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Oscillatory and Asymptotic Behaviour of Solutions of Two Nonlinear Dimensional Difference Systems

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Abstract

This paper deals with the some oscillation criteria for the two dimensional difference system of the form: $\begin{aligned}\Delta x_n &= b_n y_n^\alpha \\ \Delta y_n &= -a_n x_n^\beta, n_0 \in N = 1,2,3,\cdots\end{aligned}$. Examples illustrating the results are inserted.

Keywords

Asymptotic, Two-Dimensional Difference Systems

1. Introduction

Consider a nonlinear two dimensional difference system of the form

$$\begin{aligned}\Delta x_n &= b_n y_n^\alpha \\ \Delta y_n &= -a_n x_n^\beta, n_0 \in N = 1,2,3,\cdots\end{aligned} \tag{1.1}$$

where $\{a_n\}$ and $\{b_n\}$ are real sequences and $n \in N(n_0)$, α and β are ratio of odd positive integers.

By a solution of Equation (1.1), we mean a real sequence $\{x_n\}$ which is defined for all $n \geq n_0$ and satisfies Equation (1.1) for all $n \in N(n_0)$.

In the last few decades there has been an increasing interest in obtaining necessary and sufficient conditions for the oscillation and nonoscillation of two dimensional difference equation. See for example [1]-[10] [11] and the references cited therein.

Further it will be assumed that $\{b_n\}$ is non-negative for all $n \geq n_0$, $u^\beta - v^\beta = \frac{u^\beta - v^\beta}{u-v}(u-v)$ for all u, v.

The oscillation criteria for system (1.1), when

$$\sum_{s=n_0}^{\infty} a_s = \infty \tag{1.2}$$

studied in [12]. Therefore in this paper we consider the other case that is

$$\sum_{s=n_0}^{\infty} a_s < \infty \tag{1.3}$$

and investigated the oscillatory behaviour of solutions of the system (1.1). Hence the results obtained in this paper complement to that of in [12].

We may introduce the function A_n defined by

$$A_n = \sum_{s=n+1}^{\infty} a_s, n \in N(n_0) \tag{1.4}$$

Throughout this paper condition (1.2) is tacitly assumed; A_n always denotes the function defined by (1.3).

In Section 2, we establish necessary and sufficient conditions for the system (1.1) to have solutions which behave asymptotically like nonzero constants or linear functions and in Section 3, we present criteria for the oscillation of all solutions of the system (1.1). Examples are inserted to illustrate some of the results in Section 4.

2. Existence of Bounded/Unbounded Solutions

In this section first we obtain necessary and sufficient conditions for the system (1.1) to have solutions which behave asymptotically like nonzero constants.

Theorem 2.1. If

$$\sum_{n=n_0}^{\infty} |A_n|^{\alpha} < \infty \tag{2.1}$$

and

$$\sum_{n=n_0}^{\infty} B_n^{\alpha} < \infty \tag{2.2}$$

are satisfied, then for any constant $c \neq 0$, system (1.1) has a solution $(\{x_n\},\{y_n\})$. such that

$$x_n = c + o\left(\sum_{s=n}^{\infty}\left(|A_s|^{\alpha} + B_s^{\alpha}\right)\right) \tag{2.3}$$

$$y_n = o\left(|A_n| + B_n\right)$$

as $n \to \infty$, where

$$B_n = \sum_{s=n+1}^{\infty} |A_s|^{\alpha+1} \tag{2.4}$$

Proof. We may assume without loss of generality that $c > 0$. Let

$$\mu = \max\left\{u^{\beta}; \frac{c}{2} \le u \le \frac{3c}{2}\right\}$$

$$\delta = \max\left\{\frac{u^{\beta}-v^{\beta}}{u-v}; \frac{c}{2} \le u, v \le \frac{3c}{2}\right\}$$

choose $\lambda > 0$, so that

$$M\delta(\mu)^{\alpha} = \frac{\lambda}{2} \tag{2.5}$$

and let $N \in \mathbb{N}(n_0)$ be large enough such that

$$M(\mu)^{\alpha}\sum_{n=N}^{\infty}|A_n|^{\alpha} \le \frac{c}{4} \tag{2.6}$$

$$M(\lambda)^{\alpha}\sum_{n=N}^{\infty}B_n^{\alpha} \le \frac{c}{4} \tag{2.7}$$

and

$$M\delta(\lambda)^{\alpha}\sum_{n=N}^{\infty}B_n^{\alpha} \le \frac{\lambda}{2} \tag{2.8}$$

Let B be the space of all real sequences $y = \{y_n\}, n \ge N$ with the topology of pointwise convergence. We now define X to be the set of sequences $x \in B$. such that

$$|x_n - c| \le \frac{c}{2}, n \ge N \tag{2.9}$$

and

$$\left|x_{n_1} - x_{n_2}\right| \le M\left((\mu)^{\alpha}\left|\bar{A}_n\right|^{\alpha} + (\lambda)^{\alpha} B_N^{\alpha}\right)|n_1 - n_2|, n_1, n_2 \ge N. \tag{2.10}$$

where $\left|\bar{A}_N\right| = \sup(|A_n| : n \ge N)$ and define Y to be the set of sequences $y \in B$. Such that

$$|y_n| \le \mu|A_n| + \lambda B_n, n \ge N. \tag{2.11}$$

Let T_1 and T_2 denote the mappings from $X \times Y \to B$ defined by

$$T_1(x,y)_n = C - \sum_{s=n}^{\infty} b_s y_s^{\alpha} n > N \tag{2.12}$$

and

$$T_2(x,y)_n = A_n x_{n+1}^{\beta} + \sum_{s=n}^{\infty} A_s b_s y_s^{\alpha} \frac{x_{s+1}^{\beta} - x_s^{\beta}}{x_{s+1} - x_s}, n \ge N. \tag{2.13}$$

Finally define $T : X \times Y \to B \times B$ by

$$T(x,y) = (T_1(x,y), T_2(x,y)), (x,y) \in X \times Y \tag{2.14}$$

Clearly $X \times Y$ is a bounded, closed and convex subset of $B \times B$.

First we show that T maps $X \times Y$ into itself. Let $(x,y) \in X \times Y$. From (2.11), we have

$$y_n^{\alpha} \le (\mu)^{\alpha}|A_n|^{\alpha} + (\lambda)^{\alpha} B_n^{\alpha}, n \ge N.$$

and so, using (2.6) and (2.7), we see that

$$\sum_{s=n}^{\infty} b_s y_s^{\alpha} \le M \sum_{s=n}^{\infty} \left((\mu)^{\alpha} |A_s|^{\alpha} + (\lambda)^{\alpha} (B_s)^{\alpha} \right)$$
$$\le M(\mu)^{\alpha} \sum_{s=n}^{\infty} |A_s|^{\alpha} + M(\lambda)^{\alpha} \sum_{s=n}^{\infty} B_s^{\alpha}$$
$$\le \frac{c}{4} + \frac{c}{4} = \frac{c}{2}, n \ge N.$$

Now from (2.12) it follows that

$$\left| T_1(x,y)_n - c \right| \le \frac{c}{2}, n \ge N.$$

Moreover,

$$\left| T_1(x,y)_{n_1} - T_1(x,y)_{n_2} \right|$$
$$= \left| \sum_{s=n_1}^{n_2-1} b_s y_s^{\alpha} \right| \le M \left| \sum_{s=n_1}^{n_2-1} (\mu)^{\alpha} |A_s|^{\alpha} + (\lambda)^{\alpha} B_s^{\alpha} \right|$$
$$\le M \left((\mu)^{\alpha} |A_N|^{\alpha} + (\lambda)^{\alpha} B_N^{\alpha} \right) |n_1 - n_2|$$

for $n_1, n_2 \ge N$. This implies that $T_1(x,y) \in X$. Next from (2.13), we have

$$\left| T_2(x,y)_n \right| \le \mu |A_n| + M\delta \sum_{s=n+1}^{\infty} |A_s| \left((\mu)^{\alpha} |A_s|^{\alpha} + (\lambda)^{\alpha} B_s^{\alpha} \right)$$
$$\le \mu |A_n| + M\delta (\mu)^{\alpha} \sum_{s=n+1}^{\infty} |A_s|^{\alpha+1} + M\delta (\lambda)^{\alpha} \sum_{s=n+1}^{\infty} |A_s| B_s^{\alpha}$$
$$\le \mu |A_n| + \frac{\lambda}{2} B_n + M\delta (\lambda)^{\alpha} B_n \left(\sum_{s=n+1}^{\infty} B_s^{\alpha} \right)$$
$$\le \mu |A_n| + \lambda B_n, n \ge N.$$

where conditions (2.5), (2.7) and (2.10) have been used. Thus $T_2(x,y) \in Y$. Hence $T(x,y) \in X \times Y$ as desired.

Now let $(x,y) = (x_n, y_n) \in X \times Y$ and for each $i = 1, 2, \cdots$. Let $(x^i, y^i) = (x_n^i, y_n^i)$ be a sequence in $X \times Y$. Such that $\lim_{i \to \infty} \left\| (x^i, y^i) - (x,y) \right\| = 0$. Then a straight forward argument $\lim_{i \to \infty} \left\| T(x_n^i, y_n^i) - T(x_n, y_n) \right\| = 0$ and hence T is continuous.

Finally, in order to apply Schauder-Tychonoff fixed point theorem, we need to show that $T(X \times Y)$ is relatively compact in $B \times B$. In view of recent result of cheng and patula [8] it suffices to show that $T(X \times Y)$ is uniformly cauchy in $B \times B$. To prove this, it is enough to show that $T_1(X \times Y)$ and $T_2(X \times Y)$ are uniformly cauchy in B. To this end, let $(x,y) = (x_n, y_n) \in X \times Y$ and observe that for any $k > n \ge N$, we have

$$\left| T_1(x,y)_k - T_1(x,y)_n \right| \le M \sum_{s=n+1}^{\infty} \left((\mu)^{\alpha} |A_s|^{\alpha} + (\lambda)^{\alpha} B_s^{\alpha} \right)$$

and

$$\left| T_2(x,y)_k - T_2(x,y)_n \right| = 2\mu |A_n| + M\delta \sum_{s=n+1}^{\infty} |A_s| \left((\mu)^{\alpha} |A_s|^{\alpha} + (\lambda)^{\alpha} B_s^{\alpha} \right)$$

It is now clear that for a given $\epsilon > 0$, we can choose $N_1 \ge N$, such that

$k > n \geq N_1$, imply $\left|T_1(x,y)_k - T_1(x,y)_n\right| < \epsilon$ and $\left|T_2(x,y)_k - T_2(x,y)_n\right| < \epsilon$. Thus $T_1(X \times Y)$ and $T_2(X \times Y)$ are uniformly cauchy and so $T(X \times Y)$ is uniformly cauchy. Thus $T(X \times Y)$ is relatively compact.

Therefore by Schauder-Tychonoff fixed point theorem, there is an element $(x,y) \in X \times Y$ such that $T(x,y) = (x,y)$. From (2.12), (2.13) and (2.14)

$$x_n = c - \sum_{s=n}^{\infty} b_s y_s^{\alpha} \tag{2.15}$$

$$y_n = A_n x_{n+1}^{\beta} + \sum_{s=n+1}^{\infty} A_s b_s y_s^{\alpha} \frac{x_{s+1}^{\beta} - x_s^{\beta}}{x_{s+1} - x_s} \tag{2.16}$$

From (2.15) and (2.16), we see that $(\{x_n\},\{y_n\})$ is a solution of then system (1.1) with the properties (2.3) and (2.4). This completes the proof of the theorem.

Corollary 2.2. Assume (2.1) and (2.2) are satisfied. Then for any $c \neq 0$ system (1.1) has a nonoscillatory solution $(\{x_n\},\{y_n\})$ such that

$$\begin{aligned} x_n &= c + o(1), \\ y_n &= o(1) \end{aligned} \tag{2.17}$$

as $n \to \infty$. The proof is left to the reader.

Before stating and proving our next results, we give a lemma which is concerned with the nonoscillatory solution of (1.1).

Lemma 2.3. Let $(\{x_n\},\{y_n\})$ be a solution of (1.1) for $n \geq N \in \mathbb{N}(n_0)$ with $x_n > 0$ for all $n \geq N$. Then

$$\sum_{i=N}^{\infty} \frac{b_i y_i^{\alpha+1} \frac{x_{n+1}^{\beta} - x_n^{\beta}}{x_{n+1} - x_n}}{x_{i+1}^{\beta} x_i^{\beta}} < \infty \tag{2.18}$$

and

$$\frac{y_n}{x_{n+1}^{\beta}} = \theta + A_n + \sum_{i=n+1}^{\infty} \frac{b_i y_i^{\alpha+1} \frac{x_{i+1}^{\beta} - x_i^{\beta}}{x_{i+1} - x_i}}{x_{i+1}^{\beta} x_i^{\beta}} \tag{2.19}$$

for $n \geq N$, where θ is a nonnegative constant.

This lemma has been proved by Graef and Thandapani [3] and is very useful in the following theorems. In our next theorem, we establish a necessary condition for the system (1.1) to have nonoscillatory solution satisfying condition (2.17).

Theorem 2.4. Assume that $A_n \geq 0$ for all $n \in \mathbb{N}(n_0)$. Then a necessary condition for the system (1.1) to have a nonoscillatory solution $(\{x_n\},\{y_n\})$ satisfying (2.17) is that

$$\sum_{n=n_0}^{\infty} b_n A_n^{\alpha} < \infty \text{ and } \sum_{n=n_0}^{\infty} b_n \left(\sum_{s=n+1}^{\infty} b_s A_s^{\alpha+1} \right)^{\alpha} < \infty. \tag{2.20}$$

Proof. Let (x_n, y_n) be a nonoscillatory solution of the system (1.1) for $n \in \mathbb{N}(n_0)$. Since b_n is not identically zero for $n \in \mathbb{N}(n_0)$. Hence x_n is nonoscillatory, without loss of generality, we may assume that x_n is eventually positive for $n \in \mathbb{N}(n_0)$. From Lemma 2.3, we have $y_n > 0$ for $n \geq N \geq n_0$ and

$$y_n \geq A_n x_{n+1}^{\beta}$$

and

$$y_n \geq x_{n+1}^{\beta} \sum_{i=n+1}^{\infty} \frac{b_i y_i^{\alpha+1} \frac{x_{i+1}^{\beta} - x_i^{\beta}}{x_{i+1} - x_i}}{x_i^{\beta} x_{i+1}^{\beta}}, n \geq N. \tag{2.21}$$

Since $A_n \to 0$ as $n \to \infty$, from the first equation of system (1.1), we obtain for $n \geq N$,

$$\Delta x_n \geq b_n \left(A_n x_{n+1}^{\beta}\right)^{\alpha} \geq b_n \left(A_n\right)^{\alpha} \left(x_{n+1}^{\beta}\right)^{\alpha}$$

and hence

$$\sum_{s=N}^{n-1} b_s A_s^{\alpha} \leq \sum_{s=N}^{n-1} \frac{\Delta x_s}{x_{s+1}^{\alpha\beta}} \tag{2.22}$$

Define $\gamma(t) = x_n + (t-n)\Delta x_n, n \leq t \leq n+1$. If $\Delta x_n \geq 0$, then $x_n \leq \gamma(t) \leq x_{n+1}$ and

$$\frac{\Delta x_n}{x_{n+1}^{\alpha\beta}} \leq \frac{\gamma'(t)}{\gamma(t)^{\alpha\beta}} \leq \frac{\Delta x_n}{x_n^{\alpha\beta}} \tag{2.23}$$

If $\Delta x_n < 0$, then $x_{n+1} \leq \gamma(t) \leq x_n$ and (2.23) again holds. From (2.22) and (2.23), we obtain

$$\sum_{s=N}^{n-1} b_s A_s^{\alpha} \leq \int_{x_N}^{x_n} \frac{\mathrm{d}s}{(s)^{\alpha\beta}}$$

which in view of the boundedness of x_n implies that

$$\sum_{n=N}^{\infty} b_n A_n^{\alpha} < \infty. \tag{2.24}$$

From the second inequality of (2.21) and the following inequality

$$\frac{b_n y_n^{\alpha+1} \frac{x_{n+1}^{\beta} - x_n^{\beta}}{x_{n+1} - x_n}}{x_n^{\beta} x_{n+1}^{\beta}} \geq \frac{b_n A_n \left(x_{n+1}\right)^{\beta} \left(A_n\right)^{\alpha} \left(x_{n+1}\right)^{\alpha\beta} \frac{x_{n+1}^{\beta} - x_n^{\beta}}{x_{n+1} - x_n}}{x_n^{\beta} x_{n+1}^{\beta}} \geq d b_n A_n^{\alpha+1}, n \geq N.$$

where "d" being the constant, we see that

$$d \sum_{s=n+1}^{\infty} b_s A_s^{\alpha+1} \leq \frac{y_n}{x_{n+1}^{\beta}}$$

Since $\sum_{s=n+1}^{\infty} b_s A_s^{\alpha+1} \to 0$ as $n \to \infty$, from the first equation of system (1.1), we obtain for $n \geq N$

$$\begin{aligned} \Delta x_n &\geq b_n \left(d \left(x_{n+1}^{\beta}\right) \sum_{s=n+1}^{\infty} b_s A_s^{\alpha+1} \right)^{\alpha} \\ &\geq b_n (d)^{\alpha} \left(x_{n+1}^{\beta}\right)^{\alpha} \left(\sum_{s=n+1}^{\infty} b_s A_s^{\alpha+1} \right)^{\alpha} \end{aligned}$$

Hence

$$(d)^{\alpha}\sum_{s=N}^{n-1} b_s\left(\sum_{i=s+1}^{\infty} b_i A_i^{\alpha+1}\right)^{\alpha} \le \sum_{s=N}^{n+1}\frac{\Delta x_s}{\left(x_{s+1}^{\beta}\right)^{\alpha}} \le \int_{x_N}^{x_n}\frac{ds}{(s)^{\alpha\beta}}$$

which in view of boundedness of x_n, implies that

$$\sum_{n=N}^{\infty} b_n\left(\sum_{i=n+1}^{\infty} b_i A_i^{\alpha+1}\right)^{\alpha} < \infty \tag{2.25}$$

The inequalities (2.24) and (2.25) clearly imply (2.20). This completes the proof.

we conclude this section with the following theorem which gives a necessary condition for the system (1.1) to have a nonoscillatory solution of the form

$$x_n = n\left(c+o(1)\right), y_n = c+o(1), \text{ as } n \to \infty. \tag{2.26}$$

Theorem 2.5. Assume $A_n \ge 0$ for $n \in \mathbb{N}(n_0)$. The system (1.1) has a solution of the type (2.26) for some $c \ne 0$, then

$$\sum_{n=n_0}^{\infty}\left|k_1^{\alpha\beta}(n+1)^{\alpha\beta}\right|\frac{k_2^{\beta}(n+1)^{\beta}-k_2^{\beta}n^{\beta}}{k_2}A_n^{\alpha+1} < \infty \tag{2.27}$$

for some $k_1, k_2 \ne 0$.

Proof. Let (x_n, y_n) be a solution of (1.1) satisfying (2.26). we may assume $c > 0$. Then there is an integer $N \in \mathbb{N}(n_0)$. such that

$$\frac{cn}{2} \le x_n \le 2cn \text{ for } n \ge N.$$

From Lemma 2.2, it follows that

$$y_n = \theta x_{n+1}^{\beta} + A_n x_{n+1}^{\beta} + x_{n+1}^{\beta}\sum_{s=n+1}^{\infty}\frac{b_s y_s^{\alpha+1}\frac{x_{s+1}^{\beta}-x_s^{\beta}}{x_{s+1}-x_s}}{x_s^{\beta}x_{s+1}^{\beta}} \tag{2.28}$$

for $n \ge N$, where θ is a nonnegative constant. Also from the second equation of (1.1), we have

$$y_n = \beta + A_n x_{n+1}^{\beta} - \sum_{s=N}^{n} A_s b_s y_s^{\alpha}\frac{x_{s+1}^{\beta}-x_s^{\beta}}{x_{s+1}-x_s} \tag{2.29}$$

where $\beta = y_{N-1} - A_{N-1}x_N^{\beta}$ combining (2.28) and (2.29), we have

$$\theta x_{n+1}^{\beta} + x_{n+1}^{\beta}\sum_{s=n+1}^{\infty}\frac{b_s y_s^{\alpha+1}\frac{x_{s+1}^{\beta}-x_s^{\beta}}{x_{s+1}-x_s}}{x_s^{\beta}x_{s+1}^{\beta}} = \beta - \sum_{s=N}^{n} A_s b_s y_s^{\alpha}\frac{x_{s+1}^{\beta}-x_s^{\beta}}{x_{s+1}-x_s} \tag{2.30}$$

since $y_n > 0$ by (2.29), (2.30) implies

$$\sum_{s=N}^{\infty} A_s b_s y_s^{\alpha}\frac{x_{s+1}^{\beta}-x_s^{\beta}}{x_{s+1}-x_s} < \infty \tag{2.31}$$

Using the inequality $y_n \ge A_n\left(x_{n+1}^{\beta}\right)$ in (2.31) we obtain

$$\sum_{n=N}^{\infty} b_n A_n^{\alpha+1}x_{n+1}^{\alpha\beta}\frac{x_{n+1}^{\beta}-x_n^{\beta}}{x_{n+1}-x_n} < \infty$$

If either $\frac{x_{n+1}^{\beta}-x_{n}^{\beta}}{x_{n+1}-x_{n}}$ is nonincreasing or nondecreasing holds, then (2.27) follows. This completes the proof of the theorem.

3. Oscillation Results

In this section we establish criteria for all solutions of the system (1.1) to be oscillatory. First, we consider the case where the composition of functions is storngly superlinear in the sense that

$$\int_{c}^{\infty}\frac{\mathrm{d}u}{(u)^{\alpha\beta}}<\infty$$

and

$$\int_{-c}^{-\infty}\frac{\mathrm{d}u}{(u)^{\alpha\beta}}<\infty \quad \text{for all } c>0. \tag{3.1}$$

Theorem 3.1. Let $A_n \geq 0$ for $n \in N(n_0)$ and (3.1) hold. If

$$\sum_{n=n_0}^{\infty} b_n\left((A_n)^{\alpha}+\sum_{s=n+1}^{\infty} b_s A_s^{\alpha+1}\right)^{\alpha}=\infty. \tag{3.2}$$

then the difference system (1.1) is oscillatory.

Proof. Assume the existence of nonoscillatory solution $(\{x_n\},\{y_n\})$ of the system (1.1) for $n \geq N \in \mathbb{N}(n_0)$. As in the proof of the Theorem 2.4, we may assume that $x_n > 0$ for all $n \geq N$. From Lemma 2.3, we have (2.22) Now following argument as in the proof of Theorem 2.5, we obtain

$$\sum_{s=N}^{n-1} b_s (A_s)^{\alpha} \leq \sum_{s=N}^{n-1}\frac{\Delta x_s}{x_{s+1}^{\alpha\beta}} \leq \int_{x_N}^{x_n}\frac{\mathrm{d}u}{(u)^{\alpha\beta}}, n \geq N.$$

Because of condition (3.1), the last inequality implies

$$\sum_{n=N}^{\infty} b_n (A_n)^{\alpha} < \infty. \tag{3.3}$$

Next from the second inequality (2.21), we have

$$y_n \geq \left(x_{n+1}^{\beta}\right)\frac{\sum_{s=n+1}^{\infty} b_s A_s^{\alpha+1}\left(x_{s+1}^{\alpha\beta}\right)\left(x_{s+1}^{\beta}-x_s^{\beta}\right)}{\left(x_{s+1}-x_s\right)x_s^{\beta}}$$

The last inequality implies

$$y_n \geq \left(x_{n+1}^{\beta}\right) d \sum_{s=n+1}^{\infty} b_s A_s^{\alpha+1}, n \geq N.$$

Again using the argument as in the proof of Theorem 2.5, we obtain

$$d^{\alpha}\sum_{s=N}^{n-1} b_s\left(\sum_{i=s+1}^{\infty} b_i A_i^{\alpha+1}\right)^{\alpha} \leq \sum_{s=N}^{n-1}\frac{\Delta x_s}{x_s^{\alpha\beta}} \leq \int_{x_N}^{x_n}\frac{\mathrm{d}u}{(u)^{\alpha\beta}}$$

for all $n \geq N$. So by condition on (3.1), we have

$$\sum_{n=N}^{\infty} b_n\left(\sum_{i=n+1}^{\infty} b_i A_i^{\alpha+1}\right)^{\alpha} < \infty. \tag{3.4}$$

The inequalities (3.3) and (3.4) thus obtained clearly contradicts (3.2). This contradiction completes the proof of the theorem.

Our final result is for the case when the composition of function is strongly sublinear in the sense that

$$\begin{gathered}\int_0^c \frac{\mathrm{d}u}{\left(\left(\lambda u\right)^{\alpha\beta}\right)^{\alpha}} < \infty \\ \int_0^{-c} \frac{\mathrm{d}u}{\left(\left(\lambda u\right)^{\alpha\beta}\right)^{\alpha}} < \infty\end{gathered} \tag{3.5}$$

for all $c > 0$ and $\lambda > 0$.

Theorem 3.2. Let $A_n \geq 0$ for $n \in \mathbb{N}(n_0)$ and (3.5) hold. If

$$\sum_{n=n_0}^{\infty} b_n A_n^{\alpha+1} R_n^{\alpha\beta} = \infty \tag{3.6}$$

where $R_n = \sum_{s=n_0}^{n} b_s$, then all solutions of the system (1.1) are oscillatory.

Proof. Let $(\{x_n\},\{y_n\})$ be a nonoscillatory solution of the system (1.1) for $n \geq N \in \mathbb{N}(n_0)$. As in the proof of Theorem 2.5, we may assume that $x_n > 0$ for $n \geq N$. From the Lemma 2.3 we have (2.21). Now summing the second equation of system (1.1) from $(n+1)$ to *j*, we obtain

$$y_n = y_j - A_j x_{J+1}^{\beta} + A_n x_{n+1}^{\beta} + \sum_{s=n+1}^{j} A_s b_s y_s^{\alpha} \frac{x_{s+1}^{\beta} - x_s^{\beta}}{x_{s+1} - x_s} \tag{3.7}$$

for $j \geq n+1 \geq N$. Note that

$$\sum_{s=n+1}^{\infty} A_s b_s y_s^{\alpha} \frac{x_{s+1}^{\beta} - x_s^{\beta}}{x_{s+1} - x_s} < \infty. \tag{3.8}$$

Since otherwise it would follow from (3.9) that $y_j - A_j x_{j+1}^{\beta} \to -\infty$ as $j \to \infty$, which contradicts the first inequality of (2.21). Therefore letting $j \to \infty$ in (3.9), we obtain

$$y_n = \eta + A_n x_{n+1}^{\beta} + \sum_{s=n+1}^{\infty} A_s b_s y_s^{\alpha} \frac{x_{s+1}^{\beta} - x_s^{\beta}}{x_{s+1} - x_s}, n \geq N. \tag{3.9}$$

where

$$\eta = \lim_{j \to \infty}\left(y_j - A_j x_{j+1}^{\beta}\right) \geq 0.$$

Define

$$\kappa_n = \sum_{s=n+1}^{\infty} b_s A_s^{\alpha+1} x_{s+1}^{\alpha\beta} \tag{3.10}$$

and in view of first inequality of (2.21) and (3.7), $\{\kappa_n\}$ is convergent.

From (3.11) and (3.12), we have $y_n \geq \lambda\kappa_n, n \geq N$.

Now substituting the value in the first equation of (1.1) and then summing the resulting inequality, we obtain

$$x_{n+1} \geq x_n \geq \sum_{s=N}^{n-1} b_s \lambda \kappa_s{}^{\alpha} \geq \left(\lambda\kappa_{n-1}\right)^{\alpha} R_{n-1},$$

Now using conditions (3.7) and (3.8)

$$\frac{b_n(A_n)^{\alpha+1}(x_{n+1})^{\alpha\beta}}{\left((\lambda\kappa_{n-1})^{\alpha\beta}\right)^{\alpha}} \ge b_n A_n^{\alpha+1}(R_n)^{\alpha\beta}$$

since $\Delta\kappa_{n-1} = -b_n(A_n)^{\alpha+1}(x_{n+1})^{\alpha\beta}$, the above inequality can be written as,

$$\frac{-\Delta\kappa_{n-1}}{\left((\lambda\kappa_{n-1})^{\alpha\beta}\right)^{\alpha}} \ge b_n A_n^{\alpha+1}(R_n)^{\alpha\beta} \tag{3.11}$$

observe that for $\kappa_n \le t \le \kappa_{n-1}$, we have $\left((\lambda\kappa_{n-1})^{\alpha\beta}\right)^{\alpha} \ge \left((\lambda t)^{\alpha\beta}\right)^{\alpha}$, and therefore

$$\frac{-\Delta\kappa_{n-1}}{\left((\lambda\kappa_{n-1})^{\alpha\beta}\right)^{\alpha}} \le \int_{\kappa_n}^{\kappa_{n-1}} \frac{\mathrm{d}t}{\left((\lambda t)^{\alpha\beta}\right)^{\alpha}} \tag{3.12}$$

Hence from (3.13) and (3.14), we obtain

$$\int_{\kappa_n}^{\kappa_N} \frac{\mathrm{d}t}{\left((\lambda t)^{\alpha\beta}\right)^{\alpha}} \ge \sum_{s=N}^{n-1} b_s A_s^{\alpha+1} R_s^{\alpha\beta}$$

which, in view of condition (3.5) and (3.8) provides a contradiction. This completes the proof of the theorem.

4. Examples

Example 4.1. Consider the system

$$\begin{aligned} \Delta x_n &= 2.3^{5n} y_n^5 \\ \Delta y_{n-1} &= \frac{-4}{3^n} x_n^5 \end{aligned} \tag{4.1}$$

Here $b_n = 2.3^{5n}$, $a_n = \frac{-4}{3^n}$, $y_n^{\alpha} = y^5$, $x_n^{\beta} = x_n^5$. All the necessary conditions of Theorem 3.1 are satisfied and hence the system (4.1) is oscillatory. Here, $(x_n, y_n) = \left((-1)^n, \left\{\frac{(-1)^{n+1}}{3^n}\right\}\right)$ is an oscillatory solution of the system (4.1).

Example 4.2. Consider the system

$$\begin{aligned} \Delta x_n &= y_n^{\frac{1}{3}} \\ \Delta y_{n-1} &= \frac{-\left(4+(-1)^n(4n+1)\right)}{n(n+1)} x_n, n \ge 1 \end{aligned} \tag{4.2}$$

Here $b_n = 1$, $a_n = \frac{-\left(4+(-1)^n(4n+1)\right)}{n(n+1)}$, $y_n^{\alpha} = y_n^{\frac{1}{3}}$ and $x_n^{\beta} = x_n$ with $A_{n-1} = \frac{4+(-1)^n}{n}, n \ge 1$. we see that all conditions of Theorem 3.2 are satisfied. Hence all solutions of the system (4.2) are oscillatory.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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