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Application of Exponential Kernel to Laplace Transform

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Abstract

In this paper, the exponential decreasing kernel is used in Laplace integral transform to transform a function from a certain domain to another domain. It is shown, in a rigorous way, that the Laplace transform of the delta function is exactly one half rather than one, as it is believed. In addition, when this kernel is used in integral transform of attractive and repulsive Coulomb potential, it yields a finite definite value at the point of singularity.

Keywords

Kernels, Integral Transforms, Laplace Transforms, Singularity

1. Introduction

Usually, kernels determine an implicit map that transforms a function or data from the input space to a feature space, and therefore determine its distribution in the latter space. This is usually accomplished through integral transforms. Some of the well-known kernels include the polynomial, exponential and Gaussian kernels. In particular, the exponential kernel through Laplace transform has been widely used over the years [1]-[6]. The Laplace transform is defined to transform a function from a space, say $x \in [0,\infty)$ to a space, say $s \in (0,\infty)$. Finding the Laplace transform of a function and its properties is normally discussed in standard mathematical physics books [7] [8]. An interesting function (more precisely a limit of some distribution) is the Dirac delta function, which has been in use in different settings [9]-[15]. The value of the Laplace transform of Delta-function can be found in mathematical physics books [8], where it is claimed that this value is one. We believe that the approach used to obtain this result is oversimplified and not rigorous. Therefore, one main object of this paper is to present a rigorous proof, through the use of a decreasing exponential kernel, and show that the correct value of Laplace transform of the delta function is exactly one half. The second part of this paper is to apply the decreasing exponential kernel to a discontinuous function. In particular, we consider a function with repulsive Coulomb-like form on the positive real axis, and with attractive Coulomb-like form on the negative real axis. This function is singular at the origin, and its right-hand and left-hand limits towards the origin are $+\infty$ and $-\infty$ respectively. It is shown, with this decreasing exponential kernel, that the value of this function is exactly zero which is the average between its limiting values at the origin. The last section of this paper is devoted for conclusion and discussion.

2. The Laplace Transform of Delta-Function

Consider the decreasing exponential kernel $e^{-s|x|}$ and the delta function $\delta(x)$. Our aim is to derive the Laplace transform of $\delta(x)$ by applying this kernel to the integral;

$$\int_{-\infty}^{\infty} e^{-s|x|}\delta(x)\,dx, \quad s>0 \tag{1}$$

Due to the well-known property of the delta-function, namely

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a), \tag{2}$$

Equation (1) becomes,

$$\int_{-\infty}^{\infty} e^{-s|x|}\delta(x)\,dx = e^{0} = 1. \tag{3}$$

Splitting the integral into two parts, we get

$$\int_{-\infty}^{\infty} e^{-s|x|}\delta(x)\,dx = \int_{-\infty}^{0} e^{-s|x|}\delta(x)\,dx + \int_{0}^{\infty} e^{-s|x|}\delta(x)\,dx. \tag{4}$$

In the first integral on the left-hand side, $|x| = -x$, and by letting $x \to -x$, we get

$$\int_{-\infty}^{0} e^{-s|x|}\delta(x) = \int_{\infty}^{0} e^{-sx}\delta(-x)(-dx) = \int_{0}^{\infty} e^{-sx}\delta(x)\,dx. \tag{5}$$

Note that, in the last step, we used the fact that $\delta(-x) = \delta(x)$, since it is even. So upon the substitution of Equation (5) into Equation (4), one gets

$$\int_{-\infty}^{\infty} e^{-s|x|}\delta(x)\,dx = 2\int_{0}^{\infty} e^{-sx}\delta(x)\,dx. \tag{6}$$

The Laplace transform of a function $f(x)$ is defined as

$$\mathcal{L}\{f(x)\} = f(s) = \int_{0}^{\infty} e^{-sx} f(x)\,dx. \tag{7}$$

Therefore, Equation (6) yields

$$\int_{-\infty}^{\infty} e^{-s|x|}\delta(x)\,dx = 2\mathcal{L}\{\delta(x)\}. \tag{8}$$

Hence, the use of Equation (3) gives the Laplace transform of $\delta(x)$, namely

$$\mathcal{L}\{\delta(x)\} = \frac{1}{2}. \tag{9}$$

The problem with the derivation of the unity value of the Laplace transform of the delta function, which is found in the literature [8], is overlooked at the lower limit ($x=0$) in the definition of the Laplace transform. The point $x=0$ separates the positive and the negative parts of the *x*-axis. So, when applying Equation (2), one must ensure that the point $x=a$ must be totally included in the range of integration. This is not satisfied for the present case, and therefore one has to examine the whole domain of the delta function. This is the main essence of our derivation.

3. Application of the Exponential Kernel to Coulomb-Like Function

Discontinuous functions arise in some physical situations and usually one has to determine the value of this function at its point of discontinuity. Examples of these problems are the electric field at charged conducting sphere [16], the energy loss in the two capacitor problem [17] and Fermi-Dirac distribution [18]. Here, we consider a Coulomb-like potential (attractive and repulsive on negative and positive real axis respectively). This kind of function is discontinuous at the origin. We will show that this function converges to its average value at its singular point ($r=0$).

In this section, we apply the decreasing exponential kernel to the Coulomb-like function which is given by

$$f(r)=\begin{cases}\dfrac{1}{r} & r>0\\ -\dfrac{1}{r} & r<0\end{cases}. \tag{10}$$

Consider the integral,

$$\int_{-\infty}^{\infty}\mathrm{e}^{-s|r|}f(r)\,\mathrm{d}r=\int_{-\infty}^{0}\mathrm{e}^{sr}\left(-\frac{1}{r}\right)\mathrm{d}r+\int_{0}^{\infty}\mathrm{e}^{-sr}\left(\frac{1}{r}\right)\mathrm{d}r. \tag{11}$$

Letting $r\to -r$ in the first integral of the left-hand side of the above equation, we get

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{e}^{-s|r|}f(r)\,\mathrm{d}r&=-\int_{\infty}^{0}\mathrm{e}^{-sr}\frac{1}{r}\mathrm{d}r+\int_{0}^{\infty}\mathrm{e}^{-sr}\frac{1}{r}\mathrm{d}r\\&=2\int_{0}^{\infty}\mathrm{e}^{-sr}\frac{1}{r}\mathrm{d}r=2\mathcal{L}\left\{\frac{1}{r}\right\}\end{aligned}. \tag{12}$$

Note that the function $f(r)$ is odd and the kernel is even so that the integral on the left-hand side of Equation (12) is zero. Two conclusions from the above equation are drawn: The first one is that the Laplace transform $\mathcal{L}(1/r)=0$. For the second conclusion, we first observe that the limit of the integral on the left-hand side of Equation (12) as $s\to\infty$, the kernel $\mathrm{e}^{-s|r|}\to 0$ except at the point $r=0$, at which it is just a constant. In this case, to ensure the vanishing of the integral on the left-hand side of Equation (12), the function $f(r)$ must vanish at the origin, *i.e.* $f(0)=0$. It is noticed that $\lim_{r\to r^+} f(r)=\infty$ and $\lim_{r\to r^-} f(r)=-\infty$, so that the average between these two limiting values is zero.

Therefore, our second conclusion is that the value of the function at its point of discontinuity converges to its average value between its two limiting values at that point.

4. Conclusion and Discussion

In this paper, a decreasing exponential kernel was used to derive the correct value of the Laplace transform of the delta function which is found to be one half. We also applied this type of kernel to a function which has a Coulomb-like form. Two conclusions of this application to such function were drawn: The first is that the Laplace transform of $\left(\frac{1}{r}\right)$ is zero and the second is that the value of this function at its point of discontinuity is the average value between its two limiting values about that point.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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