ are linearly independent. Hence $\blacktriangle\ne 0$.

Let $\gamma=h+\dfrac{g-h}{\|f-h,g-h\|_{\alpha}}$, then $\|f-h,\gamma-h\|_{\alpha}=1$, and ψ satisfies AOPP, so

$$\left\|\psi(f)-\psi(h),\psi(\gamma)-\psi(h)\right\|_{\beta}=1$$

we can see $\psi(h)\ne\psi(f)$. So the mapping ψ is injective.

2) Let f,g,h mutually disjoint elements of $\Im(X)$ and $f=\dfrac{g+h}{2}$, so $f-h=g-f \quad (\star)$. Since ψ is injective and preserves collinear, there exist $s\ne 0$ such that

$$\psi(g)-\psi(f)=s\left(\psi(h)-\psi(f)\right)$$

Since dim $\Im(X)\ge 2$, there exist an element $f_1\in\Im(X)$ such that

$\|g-f,f_1-f\|_\alpha \neq 0$. Let $\eta = f + \frac{f_1 - f}{\|g-f,f_1-f\|_\alpha}$, then $\|g-f,\eta-f\|_\alpha = 1$ and

$$\|\psi(g)-\psi(f),\psi(\eta)-\psi(f)\|_\beta = 1.$$

So,

$$\|\psi(h)-\psi(f),\psi(\eta)-\psi(f)\|_\beta = \left|\frac{1}{s}\right|.$$

Since $(\star)$, we get $\|h-f,\eta-f\|_\alpha = 1$ and

$$\|\psi(h)-\psi(f),\psi(\eta)-\psi(f)\|_\beta = 1.$$

According to the mapping ψ is injective, so $s=-1$, and

$$\psi\left(\frac{g+h}{2}\right) = \frac{\psi(g)+\psi(h)}{2}$$

Let $\phi(f) = \psi(f)-\psi(0)$, so we have

$$\phi\left(\frac{g+h}{2}\right) = \frac{\phi(g)+\phi(h)}{2}$$

Therefore

$$\phi\left(\frac{f}{2}\right) = \phi\left(\frac{f+0}{2}\right) = \frac{\phi(f)}{2}$$

and

$$\phi(f+g) = \phi\left(\frac{2f+2g}{2}\right) = \frac{\phi(2f)}{2} + \frac{\phi(2g)}{2} = \phi(f)+\phi(g)$$

So ϕ is additive.

From the lemma 2.4, we know that if $\blacktriangle \le 1$, then ϕ satisfies 2-isometry.

$$0 = \|\lambda f, f\|_\alpha = \|\psi(\lambda f)-\psi(0),\psi(f)-\psi(0)\|_\beta = \|\phi(\lambda f),\phi(f)\|_\beta$$

so $\phi(\lambda f)$ and $\phi(f)$ is linearly dependent *i.e.* $\phi(\lambda f) = s\phi(f)$.

Next we assume $\|f,g\|_\alpha = \lambda$,

$$\frac{1}{\lambda}\|f,g\|_\alpha = \left\|\frac{f}{\lambda}-0, g-0\right\|_\alpha = 1$$

and

$$\begin{aligned} 1 &= \left\|\phi\left(\frac{f}{\lambda}\right)-\phi(0),\phi(g)-\phi(0)\right\|_\beta \\ &= \left\|\phi\left(\frac{f}{\lambda}\right),\phi(g)\right\|_\beta \\ &= \frac{1}{|s|}\|\phi(f),\phi(g)\|_\beta \\ &= \frac{1}{|s|}\|f,g\|_\alpha \end{aligned}$$

Thus $\lambda = |s|$, if $s=-\lambda$, then $\phi(\lambda f) = -\lambda\phi(f)$, but

$$\begin{aligned}|\lambda-1|\|f,g\|_\alpha &= \|\lambda f-f,g-0\|_\alpha \\ &= \|\phi(\lambda f)-\phi(f),\phi(g)-\phi(0)\|_\beta \\ &= \|-\lambda\phi(f)-\phi(f),\phi(g)-\phi(0)\|_\beta \\ &= (\lambda+1)\|\phi(f),\phi(g)\|_\beta \\ &= (\lambda+1)\|f,g\|_\alpha\end{aligned}$$

so $|\lambda-1|=(\lambda+1)$. It contradicts with $0<\lambda<1$. Thus $\phi(\lambda f)=\lambda\phi(f)$. □

Lemma 2.6. *Let* $\Im(X)$ *and* $\Im(Y)$ *be FNA-2. If* $\blacktriangle\le 1$, *a mapping* $\psi:\Im(X)\to\Im(Y)$ *satisfies* $(*)$ *and AOPP, then we can get for all* $f,g,h\in\Im(X)$, *we can get* (∇).

Proof: From lemma 2.4, we know ψ preserves collinear.

For any $f,g,h\in\Im(X)$, there exist two numbers $m,n\in\mathbb{N}^*$ such that $\blacktriangle\le\frac{m}{n}$.

So,

$$\left\|\psi\left(\frac{f}{m}\right)-\psi\left(\frac{h}{m}\right),\psi(g)-\psi(h)\right\|_\beta \le \left\|\frac{f-h}{m},g-h\right\|_\alpha \le \frac{1}{n}$$

and

$$\left\|\left(\frac{f}{m}\right)-\phi\left(\frac{h}{m}\right),\phi(g)-\phi(h)\right\|_\beta \le \frac{1}{n}$$

By lemma 2.5, we have

$$\left\|\frac{1}{m}(\phi(f)-\phi(h)),\phi(g)-\phi(h)\right\|_\beta \le \frac{1}{n}$$

$$\|\phi(f)-\phi(h),\phi(g)-\phi(h)\|_\beta \le \frac{m}{n}$$

Thus

$$\|\psi(f)-\psi(h),\psi(g)-\psi(h)\|_\beta \le \frac{m}{n}$$

□

Lemma 2.7. *Let* $\Im(X)$ *and* $\Im(Y)$ *be two FNA-2. If a mapping* $\psi:\Im(X)\to\Im(Y)$ *satisfies AOPP and* $(*)$ *for all* $f,g,h\in\Im(X)$ *with* $\blacktriangle\le 1$, *then* ψ *satisfies AnPP.*

Proof: Let $f,g,h\in\Im(X)$ and $n\in\mathbf{N}$. Let

$$\blacktriangle=n,\ \ g_i=h+\frac{i}{n}(g-h)$$

and

$$\|f-h,g_{i+1}-g_i\|_\alpha=1,\ \ i=0,1,\cdots,n-1.$$

So,

$$\|\psi(f)-\psi(h),\psi(g_{i+1})-\psi(g_i)\|_\beta=1,\ \ i=0,1,\cdots,n-1.$$

We know ψ preserves collinear. So there exist a number $t\in\mathbb{R}$ such that

$$\psi(g_2)-\psi(g_1)=t(\psi(g_1)-\psi(g_0))$$

Therefore

Then we have $t=\pm 1$. By lemma 2.5, $t=1$, so

$$\psi(g_2)-\psi(g_1)=\psi(g_1)-\psi(g_0).$$

In the same way, we can get

$$\psi(g_{i+1})-\psi(g_i)=\psi(g_i)-\psi(g_{i-1}),\ i=0,1,\cdots,n-1.$$

Hence

$$\begin{aligned}\psi(g)-\psi(h)&=\psi(g_n)-\psi(g_0)\\&=\psi(g_n)-\psi(g_{n-1})+\psi(g_{n-1})-\psi(g_{n-2})+\cdots+\psi(g_1)-\psi(g_0)\\&=n(\psi(g_1)-\psi(g_0))\end{aligned}$$

Therefore

$$\begin{aligned}\blacktriangledown&=\left\|\psi(f)-\psi(h),n(\psi(g_1)-\psi(g_0))\right\|_\beta\\&=n\left\|\psi(f)-\psi(h),\psi(g_1)-\psi(g_0)\right\|_\beta=n\end{aligned}$$

□

Theorem 2.8. *Let* $\Im(X)$ *and* $\Im(Y)$ *be two FNA-2. If a mapping* $\psi:\Im(X)\to\Im(Y)$ *satisfies AOPP and* $(*)$ *for all* $f,g,h\in\Im(X)$ *with* $\blacktriangle\le 1$, *then* ψ *is* 2*-isometry.*

Proof: Since lemma 2.4, we just need to prove that (∇) with $\blacktriangle>1$.

We can assume that when $\blacktriangle>1$, for all $f,g,h\in\Im(X)$, we have $\blacktriangledown<n_0+1$. and there exist a number $n_0\in\mathbb{N}^*$ such that

Let $\tau=f+\frac{n_0+1}{\|f-h,g-h\|_\alpha}(f-h)$, then

$$\|\tau-f,g-h\|_\alpha=n_0+1$$

and

$$\|\tau-h,g-h\|_\alpha=n_0+1-\blacktriangle$$

Since ψ preserves collinear, there exist a number $c\in\mathbb{R}$ such that

$$\psi(\tau)-\psi(f)=c(\psi(h)-\psi(f))$$

Since 2),

$$\begin{aligned}n_0+1&=\left\|\psi(\tau)-\psi(f),\psi(g)-\psi(h)\right\|_\beta\\&=|c|\blacktriangledown\\&\le|c-1|\blacktriangledown+\blacktriangledown\\&=\left\|\psi(\tau)-\psi(h),\psi(g)-\psi(h)\right\|_\beta+\blacktriangledown\\&<n_0+1-\blacktriangle+\blacktriangle=n_0+1\end{aligned}$$

which is contradiction, so

$$\blacktriangledown \geq n_0 + 1$$

Therefore, we get (∇) with $\blacktriangle > 1$. Hence

$$\left\| \psi(f) - \psi(h), \psi(g) - \psi(h) \right\|_\beta = \left\| f - h, g - h \right\|_\alpha$$

for all $f, g, h \in \Im(X)$. □

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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