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N-Order Fixed Point Theory for *N*-Order Generalized Meir-Keeler Type Contraction in Partially Ordered Metric Spaces

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Abstract

This paper concerns *N*-order fixed point theory in partially ordered metric spaces. For the sake of simplicity, we start our investigations with the tripled case. We define tripled generalized Meir-Keeler type contraction which extends the definition of [Bessem Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. 72 (2010), 4508-4517]. We then discuss the existence and uniqueness of tripled fixed point theorems in partially ordered metric spaces. For general cases, we generalized our results to the *N*-order case. The results will promote the study of *N*-order fixed point theory.

Keywords

Tripled Fixed Point, Meir-Keeler Type Contraction, Partially Ordered Set, *N*-Order Fixed Point

1. Introduction and Preliminaries

Banach contraction principle [1] is classical and powerful in fixed point theory. It has been widely generalized (see [2] [3] [4] and others). Recently, fixed point theory in partially ordered metric spaces has been presented by many scholars: Ran and Reurings [5], Agarwal *et al.* [6], Bhsakar and Lakshmikantham [7], Samet [8], Berinde and Borcut [9], Amini-Harandi [10], etc., considered some coupled and tripled fixed point theorems. For more fixed point theorems in partially ordered metric spaces, one can refer to [11] [12] [13] and others.

This paper focuses on the tripled and *N*-order fixed point theory. For convenience, we denote $\mathcal{N}^+=\{1,2,\cdots,n,\cdots\}$. Let $(X,\leq,d)$ denote a partially ordered set $(X,\leq)$ endowed a metric d (*i.e.*, (X,d) is a metric space). Our work is carried out on the following two preliminaries: a result about fixed point in partially ordered metric space in [6] and a definition of generally Meir-Keeler type function for the case of coupled fixed points in [8].

Lemma 1.1 ([6]). *Let* $(X,\leq,d)$ *be a partially ordered metric space and suppose the metric space* (X,d) *is complete. Assume there is a nondecreasing function* $\varphi:[0,\infty)\to[0,\infty)$ *with* $\varphi^n(t)\to 0$ *as* $n\to\infty$ *for each* $t>0$. *If* $f:X\to X$ *is a nondecreasing mapping with*

$$d\left(f(x),f(y)\right)\leq\varphi\left(d(x,y)\right),\forall x\geq y.$$

Assume that either

1) f *is continuous or,*

2) *If a nondecreasing sequence* $x_n\to x$, *then* $x_n\leq x,\forall n\in\mathcal{N}^+$.

If $x_0\in X$ *with* $x_0\leq f(x_0)$ *then* f *has a fixed point. If for each* $x,y\in X$, *there exists* $z\in X$ *which is comparable to x and y, then the fixed point of* f *is unique.*

Definition 1 ([8]) *Let* $(X,\leq,d)$ *be a partially ordered metric space and* $F:X\times X\to X$ *be a mapping. F is called generalized Meir-Keeler type function if for all* $\varepsilon>0$ *there exists* $\delta(\varepsilon)>0$ *such that*

$$\begin{aligned}&x\geq u,y\leq v,\varepsilon\leq\frac{1}{2}\left(d(x,u)+d(y,v)\right)<\varepsilon+\delta(\varepsilon)\\&\Rightarrow d\left(F(x,y),F(u,v)\right)<\varepsilon.\end{aligned}\tag{1.1}$$

Let $(X,\leq)$ be a partially ordered set with a metric *d* on *X*, $\mathcal{M}=X\times X\times X$ and $F:\mathcal{M}\to X$ be a given mapping. Let $\preceq$ be the partial order on $\mathcal{M}$: $(x,y,z)\preceq(u,v,w)\Leftrightarrow x\leq u,y\leq v,z\leq w$. We employ the notion of tripled fixed point introduced by Samet and Vetro which is investigated by Amini-Harandi [10].

Definition 2 ([11]) *An element* $x,y,z\in X$ *is called a tripled fixed point of* $F:\mathcal{M}\to X$ *if*

$$F(x,y,z)=x,F(y,z,x)=y,F(z,y,x)=z.$$

In this paper, we first define *N*-order generalized Meir-Keeler type contraction by adding some parameters (see Definition 3 and Definition 5), which is an extension of Definition 1. Then we use a simple approach introduced by [10] to discuss *N*-order fixed point theorems. We start our discussions with the tripled case. Section 2 devotes to tripled fixed point theorems. Section 3 devotes to *N*-order fixed point theory. Section 4 gives two examples to illustrate the results obtained in Section 2.

2. Tripled Fixed Point Theory

Recalling that $(X,\leq,d)$ is a partially ordered set with a metric *d* on *X* and $\mathcal{M}=X\times X\times X$. Let ρ be the metric and $\preceq$ be the partially order on $\mathcal{M}$. For each $(x,y,z),(u,v,w)\in\mathcal{M}$, we define

$$\rho\left((x,y,z),(u,v,w)\right)=d(x,u)+d(y,v)+d(z,w)$$

$$(x,y,z)\preceq(u,v,w)\Leftrightarrow x\le u, y\le v, z\le w$$

and

$(x,y,z)\prec(u,v,w)\Leftrightarrow$ at least one of the inequalities $x<u, y<v$ and $z<w$ hold.

Now, we define tripled generalized Meir-Keeler type contraction which is a useful tool for the following theorems in this section.

Definition 3 *Let* $(X,\le,d)$ *be a partially ordered metric space and* $F:\mathcal{M}\to X$ *be a mapping. F is called a tripled generalized Meir-Keeler type contraction if for all* $\varepsilon>0$ *there exists* $\delta(\varepsilon)>0$ *such that*

$$\begin{aligned}&(x,y,z)\preceq(u,v,w),\varepsilon\le\frac{1}{3}\left[ld(x,u)+kd(y,v)+jd(z,w)\right]<\varepsilon+\delta(\varepsilon)\\&\Rightarrow d\left(F(x,y,z),F(u,v,w)\right)<\varepsilon\end{aligned}\tag{2.1}$$

where l,k,j *are constants with* $0<l+k+j<3$.

Theorem 2.1 *Let* $(X,\le,d)$ *be a partially ordered metric space. Let* l,k,j *be the given constants with* $0<l+k+j<3$. *If* $F:\mathcal{M}\to X$ *is a tripled generalized Meir-Keeler contraction mapping, then*

$$d\left(F(x,y,z),F(u,v,w)\right)<\frac{1}{3}\left[ld(x,u)+kd(y,v)+jd(z,w)\right]$$

for all $(x,y,z)\prec(u,v,w)$.

Proof. Let $(x,y,z),(u,v,w)\in\mathcal{M}$ such that $(x,y,z)\prec(u,v,w)$. Then it follows that

$$\frac{1}{3}\left[ld(x,u)+kd(y,v)+jd(z,w)\right]>0.$$

Setting

$$\varepsilon=\frac{1}{3}\left[ld(x,u)+kd(y,v)+jd(z,w)\right],$$

we have

$$0<\varepsilon=\frac{1}{3}\left[ld(x,u)+kd(y,v)+jd(z,w)\right]<\varepsilon+\delta(\varepsilon).$$

By $F:\mathcal{M}\to X$ being a tripled generalized Meir-Keeler type contraction, then

$$d\left(F(x,y,z),F(u,v,w)\right)<\varepsilon=\frac{1}{3}\left[ld(x,u)+kd(y,v)+jd(z,w)\right].\qquad\square$$

Let $F:\mathcal{M}\to X$ be a mapping. We say F is nondecreasing in each of its variables if

$$x_1,x_2\in X, x_1<x_2\Rightarrow F(x_1,y,z)<F(x_2,y,z),\forall y,z\in X,$$
$$y_1,y_2\in X, y_1<y_2\Rightarrow F(x,y_1,z)<F(x,y_2,z),\forall x,z\in X,$$

and

$$z_1,z_2\in X, z_1<z_2\Rightarrow F(x,y,z_1)<F(x,y,z_2),\forall x,y\in X.$$

By the monotone property of *F*, we can get

$$(x,y,z),(u,v,w)\in\mathcal{M},(x,y,z)\prec(u,v,w)\Rightarrow F(x,y,z)<F(u,v,w). \quad (2.2)$$

For all $n\in\mathcal{N}^{+}$, $n>1$, we define:

$$F^{n}(x,y,z)=F\left(F^{n-1}(x,y,z),F^{n-1}(y,z,x),F^{n-1}(z,x,y)\right) \quad (2.3)$$

with $F^{1}=F$.

In order to investigate the tripled fixed point of *F*, we introduce a mapping $T:\mathcal{M}\to\mathcal{M}$ which is defined by

$$T(x,y,z)=\left(F(x,y,z),F(y,z,x),F(z,x,y)\right). \quad (2.4)$$

Obviously, by the definition of ρ, we have

$$\begin{aligned}&\rho\left(T(x,y,z),T(u,v,w)\right)\\&=d\left(F(x,y,z),F(u,v,w)\right)+d\left(F(y,z,x),F(v,w,u)\right)\\&\quad+d\left(F(z,x,y),F(w,u,v)\right).\end{aligned} \quad (2.5)$$

Simultaneously, by (2.3) and (2.4), we have

$$T^{n}(x,y,z)=\left(F^{n}(x,y,z),F^{n}(y,z,x),F^{n}(z,x,y)\right)$$

with $T^{1}=T$, and we have

$$F^{n}(x,y,z)=F\left(T^{n-1}(x,y,z)\right).$$

Theorem 2.2 *Let* $(X,\leq,d)$ *be a partially ordered metric space and* l,k,j *be the given constants with* $0<l+k+j<3$. *Let* $F:\mathcal{M}\to X$ *be nondecreasing in each of its variables and be a tripled generalized Meir-Keeler type contraction. There exist* $(x,y,z),(u,v,w)\in\mathcal{M}$ *with* $(x,y,z)\prec(u,v,w)$. *Then, for* $n\in\mathcal{N}^{+}$, *we have*

1) $T^{n}(x,y,z)\prec T^{n}(u,v,w)$;
2) $\rho\left(T^{n+1}(x,y,z),T^{n+1}(u,v,w)\right)<\rho\left(T^{n}(x,y,z),T^{n}(u,v,w)\right)$;
3) $\rho\left(T^{n}(x,y,z),T^{n}(u,v,w)\right)\to 0,n\to\infty$.

Proof. We first prove 1). Since $(x,y,z)\prec(u,v,w)$, due to the monotone property of *F* and (2.2), we have $F(x,y,z)<F(u,v,w)$, $F(y,z,x)<F(v,w,u)$ and $F(z,x,y)<F(w,u,v)$. By $T^{1}=T$ and (2.4), 1) holds for $n=1$. Now we assume 1) holds for $n\in\mathcal{N}^{+}$, *i.e.*

$$\begin{aligned}&\left(F^{n}(x,y,z),F^{n}(y,z,x),F^{n}(z,x,y)\right)\\&=T^{n}(x,y,z)\prec T^{n}(u,v,w)\\&=\left(F^{n}(u,v,w),F^{n}(v,w,u),F^{n}(w,u,v)\right).\end{aligned}$$

Then, we obtain

$$F\left(F^{n}(x,y,z),F^{n}(y,z,x),F^{n}(z,x,y)\right)<F\left(F^{n}(u,v,w),F^{n}(v,w,u),F^{n}(w,u,v)\right)$$

which means $F^{n+1}(x,y,z)<F^{n+1}(u,v,w)$. Using the same strategy, we have $F^{n+1}(y,z,x)<F^{n+1}(v,w,u)$ and $F^{n+1}(z,x,y)<F^{n+1}(w,u,v)$. Hence we have $T^{n+1}(x,y,z)\prec T^{n+1}(u,v,w)$, that is, 1) holds for $n+1$. Simultaneously, we can also obtain that $T^{n}(y,z,x)\prec T^{n}(v,w,u)$ and $T^{n}(z,x,y)\prec T^{n}(w,u,v)$.

Now, we prove 2). We consider

$$\rho\left(T^{n+1}(x,y,z),T^{n+1}(u,v,w)\right)$$
$$=d\left(F^{n+1}(x,y,z),F^{n+1}(u,v,w)\right)+d\left(F^{n+1}(y,z,x),F^{n+1}(v,w,u)\right)$$
$$+d\left(F^{n+1}(z,x,y),F^{n+1}(w,u,v)\right).$$

It follows from Theorem 2.1 and 1) that

$$d\left(F^{n+1}(x,y,z),F^{n+1}(u,v,w)\right)$$
$$=d\left(F\left(T^{n}(x,y,z)\right),F\left(T^{n}(u,v,w)\right)\right)$$
$$<\frac{1}{3}\Big[ld\left(F^{n}(x,y,z),F^{n}(u,v,w)\right)+kd\left(F^{n}(y,z,x),F^{n}(v,w,u)\right)$$
$$+jd\left(F^{n}(z,x,y),F^{n}(w,u,v)\right)\Big],$$

$$d\left(F^{n+1}(y,z,x),F^{n+1}(v,w,u)\right)$$
$$=d\left(F\left(T^{n}(y,z,x)\right),F\left(T^{n}(v,w,u)\right)\right)$$
$$<\frac{1}{3}\Big[ld\left(F^{n}(y,z,x),F^{n}(v,w,u)\right)+kd\left(F^{n}(z,x,y),F^{n}(w,u,v)\right)$$
$$+jd\left(F^{n}(x,y,z),F^{n}(u,v,w)\right)\Big]$$

and

$$d\left(F^{n+1}(z,x,y),F^{n+1}(w,u,v)\right)$$
$$=d\left(F\left(T^{n}(z,x,y)\right),F\left(T^{n}(w,u,v)\right)\right)$$
$$<\frac{1}{3}\Big[ld\left(F^{n}(z,x,y),F^{n}(w,u,v)\right)+kd\left(F^{n}(x,y,z),F^{n}(u,v,w)\right)$$
$$+jd\left(F^{n}(y,z,x),F^{n}(v,w,u)\right)\Big].$$

Thus,

$$\rho\left(T^{n+1}(x,y,z),T^{n+1}(u,v,w)\right)$$
$$<\frac{1}{3}(l+k+j)\Big[d\left(F^{n}(x,y,z),F^{n}(u,v,w)\right)+d\left(F^{n}(y,z,x),F^{n}(v,w,u)\right)$$
$$+d\left(F^{n}(z,x,y),F^{n}(w,u,v)\right)\Big]$$
$$<\rho\left(T^{n}(x,y,z),T^{n}(u,v,w)\right).$$

Last, we prove 3). From 2), we know that $\lim_{n\to\infty}\rho\left(T^{n}(x,y,z),T^{n}(u,v,w)\right)$ exists. If $\lim_{n\to\infty}\rho\left(T^{n}(x,y,z),T^{n}(u,v,w)\right)\neq 0$, we suppose that

$$\lim_{n\to\infty}\frac{1}{3}\rho\left(T^{n}(x,y,z),T^{n}(u,v,w)\right)=\varepsilon>0. \tag{2.6}$$

Then it follows that

$$\frac{1}{3}\rho\left(T^{n}(x,y,z),T^{n}(u,v,w)\right)\geq\varepsilon,n\in\mathcal{N}^{+}.$$

By (2.6), we have

$$\lim_{n\to\infty}\frac{1}{3}\Big[ld\left(F^n(x,y,z),F^n(u,v,w)\right)+kd\left(F^n(y,z,x),F^n(v,w,u)\right)$$
$$+jd\left(F^n(z,x,y),F^n(w,u,v)\right)\Big]=\varepsilon$$

which implies that there exists $m_0\in\mathcal{N}^+$ such that

$$\begin{aligned}\varepsilon&\le\frac{1}{3}\Big[ld\left(F^{m_0}(x,y,z),F^{m_0}(u,v,w)\right)+kd\left(F^{m_0}(y,z,x),F^{m_0}(v,w,u)\right)\\&\quad+jd\left(F^{m_0}(z,x,y),F^{m_0}(w,u,v)\right)\Big]\\&<\varepsilon+\delta(\varepsilon).\end{aligned}\tag{2.7}$$

Since F is a tripled generalized Meir-Keeler type contraction, we get

$$\begin{aligned}\varepsilon&>d\Big(F\left(F^{m_0}(x,y,z),F^{m_0}(y,z,x),F^{m_0}(z,x,y)\right),\\&\qquad F\left(F^{m_0}(u,v,w),F^{m_0}(v,w,u),F^{m_0}(w,u,v)\right)\Big)\\&=d\left(F^{m_0+1}(x,y,z),F^{m_0+1}(u,v,w)\right).\end{aligned}\tag{2.8}$$

By (2.7), we also have

$$\begin{aligned}\varepsilon&\le\frac{1}{3}\Big[kd\left(F^{m_0}(y,z,x),F^{m_0}(v,w,u)\right)+jd\left(F^{m_0}(z,x,y),F^{m_0}(w,u,v)\right)\\&\quad+ld\left(F^{m_0}(x,y,z),F^{m_0}(u,v,w)\right)\Big]\\&<\varepsilon+\delta(\varepsilon),\end{aligned}$$

and

$$\begin{aligned}\varepsilon&\le\frac{1}{3}\Big[jd\left(F^{m_0}(z,x,y),F^{m_0}(w,u,v)\right)+ld\left(F^{m_0}(x,y,z),F^{m_0}(u,v,w)\right)\\&\quad+kd\left(F^{m_0}(y,z,x),F^{m_0}(v,w,u)\right)\Big]\\&<\varepsilon+\delta(\varepsilon).\end{aligned}$$

Then, we get

$$\varepsilon>d\left(F^{m_0+1}(y,z,x),F^{m_0+1}(v,w,u)\right)\tag{2.9}$$

and

$$\varepsilon>d\left(F^{m_0+1}(z,x,y),F^{m_0+1}(w,u,v)\right).\tag{2.10}$$

From (2.8)-(2.10), we get

$$\begin{aligned}&\frac{1}{3}\rho\left(T^{m_0+1}(x,y,z),T^{m_0+1}(u,v,w)\right)\\&=\frac{1}{3}\Big[d\left(F^{m_0+1}(x,y,z),F^{m_0+1}(u,v,w)\right)+d\left(F^{m_0+1}(y,z,x),F^{m_0+1}(v,w,u)\right)\\&\quad+d\left(F^{m_0+1}(z,x,y),F^{m_0+1}(w,u,v)\right)\Big]\\&<\varepsilon.\end{aligned}$$

This is a contradiction. The proof is completed.

From the definition of T, we observe that the fixed point of T is exactly the tripled fixed point of F, that is,

$$(x,y,z)=T(x,y,z)\Leftrightarrow x=F(x,y,z),y=F(y,z,x),z=F(z,x,y).$$

We will obtain the tripled fixed point theorems by investigating the fixed point of *T*.

Theorem 2.3 *Let* $(X,\le,d)$ *be a partially ordered metric space and* (X,d) *is a complete metric space. Let* l,k,j *be the given constants with* $0<l+k+j<3$. *Let* $F:\mathcal{M}\to X$ *be nondecreasing in each of its variables and be a tripled generalized Meir-Keeler contraction.* $T:\mathcal{M}\to\mathcal{M}$ *be a mapping defined as* (2.4) *satisfying that there exists* $(x_0,y_0,z_0)\in\mathcal{M}$ *with* $(x_0,y_0,z_0)\preceq T(x_0,y_0,z_0)$. *Then, there exists* $(x^*,y^*,z^*)\in\mathcal{M}$ *which is a tripled fixed point of F, if either*

1) *F is continuous or*

2) *a nondecreasing sequence* $(x_n,y_n,z_n)\to(x,y,z)$, *then* $(x_n,y_n,z_n)\preceq(x,y,z),\forall n\in\mathcal{N}^+$.

Furthermore, if

3) *for* $(x,y,z),(u,v,w)\in\mathcal{M}$, *there exists* $(a,b,c)\in\mathcal{M}$ *that is comparable to* (x,y,z) *and* (u,v,w), *we get the uniqueness of tripled fixed point of F and* $x^*=y^*=z^*$.

Proof. Since (X,d) is a complete metric space, it is obvious that the metric space (M,ρ) is complete. By Theorem 2.2, *T* is non-decreasing. Meanwhile, by Theorem 2.1 and (2.5), for each $(x,y,z),(u,v,w)\in\mathcal{M}$ with $(x,y,z)\preceq(u,v,w)$, we have

$$\begin{aligned}&\rho\big(T(x,y,z),T(u,v,w)\big)\\&=d\big(F(x,y,z),F(u,v,w)\big)+d\big(F(y,z,x),F(v,w,u)\big)+d\big(F(z,x,y),F(w,u,v)\big)\\&<\frac{1}{3}(l+k+j)\rho\big((x,y,z),(u,v,w)\big).\end{aligned}$$

By Lemma 1.1, we deduce that *T* has a unique fixed point denoted by (x^*,y^*,z^*), then (x^*,y^*,z^*) is the unique tripled fixed point of *F*.

However, we can check that (y^*,z^*,x^*) is also a tripled fixed point of *F*. In fact, since (x^*,y^*,z^*) is the tripled fixed point of *F*, *i.e.*, $x^*=F(x^*,y^*,z^*),y^*=F(y^*,z^*,x^*),z^*=F(z^*,x^*,y^*)$, we have

$$y^*=F(y^*,z^*,x^*),z^*=F(z^*,x^*,y^*),x^*=F(x^*,y^*,z^*)$$

which implies that (y^*,z^*,x^*) is also a tripled fixed point of *F*. By the uniqueness, we get $x^*=y^*=z^*$. □

Corollary 1 *Suppose that all the hypotheses of Theorem* 2.3 *are satisfied, then the tripled fixed point* (x^*,y^*,z^*) *can be deduced by*

$$F^n(x_0,y_0,z_0)\to x^*,F^n(y_0,z_0,x_0)\to y^*,F^n(z_0,x_0,y_0)\to z^*,\text{ as } n\to\infty. \quad (2.11)$$

Proof. By examining the proof of Theorem 2.3, (x^*,y^*,z^*) is actually the fixed point of *T* on $\mathcal{M}$. According to the proof of Lemma 1.1 in [6], we have

$$\lim_{n\to\infty}T^n(x_0,y_0,z_0)=(x^*,y^*,z^*).$$

By the definition of T^n, we can easily get (2.11). □

Theorem 2.4 *In addition to the hypotheses of Theorem* 2.3 *except* (3), *we have* $x^*=y^*=z^*$ *by adding the hypotheses* (3^*): x_0,y_0,z_0 *in X are compara-*

ble.

Proof. Without the restriction of the generality, we assume that $x_0 \le y_0 \le z_0$. Setting $(x_1,y_1,z_1)=(x_0,y_0,z_0)$ and $(u_1,v_1,w_1)=(y_0,z_0,z_0)$, it's easy to see that $(x_1,y_1,z_1)\preceq(u_1,v_1,w_1)$. From Theorem 1.1, we have $\rho\left(T^n(x_1,y_1,z_1),T^n(u_1,v_1,w_1)\right)\to 0$ as $n\to\infty$, which implies that

$$d\left(F^n(x_1,y_1,z_1),F^n(u_1,v_1,w_1)\right)\to 0, n\to\infty,$$

i.e.,

$$d\left(F^n(x_0,y_0,z_0),F^n(y_0,z_0,z_0)\right)\to 0, n\to\infty. \quad (2.12)$$

By the similar strategy, setting $(x_2,y_2,z_2)=(y_0,z_0,x_0)$ and $(u_2,v_2,w_2)=(y_0,z_0,z_0)$, we can get

$$d\left(F^n(y_0,z_0,x_0),F^n(y_0,z_0,z_0)\right)\to 0, n\to\infty. \quad (2.13)$$

It follows from the triangular inequality that

$$\begin{aligned} d\left(x^*,y^*\right) \le\, & d\left(x^*,F^n(x_0,y_0,z_0)\right)+d\left(F^n(x_0,y_0,z_0),F^n(y_0,z_0,z_0)\right) \\ & +d\left(F^n(y_0,z_0,z_0),F^n(y_0,z_0,x_0)\right)+d\left(F^n(y_0,z_0,x_0),y^*\right). \end{aligned}$$

Taking the limit as $n\to\infty$, by (2.11), (2.12) and (2.13), we get $x^*=y^*$. Similarly, by setting

$$(x_3,y_3,z_3)=(y_0,z_0,x_0),(u_3,v_3,w_3)=(z_0,z_0,y_0)$$

and

$$(x_4,y_4,z_4)=(z_0,x_0,y_0),(u_4,v_4,w_4)=(z_0,z_0,y_0),$$

we can get two equalities,

$$d\left(F^n(y_0,z_0,x_0),F^n(z_0,z_0,y_0)\right)\to 0, n\to\infty \quad (2.14)$$

and

$$d\left(F^n(z_0,x_0,y_0),F^n(z_0,z_0,y_0)\right)\to 0, n\to\infty \quad (2.15)$$

respectively. Then it follows from (2.11), (2.14) and (2.15) that

$$\begin{aligned} d\left(y^*,z^*\right) \le\, & d\left(y^*,F^n(y_0,z_0,x_0)\right)+d\left(F^n(y_0,z_0,x_0),F^n(z_0,z_0,y_0)\right) \\ & +d\left(F^n(z_0,z_0,y_0),F^n(z_0,x_0,y_0)\right)+d\left(F^n(z_0,x_0,y_0),z^*\right) \\ \to\, & 0. \end{aligned}$$

We get $y^*=z^*$. Hence we have $x^*=y^*=z^*$. □

3. *N*-Order Fixed Point Theorems

Let $(X,\le,d)$ be a partially ordered set with a metric d on X. Let $\mathcal{K}=X^N$, η be the metric on $\mathcal{K}$ and $\preceq$ be the partially order. For each $x=(x_1,\cdots,x_N), y=(y_1,\cdots,y_N)\in\mathcal{K}$, we define

$$\eta(x,y)=d(x_1,y_1)+\cdots+d(x_N,y_N)$$

$$x\preceq y \Leftrightarrow x_1\le y_1,\cdots,x_N\le y_N$$

and

$$x \prec y \Leftrightarrow \text{ there exists } 1 \le i \le N, \text{ such that } x_i < y_i.$$

Definition 4 [11] *Let X be a non-empty set and* $F : \mathcal{K} \to X$ *be a given mapping. An element* $x \in \mathcal{K}$ *is called a N-order fixed point of F if*

$$x_1 = F(x_1, \cdots, x_N), x_2 = F(x_2, \cdots, x_N, x_1), \cdots, x_N = F(x_N, x_1, \cdots, x_{N-1}).$$

We introduce generally *N*-order generalized Meir-Keeler type contraction.

Definition 5 *Let* $(X, \le, d)$ *be a partially ordered metric space and* $F : \mathcal{K} \to X$ *be a mapping. F is called a N-order generalized Meir-Keeler contraction if for all* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that for* $x, y \in \mathcal{K}$

$$\begin{aligned} & x \preceq y, \varepsilon \le \frac{k_1 d(x_1, y_1) + \cdots + k_N d(x_N, y_N)}{N} < \varepsilon + \delta(\varepsilon) \\ & \Rightarrow d(F(x), F(y)) < \varepsilon \end{aligned} \tag{3.16}$$

where $k_1, \cdots, k_N$ are constants with $0 < k_1 + \cdots + k_N < N$.

Substituting the tripled case with *N*-order case in the discussions of Section 3, by the similar strategy, we can obtain the same results with Theorem 2.1, Theorem 2.2, Theorem 2.3, Corollary 1 and Theorem 2.4.

4. The Examples

This section provides two examples to illustrate Theorem 2.3 and Theorem 2.4.

Example 1 This example is aroused by [13]. Let $X = R$, $d(x, y) = |x - y|$ and $F : \mathcal{M} \to X$, defined by

$$F = \frac{4x - 4y + 3z + 1}{15}.$$

It is easy to check that *F* satisfies all the hypotheses of Theorem 2.3 with

$$l = 1, k = 1, j = \frac{3}{4}, \delta(\varepsilon) = \frac{1}{4}\varepsilon$$

and $(x^*, y^*, z^*) = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$ is the unique tripled fixed point of *F*.

Example 2 Let

$$X = \{(0,1), (0,2), (1,3), (1,0), (2,0), (3,0)\}.$$

For $x = (x_1, x_2), y = (y_1, y_2) \in X$, $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ and $x \le y \Leftrightarrow x_1 \le y_1, x_2 \le y_2$. $F : \mathcal{M} \to X$ is defined by

$$F(x, y, z) = \begin{cases} (0,3), & x, y, z \in \{(0,1), (0,2), (0,3)\} \\ (3,3), & x, y, z \in \{(1,1), (2,0), (3,0)\} \end{cases} \tag{4.1}$$

It is easy to check that:

1) *F* is continues on $\mathcal{M}$;

2) *F* is a tripled generally Meir-Keeler type contraction. In fact, we can deduce that

$$d(F(x, y, z), F(u, v, w)) = 0 \text{ for each } (x, y, z) \preceq (u, v, w);$$

3) Setting $x_0 = y_0 = z_0 = (0,1)$, then we have

$F(x_0, y_0, z_0) = F(y_0, z_0, x_0) = F(z_0, x_0, y_0) = (0,3)$. Clearly, we have $(x_0, y_0, z_0) \prec T(x_0, y_0, z_0)$;

4) Setting $(x, y, z) = (0,1,0,2,0,3), (u, v, w) = (1,0,2,0,3,0)$, there are no elements in $\mathcal{M}$ which are comparable to (x, y, z) and (u, v, w).

The above 4) implies that F doesn't satisfy all the hypotheses of Theorem 2.3. However, the above 1)-3) imply that F satisfies all the hypotheses of Theorem 2.4, then F has the unique tripled fixed point (x^*, y^*, z^*) with $x^* = y^* = z^* = (0,3)$.

5. Conclusion

In this paper, we extend the definition generalized Meir-Keeler type contraction to N-ordered case. And we use it to discuss N-order fixed point theorems. In future work, we will study N-ordered fixed point theory with invariant set.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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