Scientific
Research
Publishing

Attractors for the Stochastic Lattice Selkov Equations with Additive Noises

Hongyan Li

College of Management, Shanghai University of Engineering Science, Shanghai, China
Email: hongyanlishu@163.com

Abstract

In this paper, by proving the pullback asymptotic compactness of the stochastic lattice Selkov equations with the cubic nonlinearity, the existence of a random attractor of the stochastic lattice reversible Selkov equations on infinite lattice with additive noises is proved.

Keywords

Random Attractor, Asymptotic Compactness, Selkov System

1. Introduction

In this paper, we study that the stochastic lattice Selkov system with the cubic nonlinearity and additive white noises on an infinite lattice is considered in [1] and [2]:

$$\begin{cases} du_i = \left[d_1 \left(u_{i+1} - 2u_i + u_{i-1} \right) - a_1 u_i + b_1 u_i^2 v_i - b_2 u_i^3 + f_{1i} \right] dt + \alpha_i dw_i, \ i \in \mathbb{Z}, t > 0, \\ dv_i = \left[d_2 \left(v_{i+1} - 2v_i + v_{i-1} \right) - a_2 v_i - b_1 u_i^2 v_i + b_2 u_i^3 + f_{2i} \right] dt + \alpha_i dw_i, \ i \in \mathbb{Z}, t > 0, \end{cases} \quad (1.1)$$

with initial conditions

$$u_i(0) = u_{i,0}, \ v_i(0) = v_{i,0}, \ i \in \mathbb{Z}, \quad (1.2)$$

where $\mathbb{Z}$ denotes the integer set, $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, $v = (v_i)_{i \in \mathbb{Z}} \in \ell^2$, $d_1, d_2, a_1, a_2, b_1, b_2$ are positive constants, $\alpha = (\alpha_i)_{i \in \mathbb{Z}} \in \ell^2$, $\{w_i \mid i \in \mathbb{Z}\}$ is independent Brownian motions.

The reversible Selkov model is derived from a set of the two reversible chemical reactions, which has been studied by [3] [4] and other authors:

$$P + 2Q \leftrightarrows 3Q, \quad Q \leftrightarrows Q_1.$$

The original Selkov model corresponds to the two irreversible reactions, where the product Q_1 is an inert product. Let u_t and v_t are respectively the

concentrations of the reactants Q and P, Equation (1.1) can be regarded as a Selkov system (see [5]) on $\mathbb{R}$:

$$\begin{cases} u_t = d_1 \Delta u - a_1 u + b_1 u^2 v - b_2 u^3 + f_1 + \alpha w_t, \\ v_t = d_2 \Delta v - a_2 v - b_1 u^2 v + b_2 u^3 + f_2 + \alpha w_t. \end{cases} \tag{1.3}$$

For the Equation (1.1), the solution mapping defines a random dynamical system, which is a parametric dynamical system, and pullback absorbing set has been proved, see [1] and [2]. Random attractors are the appropriate objects for describing asymptotic dynamics of such a parametric dynamical system. Therefore, in this paper, we would prove the existence of a random attractor for the stochastic lattice Selkov Equation (1.1).

This paper is organized as follows. In the next section, we recall basic concepts and results related to random attractors. In Section 3, using the transformation of Ornstein-Uhlenbeck process, the stochastic Selkov equation with white noise is transformed into a noiseless determined Selkov equation with random variables as parameters. In Section 4, we prove the pullback asymptotic compactness for the random dynamical system. Then the existence of a random attractor is proved.

2. Preliminaries

Firstly, we introduce the relevant definitions of random attractor, which can be taken from [6] [7] [8].

Let (H, d) be a complete separable metric space, $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, $\mathbb{R}^+ = [0, \infty)$.

Definition 2.1. $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$ measurable, $\theta_0 = I, \theta_{s+t} = \theta_s \circ \theta_t$ for all $s, t \in \mathbb{R}$, and $\theta_t \mathcal{P} = \mathcal{P}$ for all $t \in \mathbb{R}$.

Definition 2.2. A continuous random dynamical system (RDS) on H over a metric dynamical system $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t \in \mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times H \to H, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(H), \mathcal{B}(H))$-measurable and satisfies, for every $\omega \in \Omega$,

1) $\varphi(0, \omega, \cdot)$ is the identity on H;

2) Cocycle property: $\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s \omega, \varphi(s, \omega, \cdot))$ for all $t, s \in \mathbb{R}^+$;

3) $\varphi(\cdot, \omega, \cdot) : \mathbb{R}^+ \times H \to H$ is strongly continuous.

Definition 2.3. Suppose $\varphi(t, \omega)$ is a random dynamical system, a random set $\mathcal{A}$ is called a random $\mathcal{D}$ attractor if the following hold:

1) $\mathcal{A}(\omega)$ is a random compact set, *i.e.*, $\omega \to d(x, \mathcal{A}(\omega))$ is measurable for every $x \in H$ and $\mathcal{A}(\omega)$ is compact for every $\omega \in \Omega$;

2) $\mathcal{A}(\omega)$ is strictly invariant, *i.e.*, for every $\omega \in \Omega$ and all $t \geq 0$ one has $\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t \omega)$;

3) $\mathcal{A}(\omega)$ attracts all sets in $\mathcal{D}$, *i.e.* for all $B \in \mathcal{D}$ and $\omega \in \Omega$ we have

$$\lim_{t \to \infty} d\left(\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)\right) = 0,$$

where $d(X,Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|_H$ is the Hausdorff semi-metric (here, $X \subseteq H, Y \subseteq H$).

Theorem 2.1 ([9], Proposition 4.1) *Let $K(\omega) \in \mathcal{D}$ be an absorbing set for the random dynamical system $\varphi(t, \theta_{-t}\omega)_{t \geq 0, \omega \in \Omega}$ which is closed and which satisfies for $\omega \in \Omega$ the following asymptotic compactness condition: each sequence $x_n \in \varphi(t_n, \theta_{-t_n}\omega, K(\theta_{-t_n}\omega))$ with $t_n \to \infty$ has a convergent subsequence in H. Then the random dynamical system φ has a unique global random attractor*

$$\mathcal{A}(\omega) = \bigcap_{t \geq t_K(\omega)} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

3. Ornstein-Uhlenbeck Process

Let $t^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}}, u_i \in \mathbb{R} : \sum_{i \in \mathbb{Z}} |u_i|^2 < +\infty \right\}$, with the inner product and norm as follows:

$$\langle u, v \rangle = \sum_{i \in \mathbb{Z}} u_i v_i, \quad \|u\|^2 = \langle u, u \rangle, \quad u = (u_i)_{i \in \mathbb{Z}}, \quad v = (v_i)_{i \in \mathbb{Z}} \in t^2.$$

Then $t^2 = (t^2, \langle \cdot, \cdot \rangle, \|\cdot\|)$ is a Hilbert space. Set $E = t^2 \times t^2$ be the product Hilbert space. In view of the cubic term $\pm u^2 v, \pm u^3$, we need $u \in t^6, v \in t^6$ to make (1.1) hold in t^2.

Introducing an Ornstein-Uhlenbeck process (O-U process) (see [10]) in t^2 on $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ given by the Wiener process:

$$y(\theta_t \omega) = -(a_1 + a_2) \int_{-\infty}^0 e^{(a_1 + a_2)s}(\theta_t \omega)(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \; \omega \in \Omega,$$

and y solve the Itô equations:

$$\mathrm{d}y + (a_1 + a_2) y \mathrm{d}t = \mathrm{d}w(t), \; t > 0.$$

There exists a θ_t-invariant set $\Omega' \subset \Omega$ of full P measure such that
1) the mappings $s \to y(\theta_s \omega)$ are continuous for each $\omega \in \Omega$;
2) the random variables $\|y(\theta_t \omega)\|$ is tempered.
Let

$$\tilde{u}(t) = u(t) - y(\theta_t \omega), \quad \tilde{v}(t) = v(t) - y(\theta_t \omega).$$

From (1.3), we have

$$\begin{cases} \tilde{u}_t = -d_1 A(\tilde{u} + y(\theta_t \omega)) - a_1 \tilde{u} + a_2 y(\theta_t \omega) + b_1 (\tilde{u} + y(\theta_t \omega))^2 (\tilde{v} + y(\theta_t \omega)) \\ \quad\quad - b_2 (\tilde{u} + y(\theta_t \omega))^3 + f_1 \\ \tilde{v}_t = -d_2 A(\tilde{v} + y(\theta_t \omega)) - a_2 \tilde{v} + a_1 y(\theta_t \omega) - b_1 (\tilde{u} + y(\theta_t \omega))^2 (\tilde{v} + y(\theta_t \omega)) \\ \quad\quad + b_2 (\tilde{u} + y(\theta_t \omega))^3 + f_2 \end{cases} \quad (3.1)$$

with the initial value condition

$$\tilde{u}(0, \omega, \tilde{u}_0) = \tilde{u}_0(\omega) = u_0 - y(\omega), \quad \tilde{v}(0, \omega, \tilde{v}_0) = \tilde{v}_0(\omega) = v_0 - y(\omega).$$

4. Pullback Asymptotic Compactness

From Theorem 2.1, to prove the existence of a random attractor for the random

dynamical system generated by (1.1), it is necessary to obtain the pullback absorbing property and the pullback asymptotic compactness. The pullback absorbing property has been obtained by [2]. For the pullback asymptotic compactness, we have the following lamma.

Lemma 4.1. *Assume the initial functions* $\left(u_0(\omega), v_0(\omega)\right) \in K(\omega)$, *where* $K(\omega)$ *is the absorbing set. Then for every* $\varepsilon > 0$, *there exist* $T(\varepsilon, \omega) > 0$ *and* $N(\varepsilon, \omega) > 0$ *such that the solution* $\left(u\left(t, \omega, u_0(\omega)\right), v\left(t, \omega, v_0(\omega)\right)\right)$ *of (1.1) satisfies*

$$\sum_{|i| \geq N(\varepsilon, \omega)} \left[\left\| u\left(t, \theta_{-t}\omega, u_0\left(\theta_{-t}\omega\right)\right) \right\|^2 + \left\| v\left(t, \theta_{-t}\omega, v_0\left(\theta_{-t}\omega\right)\right) \right\|^2 \right] < \varepsilon,$$

for all $t \geq T(\varepsilon, \omega) > 0$.

Proof. We choose a smooth function ρ such that $0 \leq \rho \leq 1$ for all $s \in \mathbb{R}$ and

$$\rho(s) = \begin{cases} 0, & \text{if } |s| < 1, \\ 1, & \text{if } |s| > 2, \end{cases} \tag{4.1}$$

and there exists a positive constant c_1, such that $|\rho'(s)| \leq c_1$ for $s \in \mathbb{R}$.

We first consider the random Equation (3.1). Let r be a fixed positive integer which will be specified later. Taking the inner product of the Equation (3.1) with $\rho\left(\frac{|i|}{r}\right)\tilde{u}$ and $\rho\left(\frac{|i|}{r}\right)\tilde{v}$ in E, respectively, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i \in \mathbb{Z}}\rho\left(\frac{|i|}{r}\right)\left|\tilde{u}_i\right|^2$$

$$= -d_1\left\langle A\tilde{u}, \rho\left(\frac{|i|}{r}\right)\tilde{u}\right\rangle - d_1\left\langle Ay\left(\theta_t\omega\right), \rho\left(\frac{|i|}{r}\right)\tilde{u}\right\rangle - a_1\sum_{i \in \mathbb{Z}}\rho\left(\frac{|i|}{r}\right)\left|\tilde{u}_i\right|^2$$

$$+ \left\langle b_1\left(\tilde{u} + y\left(\theta_t\omega\right)\right)^2\left(\tilde{v} + y\left(\theta_t\omega\right)\right), \rho\left(\frac{|i|}{r}\right)\tilde{u}\right\rangle - \left\langle b_2\left(\tilde{u} + y\left(\theta_t\omega\right)\right)^3, \rho\left(\frac{|i|}{r}\right)\tilde{u}\right\rangle$$

$$+ \left\langle f_1, \rho\left(\frac{|i|}{r}\right)\tilde{u}\right\rangle + a_2\left\langle y\left(\theta_t\omega\right), \rho\left(\frac{|i|}{r}\right)\tilde{u}\right\rangle,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i \in \mathbb{Z}}\rho\left(\frac{|i|}{r}\right)\left|\tilde{v}_i\right|^2$$

$$= -d_2\left\langle\left\langle A\tilde{v}, \rho\left(\frac{|i|}{r}\right)\tilde{v}\right\rangle - d_2\left\langle Ay\left(\theta_t\omega\right), \rho\left(\frac{|i|}{r}\right)\tilde{v}\right\rangle - a_2\sum_{i \in \mathbb{Z}}\rho\left(\frac{|i|}{r}\right)\left|\tilde{v}_i\right|^2$$

$$- \left\langle b_1\left(\tilde{u} + y\left(\theta_t\omega\right)\right)^2\left(\tilde{v} + y\left(\theta_t\omega\right)\right), \rho\left(\frac{|i|}{r}\right)\tilde{v}\right\rangle + \left\langle b_2\left(\tilde{u} + y\left(\theta_t\omega\right)\right)^3, \rho\left(\frac{|i|}{r}\right)\tilde{v}\right\rangle \tag{4.2}$$

$$+ \left\langle f_2, \rho\left(\frac{|i|}{r}\right)\tilde{v}\right\rangle + a_1\left\langle y\left(\theta_t\omega\right), \rho\left(\frac{|i|}{r}\right)\tilde{v}\right\rangle.$$

By $|\rho'(s)| \leq c_1$, we have

$$\left\langle A\tilde{u}, \rho\left(\frac{|i|}{r}\right)\tilde{u}\right\rangle = \sum_{i \in Z}\left(B\tilde{u}\right)_i\left(B\rho\left(\frac{|i|}{r}\right)\tilde{u}\right)_i$$

$$= \sum_{i \in \mathbb{Z}} \left(B\tilde{u} \right)_i \left[\rho\left(\frac{|i+1|}{r} \right) \tilde{u}_{i+1} - \rho\left(\frac{|i|}{r} \right) \tilde{u}_i \right]$$

$$= \sum_{i \in \mathbb{Z}} \left(\tilde{u}_{i+1} - \tilde{u}_i \right) \left[\rho\left(\frac{|i|}{r} \right) \left(\tilde{u}_{i+1} - \tilde{u}_i \right) + \left(\rho\left(\frac{|i+1|}{r} \right) - \rho\left(\frac{|i|}{r} \right) \right) \tilde{u}_{i+1} \right]$$

$$\geq \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{r} \right) \left| \tilde{u}_{i+1} - \tilde{u}_i \right|^2 - \sum_{i \in \mathbb{Z}} \frac{\rho'(\xi)}{r} \left(\tilde{u}_{i+1} - \tilde{u}_i \right) \tilde{u}_{i+1}$$

$$\geq -\frac{c_1}{r} \sum_{i \in \mathbb{Z}} \left(\left| \tilde{u}_{i+1} \right|^2 + \left| \tilde{u}_i \right| \left| \tilde{u}_{i+1} \right| \right) \geq -\frac{2c_1}{r} \| \tilde{u} \|^2 .$$

(4.3)

And

$$\left\langle A\tilde{v}, \rho\left(\frac{|i|}{r} \right) \tilde{v} \right\rangle = \sum_{i \in Z} \left(B\tilde{v} \right)_i \left(B\rho\left(\frac{|i|}{r} \right) \tilde{v} \right)_i \geq -\frac{2c_1}{r} \| \tilde{v} \|^2 .$$

(4.4)

From (4.2), we have

$$b_1 \left\langle \left(\tilde{u} + y(\theta_t \omega) \right)^2 \left(\tilde{v} + y(\theta_t \omega) \right), \rho\left(\frac{|i|}{r} \right) \tilde{u} \right\rangle - b_2 \left\langle \left(\tilde{u} + y(\theta_t \omega) \right)^3, \rho\left(\frac{|i|}{r} \right) \tilde{u} \right\rangle$$

$$- b_1 \left\langle \left(\tilde{u} + y(\theta_t \omega) \right)^2 \left(\tilde{v} + y(\theta_t \omega) \right), \rho\left(\frac{|i|}{r} \right) \tilde{v} \right\rangle + b_2 \left\langle \left(\tilde{u} + y(\theta_t \omega) \right)^3, \rho\left(\frac{|i|}{r} \right) \tilde{v} \right\rangle$$

$$= b_1 \left\langle \left(\tilde{u} + y(\theta_t \omega) \right)^2 \left(\tilde{v} + y(\theta_t \omega) \right), \rho\left(\frac{|i|}{r} \right) \left(\tilde{u} - \tilde{v} \right) \right\rangle$$

$$- b_2 \left\langle \left(\tilde{u} + y(\theta_t \omega) \right)^3, \rho\left(\frac{|i|}{r} \right) \left(\tilde{u} - \tilde{v} \right) \right\rangle$$

$$\leq \max \{ b_1, b_2 \} \left\langle \left(\tilde{u} + y(\theta_t \omega) \right)^2 \left(\tilde{v} - \tilde{u} \right), \rho\left(\frac{|i|}{r} \right) \left(\tilde{u} - \tilde{v} \right) \right\rangle$$

$$= -\max \{ b_1, b_2 \} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{r} \right) \left(\tilde{u}_i + y_i(\theta_t \omega) \right)^2 \left(\tilde{u}_i - \tilde{v}_i \right)^2 \leq 0 .$$

(4.5)

Then from (4.2)-(4.5), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{r} \right) \left[\left| \tilde{u}_i \right|^2 + \left| \tilde{v}_i \right|^2 \right] + 2a_1 \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{r} \right) \left| \tilde{u}_i \right|^2 + 2a_2 \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{r} \right) \left| \tilde{v}_i \right|^2$$

$$\leq \frac{4c_1 d_1}{r} \| \tilde{u} \|^2 + \frac{4c_1 d_2}{r} \| \tilde{v} \|^2 + 2 \left\langle f_1, \rho\left(\frac{|i|}{r} \right) \tilde{u} \right\rangle + 2 \left\langle f_2, \rho\left(\frac{|i|}{r} \right) \tilde{v} \right\rangle$$

$$- 2d_1 \left\langle Ay(\theta_t \omega), \rho\left(\frac{|i|}{r} \right) \tilde{u} \right\rangle - 2d_2 \left\langle Ay(\theta_t \omega), \rho\left(\frac{|i|}{r} \right) \tilde{v} \right\rangle$$

$$+ 2a_2 \left\langle y(\theta_t \omega), \rho\left(\frac{|i|}{r} \right) \tilde{u} \right\rangle + 2a_1 \left\langle y(\theta_t \omega), \rho\left(\frac{|i|}{r} \right) \tilde{v} \right\rangle .$$

(4.6)

For the third term and forth term in the right-hand side of (4.6), we have

$$2 \left\langle f_1, \rho\left(\frac{|i|}{r} \right) \tilde{u} \right\rangle = \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{r} \right) f_{1i} \tilde{u}_i = \sum_{|i| \geq r} \rho\left(\frac{|i|}{r} \right) f_{1i} \tilde{u}_i$$

$$\leq \frac{a_1}{3} \sum_{|i| \geq r} \rho\left(\frac{|i|}{r} \right) \left| \tilde{u}_i \right|^2 + \frac{3}{a_1} \sum_{|i| \geq r} \left| f_{1i} \right|^2 ,$$

$$2\left\langle f_2, \rho\left(\frac{|i|}{r}\right)\tilde{v}\right\rangle = \sum_{|i|\geq r}\rho\left(\frac{|i|}{r}\right)f_{2i}\tilde{v}_i \leq \frac{a_2}{3}\sum_{|i|\geq r}\rho\left(\frac{|i|}{r}\right)|\tilde{v}_i|^2 + \frac{3}{a_2}\sum_{|i|\geq r}|f_{2i}|^2. \tag{4.7}$$

For the fifth term and sixth term in the right-hand side of (4.6), we have

$$-2d_1\left\langle Ay, \rho\left(\frac{|i|}{r}\right)\tilde{u}\right\rangle = -2d_1\left\langle By, B\left(\rho\left(\frac{|i|}{r}\right)\tilde{u}\right)\right\rangle$$

$$= -2d_1\sum_{i\in\mathbb{Z}}(y_{i+1}-y_i)\left(\rho\left(\frac{|i+1|}{r}\right)\tilde{u}_{i+1} - \rho\left(\frac{|i|}{r}\right)\tilde{u}_i\right)$$

$$= -2d_1\sum_{|i|\geq r-1}\rho\left(\frac{|i+1|}{r}\right)(y_{i+1}-y_i)\tilde{u}_{i+1} + 2d_1\sum_{|i|\geq r}\rho\left(\frac{|i|}{r}\right)(y_{i+1}-y_i)\tilde{u}_i$$

$$\leq \frac{a_1}{3}\sum_{|i|\geq r}\rho\left(\frac{|i|}{r}\right)|\tilde{u}_i|^2 + C_1\sum_{|i|\geq r-1}|y_i|^2,$$

$$-2d_2\left\langle Ay, \rho\left(\frac{|i|}{r}\right)\tilde{v}\right\rangle \leq \frac{a_2}{3}\sum_{|i|\geq r}\rho\left(\frac{|i|}{r}\right)|\tilde{v}_i|^2 + C_2\sum_{|i|\geq r-1}|y_i|^2, \tag{4.8}$$

where C_j, $j=1,2$ are positive constants depending only on d_1, d_2, a_1, a_2. For the last two terms in the right-hand side of (4.6),

$$2a_2\left\langle y(\theta_t\omega), \rho\left(\frac{|i|}{r}\right)\tilde{u}\right\rangle \leq \frac{a_1}{3}\sum_{|i|\geq r}\rho\left(\frac{|i|}{r}\right)|\tilde{u}_i|^2 + \frac{3a_2^2}{a_1}\sum_{i\in Z}\rho\left(\frac{|i|}{r}\right)|y_i(\theta_t\omega)|^2,$$

$$2a_1\left\langle y(\theta_t\omega), \rho\left(\frac{|i|}{r}\right)\tilde{v}\right\rangle \leq \frac{a_2}{3}\sum_{i\in Z}\rho\left(\frac{|i|}{r}\right)|\tilde{v}_i|^2 + \frac{3a_1^2}{a_2}\sum_{i\in Z}\rho\left(\frac{|i|}{r}\right)|y_i(\theta_t\omega)|^2. \tag{4.9}$$

From (4.6)-(4.9), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{r}\right)\left[|\tilde{u}_i|^2 + |\tilde{v}_i|^2\right] + a_1\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{r}\right)|\tilde{u}_i|^2 + a_2\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{r}\right)|\tilde{v}_i|^2$$

$$\leq \frac{4c_1d_1}{r}\|\tilde{u}\|^2 + \frac{4c_1d_2}{r}\|\tilde{v}\|^2 + \frac{3}{a_1}\sum_{|i|\geq r}|f_{1i}|^2 + \frac{3}{a_2}\sum_{|i|\geq r}|f_{2i}|^2 \tag{4.10}$$

$$+ (C_1+C_2)\sum_{|i|\geq r-1}|y_i(\theta_t\omega)|^2 + \left(\frac{3a_2^2}{a_1} + \frac{3a_1^2}{a_2}\right)\sum_{|i|\geq r}|y_i(\theta_t\omega)|^2.$$

Let $a_3 = \min\{a_1, a_2\}$. By Gronwall's inequality in [11], we have that for $t \geq T_k = T_k(\omega) \geq 0$,

$$\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{r}\right)\left[|\tilde{u}_i(t,\omega,\tilde{u}_0(\omega))|^2 + |\tilde{v}_i(t,\omega,\tilde{v}_0(\omega))|^2\right]$$

$$\leq \mathrm{e}^{-a_3(t-T_k)}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{r}\right)\left[|\tilde{u}_i(T_k,\omega,\tilde{u}_0(\omega))|^2 + |\tilde{v}_i(T_k,\omega,\tilde{v}_0(\omega))|^2\right]$$

$$+ \int_{T_k}^t \left(\frac{4c_1d_1}{r}\|\tilde{u}\|^2 + \frac{4c_1d_2}{r}\|\tilde{v}\|^2\right)\mathrm{e}^{a_3(\tau-t)}\mathrm{d}\tau$$

$$+ \frac{1}{a_3}\left(\frac{3}{a_1}\sum_{|i|\geq r}|f_{1i}|^2 + \frac{3}{a_2}\sum_{|i|\geq r}|f_{2i}|^2\right)$$

$$+ (C_1+C_2)\int_{T_k}^t \mathrm{e}^{a_3(\tau-t)}\sum_{|i|\geq r-1}|y_i(\theta_\tau\omega)|^2\mathrm{d}\tau$$

$$+\left(\frac{3a_2^2}{a_1}+\frac{3a_1^2}{a_2}\right)\int_{T_k}^{t}e^{a_3(\tau-t)}\sum_{|i|\geq r}\left|y_i\left(\theta_\tau\omega\right)\right|^2 d\tau. \tag{4.11}$$

Replace ω by $\theta_{-t}\omega$. From (4.11) in [2], with t replaced by T_k and ω by $\theta_{-t}\omega$, it follows that

$$e^{a_3(t-T_k)}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{r}\right)\left[\left|\tilde{u}_i\left(T_k,\theta_{-t}\omega,\tilde{u}_0\left(\theta_{-t}\omega\right)\right)\right|^2+\left|\tilde{v}_i\left(T_k,\theta_{-t}\omega,\tilde{v}_0\left(\theta_{-t}\omega\right)\right)\right|^2\right]$$

$$\leq e^{a_3(t-T_k)}\left(e^{-a_3 T_k}\left[\left\|\tilde{u}_0\left(\theta_{-t}\omega\right)\right\|^2+\left\|\tilde{v}_0\left(\theta_{-t}\omega\right)\right\|^2\right]+\frac{C_4}{a_3}\left(\left\|f_1\right\|^2+\left\|f_2\right\|^2\right)\right.$$

$$\left.+C_4\int_0^{T_k}e^{-a_3(T_k-s)}\left(\left\|y\left(\theta_{s-t}\omega\right)\right\|^2+\left\|Ay\left(\theta_{s-t}\omega\right)\right\|^2\right)ds\right) \tag{4.12}$$

$$\leq e^{-a_3 t}\left[\left\|\tilde{u}_0\left(\theta_{-t}\omega\right)\right\|^2+\left\|\tilde{v}_0\left(\theta_{-t}\omega\right)\right\|^2\right]+e^{-a_3(t-T_k)}\frac{C_4}{a_3}\left(\left\|f_1\right\|^2+\left\|f_2\right\|^2\right)$$

$$+\frac{2}{a_3}C_4 l(\omega)e^{-\frac{a_3}{2}(t-T_k)}.$$

Thus, there exists a $T_1(\varepsilon,\omega)>T_k(\omega)$ such that if $t>T_1(\varepsilon,\omega)$, then

$$e^{-a_3(t-T_k)}\sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{r}\right)\left[\left|\tilde{u}_i\left(T_k,\theta_{-t}\omega,\tilde{u}_0\left(\theta_{-t}\omega\right)\right)\right|^2+\left|\tilde{v}_i\left(T_k,\theta_{-t}\omega,\tilde{v}_0\left(\theta_{-t}\omega\right)\right)\right|^2\right]<\frac{1}{4}\varepsilon. \tag{4.13}$$

From (4.11), we have

$$\int_{T_k}^{t}\left(\frac{4c_1 d_1}{r}\left\|\tilde{u}\left(\tau,\theta_{-t}\omega,\tilde{u}_0\left(\theta_{-t}\omega\right)\right)\right\|^2+\frac{4c_1 d_2}{r}\left\|\tilde{v}\left(\tau,\theta_{-t}\omega,\tilde{u}_0\left(\theta_{-t}\omega\right)\right)\right\|^2\right)e^{a_3(\tau-t)}d\tau$$

$$\leq\frac{4c_1 d_3}{r}\int_{T_k}^{t}e^{a_3(\tau-t)}\left(e^{-a_3\tau}\left[\left\|\tilde{u}_0\left(\theta_{-t}\omega\right)\right\|^2+\left\|\tilde{v}_0\left(\theta_{-t}\omega\right)\right\|^2\right]+\frac{C_4}{a_3}\left(\left\|f_1\right\|^2+\left\|f_2\right\|^2\right)\right.$$

$$\left.+C_4\int_0^\tau e^{a_3(s-\tau)}\left(\left\|y\left(\theta_{s-t}\omega\right)\right\|^2+\left\|Ay\left(\theta_{s-t}\omega\right)\right\|^2\right)ds\right)d\tau \tag{4.14}$$

$$\leq\frac{4c_1 d_3}{r}\left(\left[\left\|\tilde{u}_0\left(\theta_{-t}\omega\right)\right\|^2+\left\|\tilde{v}_0\left(\theta_{-t}\omega\right)\right\|^2\right](t-T_k)e^{-a_3 t}\right.$$

$$\left.+\frac{C_4}{a_3^2}\left(\left\|f_1\right\|^2+\left\|f_2\right\|^2\right)+\frac{4}{a_3}C_4 l(\omega)\right),$$

where $d_3=\max\{d_1,d_2\}$. Recall that $\left(\tilde{u}_0\left(\theta_{-t}\omega\right),\tilde{v}_0\left(\left(\theta_{-t}\omega\right)\right)\in K\left(\theta_{-t}\omega\right)$, which implies that $\left\|\tilde{u}_0\left(\theta_{-t}\omega\right)\right\|^2+\left\|\tilde{v}_0\left(\theta_{-t}\omega\right)\right\|^2\leq R^2\left(\theta_{-t}\omega\right)$, and $R(\omega)$ is tempered in [2]. Thus there exists $T_2(\varepsilon,\omega)>T_k(\omega)$ and $N_1(\varepsilon,\omega)>0$ such that for $t>T_2(\varepsilon,\omega)$ and $r>N_1(\varepsilon,\omega)$, we have

$$\int_{T_k}^{t}\left(\frac{4c_1 d_1}{r}\left\|\tilde{u}\left(\tau,\theta_{-t}\omega,\tilde{u}_0\left(\theta_{-t}\omega\right)\right)\right\|^2+\frac{4c_1 d_2}{r}\left\|\tilde{v}\left(\tau,\theta_{-t}\omega,\tilde{u}_0\left(\theta_{-t}\omega\right)\right)\right\|^2\right)e^{a_3(\tau-t)}d\tau<\frac{1}{4}\varepsilon. \tag{4.15}$$

Since $f_1,f_2\in l^2$, there exists $N_2(\varepsilon,\omega)>0$ such that for $r>N_2(\varepsilon,\omega)$,

$$\frac{1}{a_3}\left(\frac{3}{a_1}\sum_{|i|\geq r}\left|f_{1i}\right|^2+\frac{3}{a_2}\sum_{|i|\geq r}\left|f_{2i}\right|^2\right)<\frac{1}{4}\varepsilon. \tag{4.16}$$

Finally, we estimate the last terms on the right-hand side of (4.11). Let $T^*>0$ to be determined later. We have for $t>T^*+T_k$

$$\left(C_1 + C_2\right)\int_{T_k}^{t} e^{a_3(\tau-t)} \sum_{|i|\geq r-1} \left|y_i\left(\theta_{\tau-t}\omega\right)\right|^2 d\tau$$

$$+\left(\frac{3a_2^2}{a_1} + \frac{3a_1^2}{a_2}\right)\int_{T_k}^{t} e^{a_3(\tau-t)} \sum_{|i|\geq r} \left|y_i\left(\theta_{\tau-t}\omega\right)\right|^2 d\tau$$

$$= \int_{T_k-t}^{0} e^{a_3 s}\left(C_3 \sum_{|i|\geq r-1}\left|y_i\left(\theta_s\omega\right)\right|^2 ds + C_4\sum_{|i|\geq r}\left|y_i\left(\theta_s\omega\right)\right|^2 ds\right) \qquad (4.17)$$

$$\leq \int_{-T^*}^{0} e^{a_3 s}\left(C_3 \sum_{|i|\geq r-1}\left|y_i\left(\theta_s\omega\right)\right|^2 ds + C_4\sum_{|i|\geq r}\left|y_i\left(\theta_s\omega\right)\right|^2 ds\right)$$

$$+ \int_{T_k-t}^{-T^*} e^{a_3 s}\left(C_3 \left\|y\left(\theta_s\omega\right)\right\|^2 ds + C_4 \left\|y\left(\theta_s\omega\right)\right\|^2 ds\right).$$

Using (4.12) in [2], we have

$$\int_{T_k-t}^{-T^*} e^{a_3 s}\left(C_3 \left\|y\left(\theta_s\omega\right)\right\|^2 + C_4 \left\|y\left(\theta_s\omega\right)\right\|^2\right)ds \leq \frac{2C_5}{a_3}l\left(\omega\right)e^{-\frac{a_3 T^*}{2}}. \qquad (4.18)$$

Thus, by choosing

$$T^* > \frac{2}{a_3}\ln\frac{16C_5 l\left(\omega\right)}{a_3 \varepsilon},$$

we have for $t > T^* + T_k$,

$$\int_{T_k-t}^{-T^*} e^{a_3 s}\left(C_3 \left\|y\left(\theta_s\omega\right)\right\|^2 + C_4 \left\|y\left(\theta_s\omega\right)\right\|^2\right)ds < \frac{\varepsilon}{8}. \qquad (4.19)$$

For the fixed T^*, from Lebesgue's theorem, there is $N_3\left(\varepsilon,\omega\right)$ such that for $r > N_3\left(\varepsilon,\omega\right)$,

$$\int_{-T^*}^{0} e^{a_3 s}\left(C_3 \sum_{|i|\geq r-1}\left|y_i\left(\theta_s\omega\right)\right|^2 ds + C_4\sum_{|i|\geq r}\left|y_i\left(\theta_s\omega\right)\right|^2 ds\right) < \frac{\varepsilon}{8}. \qquad (4.20)$$

Therefore, by letting

$$T\left(\varepsilon,\omega\right) = \max\left\{T_1\left(\varepsilon,\omega\right), T_2\left(\varepsilon,\omega\right), T^*\left(\varepsilon,\omega\right) + T_k\left(\omega\right)\right\},$$

$$\tilde{N}\left(\varepsilon,\omega\right) = \max\left\{N_1\left(\varepsilon,\omega\right), N_2\left(\varepsilon,\omega\right), N_3\left(\varepsilon,\omega\right)\right\}, \qquad (4.21)$$

for $t > T\left(\varepsilon,\omega\right)$ and $N > \tilde{N}\left(\varepsilon,\omega\right)$, we have

$$\sum_{|i|>2r}\left[\left|\tilde{u}_i\left(t,\theta_{-t}\omega,\tilde{u}_0\left(\theta_{-t}\omega\right)\right)\right|^2 + \left|\tilde{v}_i\left(t,\theta_{-t}\omega,\tilde{v}_0\left(\theta_{-t}\omega\right)\right)\right|^2\right]$$

$$\leq \sum_{i\in\mathbb{Z}}\rho\left(\frac{|i|}{r}\right)\left[\left|\tilde{u}_i\left(t,\theta_{-t}\omega,\tilde{u}_0\left(\theta_{-t}\omega\right)\right)\right|^2 + \left|\tilde{v}_i\left(t,\theta_{-t}\omega,\tilde{v}_0\left(\theta_{-t}\omega\right)\right)\right|^2\right] \qquad (4.22)$$

$$< \varepsilon,$$

which implies that

$$\sum_{|i|\geq N(\varepsilon,\omega)}\left[\left|u_i\left(t,\theta_{-t}\omega,u_0\left(\theta_{-t}\omega\right)\right)\right|^2 + \left|v_i\left(t,\theta_{-t}\omega,v_0\left(\theta_{-t}\omega\right)\right)\right|^2\right]$$

$$= \sum_{|i|\geq N(\varepsilon,\omega)}\left(\left|\tilde{u}_i\left(t,\theta_{-t}\omega,\tilde{u}_0\left(\theta_{-t}\omega\right)\right) + y\left(\theta_{-t}\omega\right)\right|^2\right.$$

$$\left. + \left|\tilde{v}_i\left(t,\theta_{-t}\omega,\tilde{v}_0\left(\theta_{-t}\omega\right)\right) + y\left(\theta_{-t}\omega\right)\right|^2\right)$$

$$\le 2 \sum_{|i| \ge N(\varepsilon,\omega)} \left(\left| \tilde{u}_i \left(t, \theta_{-t}\omega, \tilde{u}_0 \left(\theta_{-t}\omega \right) \right) \right|^2 + \left| \tilde{v}_i \left(t, \theta_{-t}\omega, \tilde{v}_0 \left(\theta_{-t}\omega \right) \right) \right|^2 \right)$$

$$+ 6 \sum_{|i| \ge N(\varepsilon,\omega)} \left| y \left(\theta_{-t}\omega \right) \right|^2 \tag{4.23}$$

$$< 8\varepsilon,$$

provided $N(\varepsilon,\omega)$ is large enough. This completes the proof of the lemma. □

We are now ready to show the pullback asymptotic compactness of the random set $K(\omega)$.

Lemma 4.2. *For* $\omega \in \Omega$, *the set* $K(\omega)$ *is pullback asymptotically compact in the sense of each sequence* $(u_n, v_n) \in \varphi\left(t_n, \theta_{-t_n}\omega, K\left(\theta_{-t_n}\omega \right) \right)$ *with* $t_n \to \infty$ *having a convergent subsequence in* $\iota^2 \times \iota^2$.

Proof. We follow the method of [9]. Let $\omega \in \Omega'$ for each sequence $\{t_n\}_{n=1}^{\infty}: t_1, t_2, \cdots, t_n \to \infty$ as $n \to \infty$, and

$$\left(u_n \left(t_n, \theta_{-t_n}\omega, x_n \right), v_n \left(t_n, \theta_{-t_n}\omega, y_n \right) \right) \in \varphi\left(t_n, \theta_{-t_n}\omega, K\left(\theta_{-t_n}\omega \right) \right);$$

this implies that there exists $(x_n, y_n) \in K\left(\theta_{-t_n}\omega \right)$ such that

$$\left(u_n \left(t_n, \theta_{-t_n}\omega, x_n \right), v_n \left(t_n, \theta_{-t_n}\omega, y_n \right) \right) = \varphi\left(t_n, \theta_{-t_n}\omega, (x_n, y_n) \right).$$

Since $K(\omega)$ is a bounded absorbing set, for large n, $\varphi\left(t_n, \theta_{-t_n}\omega, (x_n, y_n) \right) \in K(\omega)$; thus there exists $(u, v) \in \iota^2 \times \iota^2$, and a subsequence $(u_n', v_n') = \varphi\left(t_n, \theta_{-t_n}\omega, (x_n, y_n) \right)$ such that

$$\left(u_n' \left(t_n, \theta_{-t_n}\omega, x_n \right), v_n' \left(t_n, \theta_{-t_n}\omega, y_n \right) \right) \to (u, v) \quad \text{weak in } \iota^2 \times \iota^2. \tag{4.24}$$

Next, we will show that (u_n', v_n') is also strongly convergent in the norm $\|\cdot\|$ in $\iota^2 \times \iota^2$, *i.e.*, for each $\epsilon > 0$ there is $N^*(\epsilon,\omega) > 0$ such that for $n \ge N^*(\epsilon,\omega)$,

$$\left\| \left(u_n' \left(t_n, \theta_{-t_n}\omega, x_n \right), v_n' \left(t_n, \theta_{-t_n}\omega, y_n \right) \right) - (u, v) \right\| \le \epsilon.$$

From Lemma 4.1, for any $\epsilon > 0$, there exists $N^*(\epsilon,\omega)$ and $K_1(\epsilon,\omega)$ such that for $n \ge N^*(\epsilon,\omega)$,

$$\sum_{|i| \ge K_1(\epsilon,\omega)} \left(\left| u_{ni}' \left(t_n, \theta_{-t_n}\omega, x_n \right) \right|^2 + \left| v_{ni}' \left(t_n, \theta_{-t_n}\omega, y_n \right) \right|^2 \right) \le \frac{1}{8}\epsilon^2. \tag{4.25}$$

Since $(u, v) \in \iota^2 \times \iota^2$, there exists $K_2(\epsilon) > 0$ such that

$$\sum_{|i| \ge K_2(\epsilon)} \left(\left| u_i \right|^2 + \left| v_i \right|^2 \right) \le \frac{1}{8}\epsilon^2. \tag{4.26}$$

Let $K(\epsilon,\omega) = \max\left\{ K_1(\epsilon,\omega), K_2(\epsilon) \right\}$. From the weak convergence (4.24), we have for each $|i| \le K(\epsilon,\omega)$ as $n \to \infty$,

$$\left(u_{ni}' \left(t_n, \theta_{-t_n}\omega, x_n \right), v_{ni}' \left(t_n, \theta_{-t_n}\omega, y_n \right) \right) \to (u_i, v_i),$$

which implies that there exists $N_2^*(\epsilon,\omega) > 0$ such that for $n \ge N_2^*(\epsilon,\omega)$,

$$\sum_{|i| \le K(\epsilon,\omega)} \left(\left| u_{ni}' \left(t_n, \theta_{-t_n}\omega, x_n \right) - u_i \right|^2 + \left| v_{ni}' \left(t_n, \theta_{-t_n}\omega, y_n \right) - v_i \right|^2 \right) \le \frac{1}{2}\epsilon^2. \tag{4.27}$$

Combining (4.25)-(4.27), we obtain that for $n \geq N^*(\epsilon, \omega)$,

$$
\begin{aligned}
& \left\| u_n'\left(t_n, \theta_{-t_n}\omega, x_n\right) - u \right\|^2 + \left\| v_n'\left(t_n, \theta_{-t_n}\omega, y_n\right) - v \right\|^2 \\
& \leq \sum_{|i| \leq K(\epsilon, \omega)} \left(\left| u_{ni}'\left(t_n, \theta_{-t_n}\omega, x_n\right) - u_i \right|^2 + \left| v_{ni}'\left(t_n, \theta_{-t_n}\omega, y_n\right) - v_i \right|^2 \right) \\
& \quad + \sum_{|i| \geq K(\epsilon, \omega)} \left(\left| u_{ni}'\left(t_n, \theta_{-t_n}\omega, x_n\right) - u_i \right|^2 + \left| v_{ni}'\left(t_n, \theta_{-t_n}\omega, y_n\right) - v_i \right|^2 \right) \\
& \leq \frac{1}{2}\epsilon^2 + 2 \sum_{|i| \geq K(\epsilon, \omega)} \left(\left| u_i \right|^2 + \left| v_i \right|^2 \right) \\
& \quad + 2 \sum_{|i| \geq K(\epsilon, \omega)} \left(\left| u_{ni}'\left(t_n, \theta_{-t_n}\omega, x_n\right) \right|^2 + \left| v_{ni}'\left(t_n, \theta_{-t_n}\omega, y_n\right) \right|^2 \right) \\
& \leq \frac{1}{2}\epsilon^2 + \frac{1}{4}\epsilon^2 + \frac{1}{4}\epsilon^2 = \epsilon^2.
\end{aligned}
\tag{4.28}
$$

Hence, we have completed the proof of Lemma 4.2. $\qquad\square$

Theorem 4.1. *The random dynamical systems* $\left\{\varphi(t, \omega)\right\}_{t \geq 0, \omega \in \Omega}$ *possess a random attractor in* $\iota^2 \times \iota^2$.

Proof. Note that random dynamical system is pullback asymptotically compact in E by Lemma 4.1 and 4.2. On the other hand, the random dynamical system has a pullback absorbing set by Lemma 4.1 in [2]. Then the existence and uniqueness of a random attractor follow from Theorem 2.1 immediately. $\qquad\square$

Acknowledgements

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Li, H. (2019) A Random Dynamical System of the Stochastic Lattice Reversible Selkov Equations. (Submitted)

[2] Li, H. (2019) Pullback Absorbing Set for the Stochastic Lattice Selkov Equations. *Asian Research Journal of Mathematics*, **14**, 1-7. https://doi.org/10.9734/arjom/2019/v14i130116

[3] Chen, H. and Tang, Y. (2019) Proof of Arts-Llibre-Valls's Conjectures for the Higgins-Selkov and the Selkov Systems. *Journal of Differential Equations*, **11**, 7638-7657. https://doi.org/10.1016/j.jde.2018.12.011

[4] Joan, C., Jaume, L. and Vallsb, C. (2018) Dynamics of the Higgins-Selkov and Selkov Systems. *Chaos, Solitons and Fractals*, **114**, 145-150. https://doi.org/10.1016/j.chaos.2018.07.007

[5] Selkov, E. (1968) Self-Oscillations in Glycolysis. I. A Simple Kinetic Model. *European Journal of Biochemistry*, **4**, 79-86.

https://doi.org/10.1111/j.1432-1033.1968.tb00175.x

[6] Huang, J. (2007) The Random Attractor of Stochastic FitzHugh-Nagumo Equations in an Infinite Lattice with White Noises. *Physica D*, **233**, 83-94. https://doi.org/10.1016/j.physd.2007.06.008

[7] Li, H. and Tu, J. (2015) Random Attractors for Stochastic Lattice Reversible Gray-Scott Systems with Additive Noise. *Electronic Journal of Differential Equations*, **2015**, 1-25.

[8] Li, H., You, Y. and Tu, J. (2015) Random Attractors and Averaging for Non-Autonomous Stochastic Wave Equations with Nonlinear Damping. *Journal of Differential Equations*, **258**, 148-190. https://doi.org/10.1016/j.jde.2014.09.007

[9] Bates, P.W., Lisei, H. and Lu, K. (2006) Attractors for Stochastic Lattice Dynamical Systems. *Stochastics and Dynamics*, **6**, 1-21. https://doi.org/10.1142/S0219493706001621

[10] Li, H. (2016) Random Attractor of the Stochastic Lattice Reversible Selkov Equations with Additive Noises. *IEEE* 13*th International Conference on e-Business Engineering*, Macau, 4-6 November 2016, 176-181. https://doi.org/10.1109/ICEBE.2016.038

[11] Témam, R. (1988) Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Applied Mathematical Sciences, Vol. 68, Springer-Verlag, New York. https://doi.org/10.1007/978-1-4684-0313-8