

Structure of Essential Spectrum and Discrete Spectrum of the Energy Operator of Three-Magnon Systems in the Isotropic Ferromagnetic Non-Heisenberg Model with Spin One and Nearest-Neighbor Interactions

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Abstract

We consider a three-magnon system in the isotropic ferromagnetic Non-Heisenberg model with spin one and with a coupling between nearest-neighbors. The structure of essential spectrum and discrete spectrum of the systems in a ν -dimensional lattice are investigated. We obtain the lower and upper estimates for the number of three-magnon bound states of the system.

Keywords

Essential Spectrum, Discrete Spectrum, Three-Magnon Bound State, Two-Magnon Bound State, Non-Heisenberg Model

1. Introduction

Two-magnon systems have attracted the attention of many researchers. Probably, such systems were first discussed by Bethe [1] in the context of one-dimensional integer-valued lattices. Bethe proved that no more than one bound state (BS) of the system can exist in the case of one-dimensional isotropic ferromagnet. N. Fukuda and M. Wortis [2] investigated the two-magnon systems in the one-dimensional Heisenberg ferromagnetic model and they have received the confirmation of Bethe results. Worts [3] examined the two-magnon system in a d -dimensional integer-valued lattice for an arbitrary d and proved that in this case, the system has $0, 1, 2, \dots, 2d$ BSs.

Majumdar [4] investigated the two-magnon system in a one-dimensional Heisenberg ferromagnet with a coupling between nearest and second nearest neighbors for the full quasi-momentum $\Lambda = \pi$. He found the spectrum and the BSs of the system numerically. In [5], such a system was examined for the case of a one-dimensional Heisenberg isotropic ferromagnet with a nearest- and second nearest-neighbor interactions for $\Lambda = \pi$ and $\Lambda = \frac{\pi}{2}$. The spectrum and the BSs of the system for these values of Λ were studied with numerical methods. Gochev [6] considered the two-magnon system in a one-dimensional Heisenberg longitudinal ferromagnet with a coupling between nearest and second nearest neighbors for an arbitrary full quasi-momentum. He investigated the spectrum and the BSs of the system analytically.

The two-magnon systems in the anisotropic Heisenberg model with a nearest-neighbor interaction were addressed in [7]. The focus in [8] was on two-magnon systems in a one-dimensional anisotropic Heisenberg ferromagnet with a interaction between nearest and second nearest neighbors. The spectrum and the BSs of such systems were investigated for all values of the full quasi-momentum.

The usual starting point for theoretical studies of magnetically organized matter is the Heisenberg exchange Hamiltonian (with an arbitrary spin s)

$$H = J \sum_{m,\tau} (\mathbf{S}_m \mathbf{S}_{m+\tau}), \quad (1)$$

where J is the bilinear exchange interaction parameter for nearest-neighbor atoms; $\mathbf{S}_m = (S_m^x, S_m^y, S_m^z)$ is the atomic spin operator of the m th node of the ν -dimensional integer-valued lattice Z^ν , and τ denotes summation over the nearest neighbors. However, the actual isotropic spin exchange Hamiltonian with an arbitrary spin s has the form [9]

$$H = \sum_{m,\tau} \sum_{n=1}^{2s} J_n (\mathbf{S}_m \mathbf{S}_{m+\tau})^n, \quad (2)$$

where J_n are the multipole exchange interaction parameters for nearest-neighbors atoms. Hamiltonian (2) coincides with Hamiltonian (1) only for $s = 1/2$, while there are terms with higher powers of $\mathbf{S}_m \mathbf{S}_{m+\tau}$ up to $(\mathbf{S}_m \mathbf{S}_{m+\tau})^{2s}$ inclusive for $s > 1/2$. These terms must be taken into account. Hamiltonian (2) is called the non-Heisenberg Hamiltonian.

Spectrum and BSs of two-magnon system in the non-Heisenberg ferromagnet with the bilinear and biquadratic exchange interactions were studied in works [10]-[17]. The spectrum and the BSs of two-magnon systems in a non-Heisenberg ferromagnet with coupling between nearest neighbors by bilinear and biquadratic interactions were investigated in [10] [11] [12] [13] [14]. Different methods, such as the Green's function method, the molecular field approximation method, the random phase approximation method, numerical methods, and the use of the creation and annihilation operators through the Holstein-Primakoff transformation, Dyson transformation, Dyson-Maleev transformation, Golghirch transformation,

and others, were applied in these works. In [15] [16], the spectrum and the BSs of this system were investigated for the case of a one-dimensional non-Heisenberg ferromagnet with $s=1$ and with a coupling between second nearest and third nearest neighbors respectively. The values of the Hamiltonian parameters for which the BSs exist were found, and the energies of these BSs were calculated. In [17], the spectrum and the BSs of two-magnon system were investigated in a ν -dimensional non-Heisenberg ferromagnet with $s=1$ and with a coupling between nearest neighbors.

In [18], the spectrum and the three-magnon BSs of three-magnon systems were investigated in a two-dimensional isotropic and anisotropic Heisenberg ferromagnet in bounded lattice with numerical methods.

In the work [19] [20] investigated the structure of essential spectrum and obtained the lower and upper estimates for the number of three-particle bound states (BS) of the energy operator of two-magnon system in a isotropic Heisenberg and Non-Heisenberg ferromagnet model with impurity in a ν -dimensional lattice Z^ν with nearest-neighbor interactions.

The spectrum and BSs of two-magnon systems in a non-Heisenberg ferromagnet with coupling between nearest-neighbors by linear and biquadratic interactions were investigated in [17].

2. Hamiltonian

In this paper, we investigate the structure of essential spectrum and we obtain the lower and upper estimates for the number of three-magnon bound states of the energy operator of three-magnon system in a isotropic Non-Heisenberg ferromagnet model with spin one and nearest-neighbor interactions in a ν -dimensional lattice Z^ν .

In this case the component S_m^z of spin operator S_m take up the values $1, 0, -1$, *i.e.* $S_m^z \varphi_0 = \varphi_0$, or $S_m^z \varphi_0 = 0$, or $S_m^z \varphi_0 = -\varphi_0$, where φ_0 be the vacuum vector. We consider these case the separately. From the beginning. We consider the case, when the spin component S_m^z take up the value 1.

The system Hamiltonian has the form

$$H = -J \sum_{m,\tau} (S_m S_{m+\tau}) - J_1 \sum_{m,\tau} (S_m S_{m+\tau})^2, \quad (3)$$

acts in the symmetrical Fock space \mathcal{H} , $S_m = (S_m^x, S_m^y, S_m^z)$ is the atomic spin $s=1$ operator in the node m , $J > 0, J_1 > 0$ are the respective the bilinear and biquadratic exchange interaction parameters for nearest-neighbor atoms of the lattice, and τ denotes summation over the nearest neighbors. We set $S_m^\pm = S_m^x \pm iS_m^y$, where S_m^- and S_m^+ are the respective magnon creation and annihilation operators at the site m . Let φ_0 be the so-called vacuum vector, which is fully determined by the conditions $S_m^+ \varphi_0 = 0$ and $S_m^z \varphi_0 = \varphi_0, \|\varphi_0\| = 1$. The vectors $S_m^- S_n^- S_l^- \varphi_0$ describe the state of the system of three magnons located at the nodes m, n and l . The vectors $S_m^- S_n^- S_l^- \varphi_0$ constitute an orthonormal system. We let \mathcal{H}_3 denote the Hilbert space spanned by these

vectors. It is called the space of three-magnon states of the operator H . The space \mathcal{H}_3 is invariant under operator H . We let H_3 denote the restriction of the operator H in the space \mathcal{H}_3 .

Theorem 1. The space \mathcal{H}_3 is invariant with respect to the operator H . The operator H_3 is a bounded self-adjoint operator. It generates the bounded self-adjoint operator \bar{H}_3 , acting in the space $l_2\left(\left(Z^\nu\right)^3\right)$ according to the formula

$$\begin{aligned}
& (\bar{H}_3 f)(p, q, r) \\
&= -J \sum_{p, q, r, \tau} \left\{ (\delta_{p, q+\tau} + \delta_{p, r+\tau} + \delta_{p+\tau, q} + \delta_{p+\tau, r} + \delta_{q, r+\tau} + \delta_{q+\tau, r} - 6) f(p, q, r) \right. \\
&+ \left(-\frac{1}{2} \delta_{p-\tau, q} - \frac{1}{2} \delta_{p-\tau, r} + 1 \right) f(p-\tau, q, r) + \left(-\frac{1}{2} \delta_{q-\tau, r} - \frac{1}{2} \delta_{p, q-\tau} + 1 \right) f(p, q-\tau, r) \\
&+ \left(-\frac{1}{2} \delta_{p, r-\tau} - \frac{1}{2} \delta_{q, r-\tau} + 1 \right) f(p, q, r-\tau) + \left(-\frac{1}{2} \delta_{p+\tau, q} - \frac{1}{2} \delta_{p+\tau, r} + 1 \right) f(p+\tau, q, r) \\
&+ \left. \left(-\frac{1}{2} \delta_{p, q+\tau} - \frac{1}{2} \delta_{q+\tau, r} + 1 \right) f(p, q+\tau, r) + \left(-\frac{1}{2} \delta_{p, r+\tau} - \frac{1}{2} \delta_{q, r+\tau} + 1 \right) f(p, q, r+\tau) \right\} \\
&- J_1 \sum_{p, q, r, \tau} \left\{ \left(-2\delta_{p-\tau, q} - 2\delta_{p-\tau, r} - 2\delta_{p+\tau, q} - 2\delta_{p+\tau, r} - 2\delta_{q, r+\tau} - 2\delta_{q+\tau, r} - \frac{3}{2} \delta_{p, q+\tau} \delta_{q, r} \right. \right. \\
&- \frac{3}{2} \delta_{p, q+\tau} \delta_{p, r} - \frac{3}{2} \delta_{p, r+\tau} \delta_{p, q} - \frac{3}{2} \delta_{p+\tau, q} \delta_{q, r} - \frac{3}{2} \delta_{p+\tau, r} \delta_{p, q} - \frac{3}{2} \delta_{p, r+\tau} \delta_{p, q} + 12 \\
&+ 2\delta_{p, q} + 2\delta_{q, r} + 2\delta_{p, r} \left. \right) f(p, q, r) + \left(\frac{1}{2} \delta_{p, q+\tau} + \frac{1}{2} \delta_{p, r+\tau} + \frac{3}{2} \delta_{p, q+\tau} \delta_{p, r} + \delta_{p, q+\tau} \delta_{q, r} \right. \\
&+ \frac{3}{2} \delta_{p, r+\tau} \delta_{p, q} - \delta_{p, q} - \delta_{p, r} - 2 \left. \right) f(p-\tau, q, r) + \left(\frac{1}{2} \delta_{q, r+\tau} + \frac{1}{2} \delta_{p+\tau, q} + \frac{3}{2} \delta_{p, r+\tau} \delta_{p, q} \right. \\
&+ \delta_{p+\tau, q} \delta_{p, r} + \frac{3}{2} \delta_{p+\tau, q} \delta_{q, r} - \delta_{p, q} - \delta_{p, r} - 2 \left. \right) f(p, q-\tau, r) + \left(\frac{1}{2} \delta_{p+\tau, r} + \frac{1}{2} \delta_{q+\tau, r} \right. \\
&+ \frac{3}{2} \delta_{p+\tau, q} \delta_{q, r} + \delta_{p+\tau, r} \delta_{p, q} + \frac{3}{2} \delta_{p, q+\tau} \delta_{p, r} - \delta_{p, r} - \delta_{q, r} - 2 \left. \right) f(p, q, r-\tau) \\
&+ \left(\frac{1}{2} \delta_{p+\tau, q} + \frac{1}{2} \delta_{p+\tau, r} + \frac{3}{2} \delta_{p+\tau, q} \delta_{p, r} + \delta_{p+\tau, q} \delta_{q, r} + \frac{3}{2} \delta_{p+\tau, r} \delta_{p, q} - \delta_{p, q} \right. \\
&- \delta_{p, r} - 2 \left. \right) f(p+\tau, q, r) + \left(\frac{1}{2} \delta_{p, q+\tau} + \frac{1}{2} \delta_{q+\tau, r} + \frac{3}{2} \delta_{p, q+\tau} \delta_{q, r} + \delta_{p, q+\tau} \delta_{p, r} \right. \\
&+ \frac{3}{2} \delta_{p+\tau, r} \delta_{p, q} - \delta_{p, q} - \delta_{q, r} - 2 \left. \right) f(p, q+\tau, r) + \left(\frac{1}{2} \delta_{p, r+\tau} + \frac{1}{2} \delta_{q, r+\tau} \right. \\
&+ \frac{3}{2} \delta_{p, q+\tau} \delta_{q, r} + \delta_{p, r+\tau} \delta_{p, q} + \frac{3}{2} \delta_{p+\tau, q} \delta_{p, r} - \delta_{p, r} - \delta_{q, r} - 2 \left. \right) f(p, q, r+\tau) \\
&+ (-\delta_{p, r+\tau} \delta_{p, q} + \delta_{p, q}) f(p-\tau, q-\tau, r) + (-\delta_{p, q+\tau} \delta_{p, r} + \delta_{p, r}) f(p-\tau, q, r-\tau) \\
&+ (-\delta_{p+\tau, q} \delta_{q, r} + \delta_{q, r}) f(p, q-\tau, r-\tau) \\
&+ \left(-\frac{3}{4} \delta_{p, r} \delta_{p, q+\tau} + \delta_{p, q+\tau} - \frac{3}{4} \delta_{q, r} \delta_{p, q+\tau} \right) f(p-\tau, q+\tau, r) \\
&+ \left(-\frac{3}{4} \delta_{p, q} \delta_{p, r+\tau} + \delta_{p, r+\tau} - \frac{3}{4} \delta_{q, r} \delta_{p, q+\tau} \right) f(p-\tau, q, r+\tau)
\end{aligned}$$

$$\begin{aligned}
 & + \left(-\frac{3}{4} \delta_{p,r} \delta_{p+\tau,q} + \delta_{p+\tau,q} - \frac{3}{4} \delta_{q,r} \delta_{p+\tau,q} \right) f(p+\tau, q-\tau, r) \\
 & + \left(-\frac{3}{4} \delta_{p,q} \delta_{p,r+\tau} + \delta_{q,r+\tau} - \frac{3}{4} \delta_{p,r} \delta_{p+\tau,q} \right) f(p, q-\tau, r+\tau) \\
 & + \left(-\frac{3}{4} \delta_{q,r} \delta_{p+\tau,q} + \delta_{p+\tau,r} - \frac{3}{4} \delta_{p,q} \delta_{p+\tau,r} \right) f(p+\tau, q, r-\tau) \\
 & + \left(-\frac{3}{4} \delta_{p,q} \delta_{p+\tau,r} + \delta_{q+\tau,r} - \frac{3}{4} \delta_{p,r} \delta_{p,q+\tau} \right) f(p, q+\tau, r-\tau) \\
 & + (-\delta_{p+\tau,r} \delta_{p,q} + \delta_{p,q}) f(p+\tau, q+\tau, r) \\
 & + (-\delta_{p+\tau,q} \delta_{p,r} + \delta_{p,r}) f(p+\tau, q, r+\tau) \\
 & + (-\delta_{p,q+\tau} \delta_{q,r} + \delta_{q,r}) f(p, q+\tau, r+\tau) \Big\},
 \end{aligned} \tag{4}$$

where $\delta_{k,j}$ is the Kronecker symbol. The operator H_3 acts on the vector $\psi \in \mathcal{H}_3$ according to the formula

$$H_3 \psi = \sum_{p,q,r} (\bar{H}_3 f)(p, q, r) S_p^- S_q^- S_r^- \varphi_0. \tag{5}$$

Proof. The proof is by direct calculation in which we use the well-known commutation relations between the operators $S_m^+, S_p^-,$ and S_n^z :

$$[S_m^+, S_p^-] = 2\delta_{m,p} S_m^z, \text{ and } [S_m^z, S_n^\pm] = \pm \delta_{m,n} S_m^\pm.$$

Lemma 1. The spectra of the operators H_3 and \bar{H}_3 coincide.

Proof. Because H_3 and \bar{H}_3 are bounded self-adjoint operators, it follows from the Weyl criterion that there exist a sequence of vectors ψ_n such that $\psi_n = \sum_{p,q,r} f_n(p, q, r) S_p^- S_q^- S_r^- \varphi_0$, $\|\psi_n\| = 1$, and

$$\lim_{n \rightarrow \infty} \|H_3 \psi_n - \lambda \psi_n\| = 0, \tag{6}$$

where $\lambda \in \sigma(H_3)$. On the other hand,

$$\begin{aligned}
 \|H_3 \psi_n - \lambda \psi_n\|^2 &= (H_3 \psi_n - \lambda \psi_n, H_3 \psi_n - \lambda \psi_n) \\
 &= \sum_{p,q,r} \left\| \bar{H}_3 f_n(p, q, r) - \lambda f_n(p, q, r) \right\|^2 \\
 &\quad \times \left(\frac{1}{\sqrt{8-8\delta_{p,q}\delta_{p,r}}} S_p^- S_q^- S_r^- \varphi_0, \frac{1}{\sqrt{8-8\delta_{p,q}\delta_{p,r}}} S_p^- S_q^- S_r^- \varphi_0 \right) \\
 &= \|\bar{H}_3 F_n - \lambda F_n\|^2 \times \left(\frac{1}{8-8\delta_{p,q}\delta_{p,r}} S_p^+ S_q^+ S_r^+ S_p^- S_q^- S_r^- \varphi_0, \varphi_0 \right) \\
 &= \|(\bar{H}_3 - \lambda) F_n\|^2 (\varphi_0, \varphi_0) = \|(\bar{H}_3 - \lambda) F_n\|^2 \rightarrow 0,
 \end{aligned}$$

$n \rightarrow \infty$. Here $F_n = (f_n(p, q, r))_{p,q,r \in \mathbb{Z}^v}$ and

$\|F_n\|^2 = \sum_{p,q,r} |f_n(p, q, r)|^2 = \|\psi_n\|^2 = 1$. It follows that $\lambda \in \sigma(\bar{H}_3)$. Consequently,

$\sigma(H_3) \subset \sigma(\bar{H}_3)$. Conversely, let $\bar{\lambda} \in \sigma(\bar{H}_3)$. Again by the Weyl criterion,

there then exist a sequence F_n such that $\|F_n\| = \sqrt{\sum_{p,q,r} |f_n(p, q, r)|^2} = 1$ and

$$\|\bar{H}_3 F_n - \bar{\lambda} F_n\| \rightarrow 0, \tag{7}$$

as $n \rightarrow \infty$.

Setting $\psi_n = \frac{1}{\sqrt{8-8\delta_{p,q}\delta_{p,r}}} \sum_{p,q,r} f_n(p,q,r) S_p^- S_q^- S_r^- \varphi_0$, we have

$\|\psi_n\| = \|F_n\| = 1$ and $\|\bar{H}_3 F_n - \bar{\lambda} F_n\| = \|H_3 \psi_n - \bar{\lambda} \psi_n\|$. This, together with Formula (7) and the Weyl criterion, implies that $\bar{\lambda} \in \sigma(H_3)$, and hence $\sigma(\bar{H}_3) \subset \sigma(H_3)$. These two relations imply that $\sigma(H_3) = \sigma(\bar{H}_3)$.

We let \mathcal{F} denote the Fourier transform:

$$\mathcal{F} : l_2\left((Z^\nu)^3\right) \rightarrow L_2\left((T^\nu)^3\right) \equiv \tilde{\mathcal{H}}_3,$$

where T^ν is a ν -dimensional torus with the normalized Lebesgue measure $d\lambda : \lambda(T^\nu) = 1$. We set $\tilde{H}_3 = \mathcal{F}\bar{H}_3\mathcal{F}^{-1}$.

Theorem 2. The Fourier transformation transforms the operator \bar{H}_3 into the bounded self-adjoint operator \tilde{H}_3 acting in the space $\tilde{\mathcal{H}}_3$ according to the formula

$$\begin{aligned} (\tilde{H}_3 f)(\lambda, \mu, \gamma) = & 4(J - 2J_1) \sum_{i=1}^{\nu} [3 - \cos \lambda_i - \cos \mu_i - \cos \gamma_i] f(\lambda, \mu, \gamma) \\ & - 2(J - J_1) \int_{T^\nu} \sum_{i=1}^{\nu} [\cos(\lambda_i - s_i) + \cos(\mu_i - s_i) - \cos s_i - \cos(\lambda_i + \mu_i - s_i)] \\ & \times f(s, \lambda + \mu - s, \gamma) ds - 2(J - J_1) \int_{T^\nu} \sum_{i=1}^{\nu} [\cos(\lambda_i - s_i) + \cos(\gamma_i - s_i) - \cos s_i \\ & - \cos(\lambda_i + \gamma_i - s_i)] f(s, \mu, \lambda + \gamma - s) ds - 2(J - J_1) \int_{T^\nu} \sum_{i=1}^{\nu} [\cos(\mu_i - s_i) \\ & + \cos(\gamma_i - s_i) - \cos s_i - \cos(\mu_i + \gamma_i - s_i)] f(\lambda, s, \mu + \gamma - s) ds \\ & - 4J_1 \int_{T^\nu} \sum_{i=1}^{\nu} [1 + \cos(\lambda_i + \mu_i) - \cos \lambda_i - \cos \mu_i] f(s, \lambda + \mu - s, \gamma) ds \\ & - 4J_1 \int_{T^\nu} \sum_{i=1}^{\nu} [1 + \cos(\lambda_i + \gamma_i) - \cos \lambda_i - \cos \gamma_i] f(s, \mu, \lambda + \gamma - s) ds \\ & - 4J_1 \int_{T^\nu} \sum_{i=1}^{\nu} [1 + \cos(\mu_i + \gamma_i) - \cos \mu_i - \cos \gamma_i] \times f(\lambda, s, \mu + \gamma - s) ds \\ & + J_1 \int_{T^\nu} \int_{T^\nu} \sum_{i=1}^{\nu} \left\{ -6[\cos(\lambda_i - s_i) + \cos(\mu_i - t_i) + \cos(\lambda_i + \mu_i - s_i - t_i)] \right. \\ & + 6[\cos(\lambda_i + \mu_i - t_i) + \cos(\mu_i - s_i - t_i) + \cos(\lambda_i + \mu_i - s_i) + \cos(\lambda_i - s_i - t_i) \\ & + \cos(\mu_i + \gamma_i - t_i) + \cos(\lambda_i + \gamma_i - s_i)] + 2[3\cos s_i + 3\cos t_i - 2\cos(s_i + t_i) \\ & - 2\cos(\lambda_i + \mu_i + \gamma_i - s_i) - 2\cos(\lambda_i + \mu_i + \gamma_i - t_i)] - 3[\cos(\lambda_i - t_i) \\ & + \cos(\mu_i - s_i) + \cos(\mu_i + \gamma_i - s_i - t_i) + \cos(\gamma_i - s_i) + \cos(\lambda_i + \gamma_i - s_i - t_i) \\ & \left. + \cos(\gamma_i - t_i)] \right\} f(s, t, \lambda + \mu + \gamma - s - t) ds dt. \end{aligned} \quad (8)$$

The following fact is important for further investigating the spectrum of the operator \tilde{H}_3 . Let the full quasi-momentum of the three-magnon system, *i.e.* sum of quasi-momentum of each three magnons $x + y + z = \Lambda \in T^\nu$ be fixed. Let $L_2(\Gamma_\Lambda)$ be the space of functions that are quadratically integrable over the manifold $\Gamma_\Lambda = \{(x, y, z) : x + y + z = \Lambda\}$. It is known [21] that the operator \tilde{H}_3

and space $\tilde{\mathcal{H}}_3$ can be expanded into the direct integrals

$\tilde{H}_3 = \oplus \int_{T^v} \tilde{H}_{3\Lambda} d\Lambda, \tilde{\mathcal{H}}_3 = \oplus \int_{T^v} \tilde{\mathcal{H}}_{3\Lambda} d\Lambda$ of the operators $\tilde{H}_{3\Lambda}$ and the spaces $\tilde{\mathcal{H}}_{3\Lambda}$ such that the spaces $\tilde{\mathcal{H}}_{3\Lambda}$ are invariant with respect to the operators $\tilde{H}_{3\Lambda}$.

In the isotropic non-Heisenberg Ferromagnet model with spin $s = 1$, the spectral properties of the considered operator of the energy of three-magnon systems are closely related to those of its two-magnon subsystems (see Formula (11)). We first study the spectrum and BSs of two-magnon systems.

3. Two-Magnon Bound States

The Hamiltonian of a two-magnon subsystem also has form (3). We let \mathcal{H}_2 denote the space of two-magnon states of the operator H . We let H_2 denote the restriction of H to the space \mathcal{H}_2 .

Theorem 3. The space \mathcal{H}_2 is invariant with respect of the operator H . The operator H_2 is a bounded self-adjoint operator. It generates the bounded self-adjoint operator \bar{H}_2 , acting in the space $l_2\left((Z^v)^2\right)$ according to the formula

$$\begin{aligned} (\bar{H}_2 f)(p, q) = & -J \sum_{p, q, \tau} \left\{ \left[\delta_{p, q+\tau} + \delta_{p+\tau, q} - 4 \right] f(p, q) - \frac{1}{2} \delta_{p-\tau, q} f(p-\tau, q) \right. \\ & - \frac{1}{2} \delta_{p, q-\tau} f(p, q-\tau) - \frac{1}{2} \delta_{p+\tau, q} f(p+\tau, q) - \frac{1}{2} \delta_{p, q+\tau} f(p, q+\tau) \\ & \left. + f(p+\tau, q) + f(p-\tau, q) + f(p, q+\tau) + f(p, q-\tau) \right\} \\ & - J_1 \sum_{p, q, \tau} \left\{ \left[2\delta_{p, q} - 2\delta_{p+\tau, q} - 2\delta_{p, q+\tau} + 8 \right] f(p, q) + \delta_{p, q+\tau} f(p-\tau, q+\tau) \right. \\ & + \delta_{p+\tau, q} f(p+\tau, q-\tau) - 2f(p-\tau, q) - 2f(p+\tau, q) + \delta_{p, q} f(p-\tau, q-\tau) \\ & + \delta_{p+\tau, q} f(p+\tau, q+\tau) + \frac{1}{2} \delta_{p, q+\tau} f(p-\tau, q) - \delta_{p, q} f(p-\tau, q) - 2f(p, q+\tau) \\ & - 2f(p, q-\tau) + \frac{1}{2} \delta_{p+\tau, q} f(p, q+\tau) - \delta_{p, q} f(p, q-\tau) + \frac{1}{2} \delta_{p, q+\tau} f(p, q+\tau) \\ & \left. + \frac{1}{2} \delta_{p+\tau, q} f(p+\tau, q) - \delta_{p, q} f(p, q+\tau) - \delta_{p, q} f(p+\tau, q) \right\}. \end{aligned} \tag{9}$$

The operator H_2 acts on the vector $\psi \in \mathcal{H}_2$ according to the formula

$$H_2 \psi = \sum_{p, q} (\bar{H}_2 f)(p, q) S_p^- S_q^- \varphi_0. \tag{10}$$

Lemma 2. The spectra of the operators H_2 and \bar{H}_2 coincide.

Theorem 4. The Fourier transformation transforms the operator \bar{H}_2 into the bounded self-adjoint operator \tilde{H}_2 acting in the space $\tilde{\mathcal{H}}_2$ according to the formula

$$(\tilde{H}_2 f)(x, y) = h(x, y) f(x, y) + \int_{T^v} h_1(x, y, t) f(t, x+y-t) dt, \tag{11}$$

where

$$h(x, y) = 8(J - 2J_1) \sum_{i=1}^v \left[1 - \cos \frac{x_i + y_i}{2} \cos \frac{x_i - y_i}{2} \right],$$

$$h_1(x, y, t) = -4(J - J_1) \sum_{i=1}^v \left[\cos \frac{x_i - y_i}{2} - \cos \frac{x_i + y_i}{2} \right] \cos \left(\frac{x_i + y_i}{2} - t_i \right) - 4J_1 \sum_{i=1}^v \left[1 - 2 \cos \frac{x_i + y_i}{2} \cos \frac{x_i - y_i}{2} + \cos(x_i + y_i) \right], x, y, t \in T^v.$$

Let the full quasi-momentum of the two-magnon system, *i.e.* sum of quasi-momentum of each two magnons $x + y = \Lambda \in T^v$ be fixed. Let $L_2(\Gamma_\Lambda)$ be the space of functions that are quadratically integrable over the manifold $\Gamma_\Lambda = \{(x, y) : x + y = \Lambda\}$. It is known [21] that the operator \tilde{H}_2 and space $\tilde{\mathcal{H}}_2$ can be expanded into the direct integrals $\tilde{H}_2 = \bigoplus_{\Lambda \in T^v} \tilde{H}_{2\Lambda} d\Lambda$, $\tilde{\mathcal{H}}_2 = \bigoplus_{\Lambda \in T^v} \tilde{\mathcal{H}}_{2\Lambda} d\Lambda$ of the operators $\tilde{H}_{2\Lambda}$ and the spaces $\tilde{\mathcal{H}}_{2\Lambda}$ such that the spaces $\tilde{\mathcal{H}}_{2\Lambda}$ are invariant with respect to the operators $\tilde{H}_{2\Lambda}$ and the operators $\tilde{H}_{2\Lambda}$ act in the space $\tilde{\mathcal{H}}_{2\Lambda}$ according to the formula

$$(\tilde{H}_{2\Lambda} f_\Lambda)(x) = h_\Lambda(x) f_\Lambda(x) + \int_{T^v} h_{1\Lambda}(x, t) f_\Lambda(t) dt, \quad (12)$$

where $h_\Lambda(x) = h(x, \Lambda - x)$, $h_{1\Lambda}(x, t) = h_1(x, \Lambda - x, t)$ and $f_\Lambda(x) = f(x, \Lambda - x)$.

It is known that the continuous spectrum of the operator $\tilde{H}_{2\Lambda}$ does not depend on the functions $h_{1\Lambda}(x, t)$ and consists of the intervals

$$G_\Lambda^v = [m_\Lambda^v, M_\Lambda^v],$$

where $m_\Lambda^v = \inf_{x \in T^v} h_\Lambda(x)$ and $M_\Lambda^v = \sup_{x \in T^v} h_\Lambda(x)$.

Definition 1. The eigenfunction $\varphi_\Lambda \in L_2(T^v \times T^v)$ of the operator $\tilde{H}_{2\Lambda}$ corresponding to the eigenvalue $z_\Lambda \notin G_\Lambda^v$ is called the bound state (BS) of the operator \tilde{H}_2 with quasi-momentum Λ , and the quantity z_Λ is called the energy of this BS.

We consider the operator $K_\Lambda(z)$, acting in the space $\tilde{\mathcal{H}}_{2\Lambda}$ according to the formula

$$K_\Lambda(z) f_\Lambda(x) = \int_{T^v} \frac{h_{1\Lambda}(x, t)}{h_\Lambda(t) - z} f_\Lambda(t) dt.$$

This operator is totally continuous in the space $\tilde{\mathcal{H}}_{2\Lambda}$ for values of $z \notin G_\Lambda^v = [m_\Lambda^v, M_\Lambda^v]$.

Let $\Delta_\Lambda^v(z) = \det D_\Lambda^v(z)$, where

$$D_\Lambda^v(z) = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \cdots & d_{1,v+1} \\ d_{2,1} & d_{2,2} & d_{2,3} & \cdots & d_{2,v+1} \\ d_{3,1} & d_{3,2} & d_{3,3} & \cdots & d_{3,v+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{v,1} & d_{v,2} & d_{v,3} & \cdots & d_{v,v+1} \\ d_{v+1,1} & d_{v+1,2} & d_{v+1,3} & \cdots & d_{v+1,v+1} \end{pmatrix},$$

and

$$d_{1,1} = 1 - 4J_1 \int_{T^v} \frac{g_\Lambda(s) ds}{h_\Lambda(s) - z},$$

$$d_{1,k+1} = -4(J - J_1) \int_{T^v} \frac{\xi_{\Lambda k}(s_k) ds}{h_\Lambda(s) - z}, k = \overline{1, v},$$

$$d_{k+1,1} = -4J_1 \int_{T^\nu} \frac{\eta_{\Lambda_k}(s_k) g_\Lambda(s) ds}{h_\Lambda(s) - z}, k = \overline{1, \nu},$$

$$d_{k+1,k+1} = 1 - 4(J - J_1) \int_{T^\nu} \frac{\eta_{\Lambda_k}(s_k) \xi_{\Lambda_k}(s_k) ds}{h_\Lambda(s) - z}, k = \overline{1, \nu},$$

$$d_{k+1,i+1} = -4(J - J_1) \int_{T^\nu} \frac{\eta_{\Lambda_k}(s_k) \xi_{\Lambda_i}(s_i) ds}{h_\Lambda(s) - z}, k = \overline{1, \nu}, i = \overline{1, \nu}, k \neq i.$$

In these formulas,

$$g_\Lambda(s) = \sum_{k=1}^\nu \left[1 + \cos \Lambda_k - 2 \cos \frac{\Lambda_k}{2} \cos \left(\frac{\Lambda_k}{2} - s_k \right) \right],$$

$$\xi_{\Lambda_k}(s_k) = \cos \left(\frac{\Lambda_k}{2} - s_k \right) - \cos \frac{\Lambda_k}{2}, \quad \eta_{\Lambda_k}(s_k) = \cos \left(\frac{\Lambda_k}{2} - s_k \right), k = \overline{1, \nu}.$$

Lemma 3. A number $z_0 \notin G_\Lambda^\nu$ be an eigenvalue of the operator $\tilde{H}_{2\Lambda}$ if and only if it is a zero of the function $D_\Lambda^\nu(z)$, i.e. $D_\Lambda^\nu(z_0) = 0$.

Proof. In the case under consideration, the equation for the eigenvalues is an integral equation with a degenerate kernel. It is therefore equivalent to a system of linear homogeneous algebraic equations. It is known that such a system has a nontrivial solution if and only if its determinant is equal to zero. In this case, the determinant of this linear homogeneous algebraic system is equal to function $\Delta_\Lambda^\nu(z)$.

Theorem 5. Let $J = 2J_1$ and ν be arbitrary. Then the operator \tilde{H}_2 has two BSs φ_1 and φ_2 (not taking the order of the energy degeneration into account) with the energy values $z_1 = -2J_1$, $z_2 = -(4\nu + 2)J_1 - 4J_1 \sum_{i=1}^\nu \cos \Lambda_i$, and z_1 is degenerate ν times, while z_2 is not degenerate, $z_i < m_\Lambda^\nu, i = 1, 2$, for all $\Lambda \in T^\nu$, i.e. the energy values of these BSs lie below the continuous spectrum domain of the operator \tilde{H}_2 .

Proof. If $J = 2J_1$, then $h_\Lambda(s) \equiv 0$, and

$$\Delta_\Lambda^\nu(z) = \left(1 + \frac{2J_1}{z} \right)^\nu \left\{ \left(1 + \frac{2J_1}{z} \right) \left[1 + \frac{4J_1}{z} \sum_{i=1}^\nu (1 + \cos \Lambda_i) \right] - \frac{16J_1^2}{z^2} \sum_{i=1}^\nu \cos^2 \frac{\Lambda_i}{2} \right\}.$$

Solving the equation $\Delta_\Lambda^\nu(z) = 0$, we prove the theorem.

Let $\tilde{\pi} = (\pi, \pi, \dots, \pi) \in T^\nu$.

Theorem 6. Let $\Lambda = \tilde{\pi}$ and $J \neq J_1$. Then the operator \tilde{H}_2 has only one BS φ with the energy value $z = 8\nu(J - 2J_1) - 2(J - J_1)$, and this energy level is degenerate ν times. In addition, if $J > J_1$, then $z < m_\Lambda^\nu$, and if $J < J_1$, then $z > M_\Lambda^\nu$. When $J = J_1$, this BS vanishes because it is incorporated into the continuous spectrum.

Proof. The proof of this theorem is based on the equality

$h_\Lambda(x) = 8\nu(J - 2J_1)$ with $\Lambda = \tilde{\pi}$ and also on the corresponding form of the

determinant
$$\Delta_\Lambda^\nu(z) = \left(1 - \frac{2(J - J_1)}{8\nu(J - 2J_1) - z} \right)^\nu.$$

From Theorem 5 and 6 and later is obviously, what the spectrum of the

Hamiltonian \mathcal{H} by different value of ν differ from one another.

In the case where $\nu = 1$, the change of the energy spectrum of the operator \tilde{H}_2 is described by the following theorems.

Theorem 7.

1) Let $J < J_1$ and $\Lambda \in]0, \pi[$ or $\Lambda \in]\pi, 2\pi[$.

a) If respectively $\cos \frac{\Lambda}{2} > -\frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda}{2} < \frac{J-J_1}{2J_1}$, then the operator \tilde{H}_2

has two BSs φ_1 and φ_2 with the corresponding energy values $z_i < m_\Lambda^1, i = 1, 2$.

b) If respectively $\cos \frac{\Lambda}{2} \leq -\frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda}{2} \geq \frac{J-J_1}{2J_1}$, then the operator \tilde{H}_2

has only one BS φ_1 with the energy value z_1 , and $z_1 < m_\Lambda^1$.

2) Let $J = J_1$ and $\Lambda \in]0, \pi[$ or $\Lambda \in]\pi, 2\pi[$.

a) If respectively $0 < \Lambda < \alpha_1$ or $\alpha_2 < \Lambda < 2\pi$, then the operator \tilde{H}_2 has only one BS φ with the energy value $z < m_\Lambda^1$.

b) If $\Lambda \in [\alpha_1, \pi[\cup]\pi, \alpha_2[$, then the operator \tilde{H}_2 has no BS. Above,

$$\alpha_1 \approx 100^\circ, \alpha_2 \approx 260^\circ, \text{ and } z = -8J_1 - \frac{8J_1 \cos^2 \frac{\Lambda}{2} \left(1 + 2\sqrt{3 + \cos^2 \frac{\Lambda}{2}} \right)}{3}.$$

3) Let $J_1 < J < 2J_1$ and $\Lambda \in]0, \pi[$ or $\Lambda \in]\pi, 2\pi[$.

a) If respectively $\cos \frac{\Lambda}{2} \leq \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda}{2} \geq -\frac{J-J_1}{2J_1}$, then the operator \tilde{H}_2

has two BSs φ_1 and φ_2 with the corresponding energy values $z_1 < m_\Lambda^1$, and $z_2 > M_\Lambda^1$.

b) If respectively $\cos \frac{\Lambda}{2} > \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda}{2} < -\frac{J-J_1}{2J_1}$, then the operator \tilde{H}_2

has three BSs φ_1 , φ_2 , and φ_3 with the corresponding energy values $z_1 < m_\Lambda^1$, and $z_i > M_\Lambda^1, i = 2, 3$.

4) Let $2J_1 < J < 3J_1$ and $\Lambda \in]0, \pi[$ or $\Lambda \in]\pi, 2\pi[$.

a) If respectively $\cos \frac{\Lambda}{2} > \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda}{2} < -\frac{J-J_1}{2J_1}$, then the operator \tilde{H}_2

has two BSs φ_1 and φ_2 with the corresponding energy values $z_i < m_\Lambda^1, i = 1, 2$.

b) If respectively $\cos \frac{\Lambda}{2} \leq \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda}{2} \geq -\frac{J-J_1}{2J_1}$, then the operator \tilde{H}_2

has only BS φ_1 with the energy value $z_1 < m_\Lambda^1$. In this case, the second BS vanishes because it is incorporated into the continuous spectrum.

5) Let $J = 3J_1$ and $\Lambda \neq 0$. Then the operator \tilde{H}_2 has only one BS φ with the energy value $z = 4J_1 \left(1 - \cos^2 \frac{\Lambda}{2} \right) < m_\Lambda^1$.

6) Let $J > 3J_1$ and $\Lambda \neq 0$. Then the operator \tilde{H}_2 has two BSs φ_1 and φ_2 with the corresponding energy values $z_1 < m_\Lambda^1$, and $z_2 > M_\Lambda^1$.

In the case where $\nu = 1$ and $\Lambda = 0$, the change of the energy spectrum is described by the following theorem.

Theorem 8.

- 1) If $J < J_1$ and $\Lambda = 0$, then the operator \tilde{H}_2 has two BSs φ_1 and φ_2 with the corresponding energy values $z_i < m_\Lambda^1, i = 1, 2$.
- 2) If $J = J_1$ and $\Lambda = 0$, then the operator \tilde{H}_2 has only one BS φ with the energy value $z = -\frac{64}{3}J_1 < m_\Lambda^1$.
- 3) If $J_1 < J < 2J_1$ and $\Lambda = 0$, then the operator \tilde{H}_2 has two BSs φ_1 and φ_2 with the corresponding energy values $z_1 < m_\Lambda^1$ and $z_2 > M_\Lambda^1$.
- 4) If $2J_1 < J < 3J_1$ and $\Lambda = 0$, then the operator \tilde{H}_2 has two BSs φ_1 and φ_2 with the corresponding energy values $z_i < m_\Lambda^1, i = 1, 2$.
- 5) If $J = 3J_1$ and $\Lambda = 0$, then the operator \tilde{H}_2 has no BS.
- 6) If $J > 3J_1$ and $\Lambda = 0$, then the operator \tilde{H}_2 has only one BS φ with the energy value $z > M_\Lambda^1$.

A sketch proof of Theorems 7-8 is given below. In the case under consideration, the equation for the eigenvalues is an integral equation with a degenerate kernel. It is therefore equivalent to a system of linear homogeneous algebraic equations. It is known that such a system has a nontrivial solution if and only if its determinant is equal to zero. In this case, the equation $\Delta_\Lambda^1(z) = 0$ is therefore equivalent to the equation stating that the determinant of the system is zero. Expressing all integrals in the equation $\Delta_\Lambda^1(z) = 0$, through the integral $J^*(z) = \int_T \frac{dt}{h_\Lambda(t) - z}$, we find that the equation $\Delta_\Lambda^1(z) = 0$ is equivalent to the equation

$$\begin{aligned}
 J^*(z) = & \left\{ 8(J - 2J_1)(J - 5J_1) \cos^2 \frac{\Lambda}{2} - (J - J_1)[z - 8(J - 2J_1)] \right\} \\
 & \times \left\{ 128J_1(J - 2J_1)^2 \cos^4 \frac{\Lambda}{2} + 8(J - 2J_1)(J + J_1) \right. \\
 & \left. \times \cos^2 \frac{\Lambda}{2} [z - 8(J - 2J_1)] + (J - J_1)[z - 8(J - 2J_1)]^2 \right\}^{-1}.
 \end{aligned} \tag{13}$$

Because $\frac{1}{h_\Lambda(t) - z}$ is a continuous function for $z \notin [m_\Lambda^1; M_\Lambda^1]$ and $[J^*(z)]' = \int_T \frac{dt}{[h_\Lambda(t) - z]^2} > 0$, the function $J^*(z)$ is an increasing function of z for $z \notin [m_\Lambda^1; M_\Lambda^1]$. Moreover, $J^*(z) \rightarrow 0$ as $z \rightarrow -\infty$, $J^*(z) \rightarrow +\infty$ as $z \rightarrow m_\Lambda^1 - 0$, $J^*(z) \rightarrow -\infty$ as $z \rightarrow M_\Lambda^1 + 0$ and $J^*(z) \rightarrow 0$ as $z \rightarrow +\infty$. Analysis of Equation (13) outside the set $G_\Lambda = [m_\Lambda^1; M_\Lambda^1]$, leads to the proof of Theorems 7-8.

The energy spectrum in the case where $\nu = 2$ for the full quasi momenta of the form $\Lambda = (\Lambda_1, \Lambda_2) = (\Lambda_0, \Lambda_0)$ is described below. It is easy to see that if the parameters J, J_1 , and Λ_0 satisfy the conditions of Theorems 7-8, the statements of the theorems are true. Only one additional BS $\tilde{\varphi}$ appears, whose energy value is \tilde{z} , moreover $\tilde{z} < m_\Lambda^2$ ($\tilde{z} > M_\Lambda^2$), if $J > J_1$ ($J < J_1$). If $J = J_1$,

the operator \tilde{H}_2 has no additional BS.

The proof of this statements is based on the fact that if $\nu = 2$ and $\Lambda = (\Lambda_0, \Lambda_0)$, then the function $\Delta_\Lambda^2(z)$ has the form

$$\Delta_\Lambda^2(z) = \left[1 - 2(J - J_1) \int_{T^2} \frac{\left[\cos\left(\frac{\Lambda_0}{2} - t_1\right) - \cos\left(\frac{\Lambda_0}{2} - t_2\right) \right]^2 dt_1 dt_2}{h_\Lambda(t_1, t_2) - z} \right] \Psi_\Lambda(z), \quad (14)$$

where

$$\begin{aligned} \Psi_\Lambda(z) = & \left\{ 1 - 8J_1 \int_{T^2} \frac{1 + \cos \Lambda_0 - \cos \frac{\Lambda_0}{2} \left[\cos\left(\frac{\Lambda_0}{2} - t_1\right) + \cos\left(\frac{\Lambda_0}{2} - t_2\right) \right]}{h_\Lambda(t_1, t_2) - z} dt \right\} \\ & \times \left[1 - 4(J - J_1) \int_{T^2} \frac{\cos\left(\frac{\Lambda_0}{2} - t_1\right) \left[\cos\left(\frac{\Lambda_0}{2} - t_1\right) + \cos\left(\frac{\Lambda_0}{2} - t_2\right) - 2\cos \frac{\Lambda_0}{2} \right]}{h_\Lambda(t_1, t_2) - z} dt \right] \\ & - 64(J - J_1) J_1 \int_{T^2} \frac{\left\{ 1 + \cos \Lambda_0 - \cos \frac{\Lambda_0}{2} \left[\cos\left(\frac{\Lambda_0}{2} - t_1\right) + \cos\left(\frac{\Lambda_0}{2} - t_2\right) \right] \right\} \cos\left(\frac{\Lambda_0}{2} - t_1\right)}{h_\Lambda(t_1, t_2) - z} dt_1 dt_2 \\ & \times \int_{T^2} \frac{\cos\left(\frac{\Lambda_0}{2} - t_1\right) - \cos \frac{\Lambda_0}{2}}{h_\Lambda(t_1, t_2) - z} dt_1 dt_2. \end{aligned}$$

The equation $\Delta_\Lambda^2(z) = 0$ is therefore equivalent to the equation

$$1 - 2(J - J_1) \int_{T^2} \frac{\left[\cos\left(\frac{\Lambda_0}{2} - t_1\right) - \cos\left(\frac{\Lambda_0}{2} - t_2\right) \right]^2 dt_1 dt_2}{h_\Lambda(t_1, t_2) - z} = 0 \quad (15)$$

and

$$\Psi_\Lambda(z) = 0. \quad (16)$$

It is easy to see that Equation (15) has a unique solution $\tilde{z} < m_\Lambda^2$ if $J > J_1$; if $J < J_1$, this solution satisfies the condition $\tilde{z} > M_\Lambda^2$. If $J = J_1$, Equation (15) has no solution. Expressing the integrals in Equation (16) through the integral

$J^*(z) = \int_{T^2} \frac{dt_1 dt_2}{h_\Lambda(t_1, t_2) - z}$, we obtain an equation of the form

$$\eta_\Lambda(z) J^*(z) = \xi_\Lambda(z)$$

where

$$\eta_\Lambda(z) = (J - J_1) \tilde{z}^2 + 16(J - 2J_1)(J + J_1) \cos^2 \frac{\Lambda_0}{2} \tilde{z} + 512J_1(J - 2J_1) \cos^4 \frac{\Lambda_0}{2},$$

and $\xi_\Lambda(z) = 16(J - 2J_1)(J - 5J_1) \cos^2 \frac{\Lambda_0}{2} - (J - J_1) \tilde{z}$. In this case,

$\tilde{z} = z - 16(J - 2J_1)$. In turn, for $\eta_\Lambda(z) \neq 0$, the latter equation is equivalent to the equation of the form

$$J^*(z) = \frac{\xi_\Lambda(z)}{\eta_\Lambda(z)}. \tag{17}$$

Analyzing Equation (17) outside the set G_Λ^2 and taking into account that the function $J^*(z)$ is monotonic for $z \notin [m_\Lambda^2; M_\Lambda^2]$, we obtain statements similar to the statements in Theorems 7-8.

For all other quasi momenta $\Lambda = (\Lambda_1, \Lambda_2), \Lambda_1 \neq \Lambda_2$, there exist sets $R_j, j = \overline{0, 5}$, of the parameters J, J_1 , and Λ such that in every set R_j the operator \tilde{H}_2 has exactly j BSs (taking the energy degeneration order into account) with the corresponding energy values $z_k \notin G_\Lambda^2, k = \overline{1, 5}$.

Indeed, in this case and for $\nu = 2$, the function $\Delta_\Lambda^2(z)$ has the form

$$\Delta_\Lambda^2(z) = \det D_\Lambda^2(z),$$

where $D_\Lambda^2(z) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$, and $a_1 = 1 - 4J_1 \int_{T^2} \frac{g_\Lambda(s) ds}{h_\Lambda(s) - z}$,

$$a_{k+1} = -4(J - J_1) \int_{T^2} \frac{f_{\Lambda_k}(s_k) ds}{h_\Lambda(s) - z}, k = 1, 2; \quad b_1 = -4J_1 \int_{T^2} \frac{\eta_{\Lambda_1}(s_1) g_\Lambda(s) ds}{h_\Lambda(s) - z},$$

$$b_2 = 1 - 4(J - J_1) \int_{T^2} \frac{\eta_{\Lambda_1}(s_1) f_{\Lambda_1}(s_1) ds}{h_\Lambda(s) - z}, \quad b_3 = -4(J - J_1) \int_{T^2} \frac{\eta_{\Lambda_1}(s_1) f_{\Lambda_2}(s_2) ds}{h_\Lambda(s) - z},$$

$$c_1 = -4J_1 \int_{T^2} \frac{\eta_{\Lambda_2}(s_2) g_\Lambda(s) ds}{h_\Lambda(s) - z}, \quad c_2 = -4(J - J_1) \int_{T^2} \frac{\eta_{\Lambda_2}(s_2) f_{\Lambda_1}(s_1) ds}{h_\Lambda(s) - z},$$

$$c_3 = 1 - 4(J - J_1) \int_{T^2} \frac{\eta_{\Lambda_2}(s_2) f_{\Lambda_2}(s_2) ds}{h_\Lambda(s) - z}.$$

In these formulas, $g_\Lambda(s) = \sum_{k=1}^2 \left[1 + \cos \Lambda_k - 2 \cos \frac{\Lambda_k}{2} \cos \left(\frac{\Lambda_k}{2} - s_k \right) \right]$,

$$f_{\Lambda_k}(s_k) = \cos \left(\frac{\Lambda_k}{2} - s_k \right) - \cos \frac{\Lambda_k}{2}, k = 1, 2, \quad \eta_{\Lambda_k}(s_k) = \cos \left(\frac{\Lambda_k}{2} - s_k \right), k = 1, 2,$$

$$\Lambda \in T^2, \quad t \in T^2.$$

Expressing all integrals in the equation $\Delta_\Lambda^2(z) = 0$ through $J^*(z)$ and rearranging algebraically, we reduce the latter equation to the form

$$\theta_\Lambda(z) J^*(z) = \chi_\Lambda(z), \tag{18}$$

where $\theta_\Lambda(z)$ is a fifth-order polynomial in z , and $\chi_\Lambda(z)$ is a lower-order polynomial in z . Analyzing Equation (18) outside the set G_Λ^2 and taking into account that the function $J^*(z)$ with $z \notin [m_\Lambda^2, M_\Lambda^2]$ is monotonic, we can easily verify that the equation has no more than five solutions outside the set G_Λ^2 .

We now consider the case of $\nu = 3$. Let the full quasi-momentum have the form $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) = (\Lambda_0, \Lambda_0, \Lambda_0)$. If the parameters Λ_0, J and J_1 satisfy the conditions in Theorems 7-8, then statements similar to those in the theorems

are true. Only one additional BS $\tilde{\varphi}$ appears, whose energy value is \tilde{z} . This energy level is twice degenerate and $\tilde{z} < m_\Lambda^3$ ($\tilde{z} > M_\Lambda^3$), if $J > J_1$ ($J < J_1$). This additional BS vanishes when $J = J_1$ because it is incorporated into the continuous spectrum.

To prove this, we note that in this case, the function $\Delta_\Lambda^3(z)$ has the form

$$\Delta_\Lambda^3(z) = \left[1 - 2(J - J_1) \int_{T^3} \frac{\left[\cos\left(\frac{\Lambda_0}{2} - t_1\right) - \cos\left(\frac{\Lambda_0}{2} - t_2\right) \right]^2 dt_1 dt_2 dt_3}{h_\Lambda(t_1, t_2, t_3) - z} \right]^2 \tilde{\Psi}_\Lambda(z), t \in T^3,$$

where

$$\begin{aligned} \tilde{\Psi}_\Lambda(z) = & \left\{ 1 - 4J_1 \int_{T^3} \frac{3 + 3\cos\Lambda_0 - 2\cos\frac{\Lambda_0}{2} \left(\sum_{i=1}^3 \cos\left(\frac{\Lambda_0}{2} - t_i\right) \right)}{h_\Lambda(t_1, t_2, t_3) - z} dt \right\} \\ & \times \left[1 - 4(J - J_1) \int_{T^3} \frac{\cos\left(\frac{\Lambda_0}{2} - t_1\right) \left[\sum_{i=1}^3 \cos\left(\frac{\Lambda_0}{2} - t_i\right) - 3\cos\frac{\Lambda_0}{2} \right]}{h_\Lambda(t_1, t_2, t_3) - z} dt \right] \\ & - 48(J - J_1) J_1 \int_{T^3} \frac{\left\{ 3 + 3\cos\Lambda_0 - 2\cos\frac{\Lambda_0}{2} \left[\sum_{i=1}^3 \cos\left(\frac{\Lambda_0}{2} - t_i\right) \right] \right\} \cos\left(\frac{\Lambda_0}{2} - t_1\right)}{h_\Lambda(t_1, t_2, t_3) - z} dt_1 dt_2 dt_3 \\ & \times \int_{T^3} \frac{\cos\left(\frac{\Lambda_0}{2} - t_1\right) - \cos\frac{\Lambda_0}{2}}{h_\Lambda(t_1, t_2, t_3) - z} dt_1 dt_2 dt_3. \end{aligned}$$

Therefore the equation $\Delta_\Lambda^3(z) = 0$ is equivalent to the equations

$$\left[1 - 2(J - J_1) \int_{T^3} \frac{\left[\cos\left(\frac{\Lambda_0}{2} - t_1\right) - \cos\left(\frac{\Lambda_0}{2} - t_2\right) \right]^2 dt_1 dt_2 dt_3}{h_\Lambda(t_1, t_2, t_3) - z} \right]^2 = 0 \quad (19)$$

and

$$\tilde{\Psi}_\Lambda(z) = 0. \quad (20)$$

It is easy to see that Equation (19) has a unique double solution z' if $J \neq J_1$ and $z' < m_\Lambda^3$ ($z' > M_\Lambda^3$), if $J > J_1$ ($J < J_1$). Expressing all integrals in Equation

(20) through $J^*(z) = \int_{T^3} \frac{dt_1 dt_2 dt_3}{h_\Lambda(t_1, t_2, t_3) - z}$, we obtain the equation

$$\tilde{\eta}_\Lambda(z) J^*(z) = \tilde{\theta}_\Lambda(z), \quad (21)$$

where

$$\tilde{\eta}_\Lambda(z) = (J - J_1) \tilde{z}^2 + 24(J + J_1)(J - 2J_1) \cos^2 \frac{\Lambda_0}{2} \tilde{z} + 1152J_1(J - 2J_1)^2 \cos^4 \frac{\Lambda_0}{2},$$

and $\tilde{\theta}_\Lambda(z) = 24(J - 2J_1)(J - 5J_1)\cos^2 \frac{\Lambda_0}{2} - (J - J_1)\tilde{z}$. Here

$\tilde{z} = z - 24(J - 2J_1)$. If $\tilde{\eta}_\Lambda(z) \neq 0$, Equation (21) is, in turn, equivalent to the equation

$$J^*(z) = \frac{\tilde{\theta}_\Lambda(z)}{\tilde{\eta}_\Lambda(z)}. \tag{22}$$

Analyzing Equation (22) outside the set G_Λ^3 and taking into account that the function $J^*(z)$ for $z \notin G_\Lambda^3$, is monotonic, we prove the statements made above.

If $\Lambda \neq (\Lambda_0, \Lambda_0, \Lambda_0)$, the system has no more than seven BSs (taking the energy degeneration order into account), and there exist sets $R_k, k = \overline{0, 7}$, of the parameters J, J_1 , and Λ such that in every set R_k the system has exactly k BSs. The energy values of these BSs lie outside the set G_Λ^3 . When passing from one of these sets to another, either some additional BSs of the operator \tilde{H}_2 appear or some existing BSs vanish.

In this case, the function $\Delta_\Lambda^3(z)$ has the form $\Delta_\Lambda^3(z) = \det D_\Lambda^3(z)$, where

$$D_\Lambda^3(z) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & a_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix},$$

and $a_1 = 1 - 4J_1 \int_{T^3} \frac{g_\Lambda(s) ds}{h_\Lambda(s) - z}$, $a_{k+1} = -4(J - J_1) \int_{T^3} \frac{f_{\Lambda_k}(s_k) ds}{h_\Lambda(s) - z}, k = 1, 2, 3$;

$b_1 = -4J_1 \int_{T^3} \frac{\eta_{\Lambda_1}(s_1) g_\Lambda(s) ds}{h_\Lambda(s) - z}$, $b_2 = 1 - 4(J - J_1) \int_{T^3} \frac{\eta_{\Lambda_1}(s_1) f_{\Lambda_1}(s_1) ds}{h_\Lambda(s) - z}$,

$b_{k+1} = -4(J - J_1) \int_{T^3} \frac{\eta_{\Lambda_1}(s_1) f_{\Lambda_k}(s_k) ds}{h_\Lambda(s) - z}, k = 2, 3$; $c_1 = -4J_1 \int_{T^3} \frac{\eta_{\Lambda_2}(s_2) g_\Lambda(s) ds}{h_\Lambda(s) - z}$,

$c_{k+1} = -4(J - J_1) \int_{T^3} \frac{\eta_{\Lambda_2}(s_2) f_{\Lambda_k}(s_k) ds}{h_\Lambda(s) - z}, k = 1, 3$;

$c_3 = 1 - 4(J - J_1) \int_{T^3} \frac{\eta_{\Lambda_2}(s_2) f_{\Lambda_2}(s_2) ds}{h_\Lambda(s) - z}$, $d_1 = -4J_1 \int_{T^3} \frac{\eta_{\Lambda_3}(s_3) g_\Lambda(s) ds}{h_\Lambda(s) - z}$,

$d_{k+1} = -4(J - J_1) \int_{T^3} \frac{\eta_{\Lambda_3}(s_3) f_{\Lambda_k}(s_k) ds}{h_\Lambda(s) - z}, k = 1, 2$;

$d_4 = 1 - 4(J - J_1) \int_{T^3} \frac{\eta_{\Lambda_3}(s_3) f_{\Lambda_3}(s_3) ds}{h_\Lambda(s) - z}$. In these formulas

$$g_\Lambda(s) = \sum_{k=1}^3 \left[1 + \cos \Lambda_k - 2 \cos \frac{\Lambda_k}{2} \cos \left(\frac{\Lambda_k}{2} - s_k \right) \right],$$

$$f_{\Lambda_k}(s_k) = \cos \left(\frac{\Lambda_k}{2} - s_k \right) - \cos \frac{\Lambda_k}{2}, k = 1, 2, 3, \quad \eta_{\Lambda_k}(s_k) = \cos \left(\frac{\Lambda_k}{2} - s_k \right), k = 1, 2, 3, .$$

$\Lambda \in T^3, \quad t \in T^3$.

Expressing all integrals in the equation $\Delta_\Lambda^3(z) = 0$ through $J^*(z)$ and rearranging algebraically, we reduce this equation to the form

$$J^*(z) = \frac{A_\Lambda(z)}{B_\Lambda(z)}$$

where $B_\Lambda(z)$ is a seventh-order polynomial in z , and $A_\Lambda(z)$ is a lower-order polynomial in z . Therefore, this equation has no more than seven solutions outside the set G_Λ^3 .

For an arbitrary $\nu > 3$ and $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_\nu) = (\Lambda_0, \Lambda_0, \dots, \Lambda_0)$, if the parameters J, J_1 and Λ_0 satisfy the conditions in Theorems 7-8, statements similar to those in the theorems are true. In this situation, the operator \tilde{H}_2 with $J \neq J_1$ has only one additional BS. The energy z of this additional BS is degenerate $(\nu - 1)$ times. Moreover, $z < m_\Lambda^\nu$ ($z > M_\Lambda^\nu$), if $J > J_1$ ($J < J_1$). For all other values of the full quasi-momentum Λ of the system, the operator \tilde{H}_2 has no more than $2\nu + 1$ BSs (taking the energy degeneration order into account) with the energy values lying outside the set G_Λ^ν .

The proof of these statements is based on finding zeros of the determinants $\Delta_\Lambda^\nu(z)$ of the operators. Expressing all integrals in $\Delta_\Lambda^\nu(z)$ through $J^*(z)$, we can bring the equation $\Delta_\Lambda^\nu(z) = 0$ to the form

$$J^*(z) = \frac{C_\Lambda(z)}{D_\Lambda(z)}, \quad (23)$$

where $D_\Lambda(z)$ is a $(2\nu + 1)$ th-order polynomial in z and $C_\Lambda(z)$ is also a polynomial in z whose order (with respect to $D_\Lambda(z)$) is lower. Analysis of Equation (23) outside the set G_Λ^ν leads to the proof of the statements made above.

Theorem 9. Let $J = J_1$ and ν be an arbitrary number. Then the operator \tilde{H}_2 has no more than one BS, and the corresponding energy level $z < m_\Lambda^\nu$ is not degenerate.

Proof. If $J = J_1$, the relations

$$h_{1\Lambda}(x, t) = -4J_1 \sum_{i=1}^{\nu} \left[1 - 2 \cos \frac{\Lambda_i}{2} \cos \left(\frac{\Lambda_i}{2} - x_i \right) + \cos \Lambda_i \right],$$

$$h_\Lambda(x) = -4J_1 \sum_{i=1}^{\nu} \left[1 - \cos \frac{\Lambda_i}{2} \cos \left(\frac{\Lambda_i}{2} - x_i \right) \right] \quad \text{hold. Using the determinant}$$

$\Delta_\Lambda^\nu(z)$ and solving the corresponding equation, we obtain the statement in Theorem 9.

4. Structure of Essential Spectrum and Discrete Spectrum of Three-Magnon Systems

We first determine the structure of the essential spectrum of the three-magnon system and then estimate the number of three-magnon BSs in this system. Comparing Formulas (8) and (11) and using tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces [22], we can verify that the operator $\tilde{H}_{3\Lambda}$ can be represented in the form

$$\tilde{H}_{3\Lambda} = \tilde{H}_{2\Lambda_1} \otimes I + I \otimes (\tilde{H}_{2\Lambda_2} + \tilde{H}_{2\Lambda_3}) + K_\Lambda, \quad (24)$$

where I is the unit operator in the space \mathcal{H}_1 , and $\tilde{H}_{2\Lambda_1}$ and $\tilde{H}_{2\Lambda_2}$ and $\tilde{H}_{2\Lambda_3}$ are the energy operator of two-magnon systems, and K_Λ are finite-dimensional operator, $\Lambda = \lambda + \mu + \gamma$, $\Lambda_1 = \lambda + \mu$, $\Lambda_2 = \lambda + \gamma$, $\Lambda_3 = \mu + \gamma$.

The spectrum of $A \otimes I + I \otimes B$, where A and B are densely defined bounded linear operators, was studied in [23] [24] [25]. Explicit formulas expressing $\sigma_{ess}(A \otimes I + I \otimes B)$ and $\sigma_{disc}(A \otimes I + I \otimes B)$ in terms of $\sigma(A), \sigma_{disc}(A), \sigma(B)$ and $\sigma_{disc}(B)$ were given in those papers:

$$\begin{aligned} &\sigma_{disc}(A \otimes I + I \otimes B) \\ &= \{(\sigma(A) \setminus \sigma_{ess}(A)) + (\sigma(B) \setminus \sigma_{ess}(B))\} \setminus \{(\sigma_{ess}(A) + \sigma(B)) \cup (\sigma(A) + \sigma_{ess}(B))\}, \\ &\sigma_{ess}(A \otimes I + I \otimes B) = (\sigma_{ess}(A) + \sigma(B)) \cup (\sigma(A) + \sigma_{ess}(B)). \end{aligned}$$

It is clear that $\sigma(A \otimes E + E \otimes B) = \{z_1 + z_2 : z_1 \in \sigma(A), z_2 \in \sigma(B)\}$.

Note that, what the problems of finite-rank perturbations for the compact operators be considered in the work [26] [27] [28].

The following theorems describe the structure of the essential spectrum of $\tilde{H}_{2\Lambda_1} \otimes I + I \otimes (\tilde{H}_{2\Lambda_2} + \tilde{H}_{2\Lambda_3})$.

Theorem 10. If $J = 2J_1$ and ν be arbitrary. Then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the set of five points:

$$\sigma_{ess}(\tilde{H}_{3\Lambda}) = \left\{ 0, -2J_1, -4J_1, -(4\nu + 2)J_1 - 4J_1 \sum_{i=1}^{\nu} \cos \Lambda_{1,i}, -2(4\nu + 2)J_1 - 4J_1 \sum_{i=1}^{\nu} (\cos \Lambda_{2,i} + \cos \Lambda_{3,i}) \right\},$$

and the inequality $4 \leq N \leq 8\nu + 4$ holds for the number N of three-magnon BSs.

Proof. It can be seen from Theorem 5, that for $J = 2J_1$ the operator \tilde{H}_2 has exactly two BSs φ_1 and φ_2 (not taking the order of the energy degeneration into account) with the energy values $z_1 = -2J_1$,

$z_2 = -(4\nu + 2)J_1 - 4J_1 \sum_{i=1}^{\nu} \cos \Lambda_{1,i}$, while the continuous spectrum of the operator \tilde{H}_2 is consists of one point $z = 0$, therefore the essential spectrum of operator $\tilde{H}_{3\Lambda}$ is consists of points $z_1 = 0$, $z_2 = -2J_1$, $z_3 = -4J_1$,

$$z_4 = -(4\nu + 2)J_1 - 4J_1 \sum_{i=1}^{\nu} \cos \Lambda_{1,i},$$

$$z_5 = -2(4\nu + 2)J_1 - 4J_1 \sum_{i=1}^{\nu} (\cos \Lambda_{2,i} + \cos \Lambda_{3,i}), \text{ i.e.}$$

$$\sigma_{ess}(\tilde{H}_{3\Lambda}) = \left\{ 0, -2J_1, -4J_1, -(4\nu + 2)J_1 - 4J_1 \sum_{i=1}^{\nu} \cos \Lambda_{1,i}, -2(4\nu + 2)J_1 - 4J_1 \sum_{i=1}^{\nu} (\cos \Lambda_{2,i} + \cos \Lambda_{3,i}) \right\}.$$

The operator $\tilde{H}_{2\Lambda_1} \otimes I + I \otimes (\tilde{H}_{2\Lambda_2} + \tilde{H}_{2\Lambda_3})$ has a eigenvalues to equally $z_1 = -6J_1$,

$$z_2 = -(4\nu + 6)J_1 - 4J_1 \sum_{i=1}^{\nu} \cos \Lambda_{1,i},$$

$$z_3 = -(8\nu + 6)J_1 - 4J_1 \sum_{i=1}^{\nu} (\cos \Lambda_{2,i} + \cos \Lambda_{3,i}), \text{ and}$$

$z_4 = -(12\nu + 6)J_1 - 4J_1 \sum_{i=1}^{\nu} (\cos \Lambda_{1,i} + \cos \Lambda_{2,i} + \cos \Lambda_{3,i})$. The operator K_Λ is a finite rank operator, to rank is equal to 8ν . Consequently, in this case the number of three-magnon BS $N \leq 8\nu + 4$.

Theorem 11. Let $\Lambda_1 = \tilde{\pi}, \Lambda_2 = \tilde{\pi}, \Lambda_3 = \tilde{\pi}$ and $J \neq J_1$. Then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the set of three points:

$\sigma_{ess}(\tilde{H}_{3\Lambda}) = \{24\nu(J-2J_1), 24\nu(J-2J_1)-4(J-J_1), 24\nu(J-2J_1)-2(J-J_1)\}$, and the inequality $1 \leq N \leq 8\nu+1$ holds for the number N of three-magnon BSs.

Proof. It can be seen from Theorem 6, that for $\Lambda = \tilde{\pi}$ and $J \neq J_1$ the operator \tilde{H}_2 has a unique BS φ with the energy value

$z = 8\nu(J-2J_1)-2(J-J_1)$, and this energy level is ν -fold degenerate. The continuous spectrum of the operator \tilde{H}_2 consists of point $z = 8\nu(J-2J_1)$,

therefore the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of points $z_1 = 24\nu(J-2J_1)$, $z_2 = 24\nu(J-2J_1)-4(J-J_1)$,

$z_3 = 24\nu(J-2J_1)-2(J-J_1)$, i.e.

$\sigma_{ess}(\tilde{H}_{3\Lambda}) = \{24\nu(J-2J_1), 24\nu(J-2J_1)-4(J-J_1), 24\nu(J-2J_1)-2(J-J_1)\}$.

The operator $\tilde{H}_{2\Lambda_1} \otimes I + I \otimes (\tilde{H}_{2\Lambda_2} + \tilde{H}_{2\Lambda_3})$ has a eigenvalues equal to $z = 24\nu(J-2J_1)-6(J-J_1)$. The operator K_Λ has a finite rank operator, with rank to equal to 8ν . Consequently, in this case, the number of three-magnon BS $N \leq 8\nu+1$.

$$\begin{aligned} \text{We let } a &= 8(J-2J_1)\left(3 - \cos\frac{\Lambda_1}{2} - \cos\frac{\Lambda_2}{2} - \cos\frac{\Lambda_3}{2}\right), \\ b &= 8(J-2J_1)\left(3 + \cos\frac{\Lambda_1}{2} + \cos\frac{\Lambda_2}{2} + \cos\frac{\Lambda_3}{2}\right), \\ c &= 8(J-2J_1)\left(2 - \cos\frac{\Lambda_2}{2} - \cos\frac{\Lambda_3}{2}\right), \quad d = 8(J-2J_1)\left(2 + \cos\frac{\Lambda_2}{2} + \cos\frac{\Lambda_3}{2}\right), \\ e &= 8(J-2J_1)\left(1 - \cos\frac{\Lambda_1}{2}\right), \quad f = 8(J-2J_1)\left(1 + \cos\frac{\Lambda_1}{2}\right). \end{aligned}$$

Theorem 12. Let $\nu=1$, $J < J_1$ and $\Lambda_1 \in]0, \pi[$, $\Lambda_2 \in]0, \pi[$, $\Lambda_3 \in]0, \pi[$ or $\Lambda_1 \in]\pi, 2\pi[$, $\Lambda_2 \in]\pi, 2\pi[$, $\Lambda_3 \in]\pi, 2\pi[$.

a) If $\cos\frac{\Lambda_1}{2} > -\frac{J-J_1}{2J_1}$, $\cos\frac{\Lambda_2}{2} > -\frac{J-J_1}{2J_1}$, $\cos\frac{\Lambda_3}{2} > -\frac{J-J_1}{2J_1}$ or $\cos\frac{\Lambda_1}{2} < \frac{J-J_1}{2J_1}$, $\cos\frac{\Lambda_2}{2} < \frac{J-J_1}{2J_1}$, $\cos\frac{\Lambda_3}{2} < \frac{J-J_1}{2J_1}$, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the union of five intervals:

$$\begin{aligned} \sigma_{ess}(\tilde{H}_{3\Lambda}) &= [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + c, z_2 + d] \\ &\cup [z_3 + z_4 + e, z_3 + z_4 + f] \cup [z_5 + z_6 + e, z_5 + z_6 + f] \end{aligned}$$

and the inequality $4 \leq N \leq 8\nu+4$ holds for the number N of three-magnon BSs.

b) If $\cos\frac{\Lambda_1}{2} \leq -\frac{J-J_1}{2J_1}$, $\cos\frac{\Lambda_2}{2} \leq -\frac{J-J_1}{2J_1}$, $\cos\frac{\Lambda_3}{2} \leq -\frac{J-J_1}{2J_1}$ or $\cos\frac{\Lambda_1}{2} \geq \frac{J-J_1}{2J_1}$, $\cos\frac{\Lambda_2}{2} \geq \frac{J-J_1}{2J_1}$, $\cos\frac{\Lambda_3}{2} \geq \frac{J-J_1}{2J_1}$, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the union of three intervals:

$\sigma_{ess}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + z_3 + e, z_2 + z_3 + f]$, and the inequality $1 \leq N \leq 8\nu+1$ holds for the number of three-magnon BSs N .

c) If $\cos\frac{\Lambda_1}{2} > -\frac{J-J_1}{2J_1}$, $\cos\frac{\Lambda_2}{2} < \frac{J-J_1}{2J_1}$, $\cos\frac{\Lambda_3}{2} < \frac{J-J_1}{2J_1}$ or

$\cos \frac{\Lambda_1}{2} < \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} > \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_3}{2} > \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda_1}{2} < -\frac{J-J_1}{2J_1}$,
 $\cos \frac{\Lambda_2}{2} > \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_3}{2} > \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda_1}{2} > \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} < \frac{J-J_1}{2J_1}$,
 $\cos \frac{\Lambda_3}{2} < \frac{J-J_1}{2J_1}$, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of

the union of four intervals:

$\sigma_{ess}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + c, z_2 + d] \cup [z_3 + z_4 + e, z_3 + z_4 + f]$, and the inequality $2 \leq N \leq 8\nu + 2$ holds for the number of three-magnon BSs N .

Theorem 13. Let $J = J_1$ and $\Lambda_1 \in]0, \pi[$, $\Lambda_2 \in]0, \pi[$, $\Lambda_3 \in]0, \pi[$ or $\Lambda_1 \in]\pi, 2\pi[$, $\Lambda_2 \in]\pi, 2\pi[$, $\Lambda_3 \in]\pi, 2\pi[$.

a) If $0 < \Lambda_1 < \alpha_1$ or $\alpha_2 < \Lambda_1 < 2\pi$, and $0 < \Lambda_2 < \alpha_1$ or $\alpha_2 < \Lambda_2 < 2\pi$, and $0 < \Lambda_3 < \alpha_1$ or $\alpha_2 < \Lambda_3 < 2\pi$, respectively, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the union of three intervals:

$\sigma_{ess}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + z_3 + e, z_2 + z_3 + f]$, and the inequality $1 \leq N \leq 8\nu + 1$ holds for the number of three-magnon BSs N .

b) If $\Lambda_1 \in [\alpha_1, \pi[\cup]\pi, \alpha_2[$ and $\Lambda_2 \in [\alpha_1, \pi[\cup]\pi, \alpha_2[$ and $\Lambda_3 \in [\alpha_1, \pi[\cup]\pi, \alpha_2[$, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of single interval: $\sigma_{ess}(\tilde{H}_{3\Lambda}) = [a, b]$, and the inequality $0 \leq N \leq 8\nu$ holds for the number of three-magnon BSs N . Here $\alpha_1 \approx 100^\circ, \alpha_2 \approx 260^\circ$.

c) If $0 < \Lambda_1 < \alpha_1$ or $\Lambda_2 \in [\alpha_1, \pi[\cup]\pi, \alpha_2[$ or $\Lambda_3 \in [\alpha_1, \pi[\cup]\pi, \alpha_2[$, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the union of two intervals: $\sigma_{ess}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d]$, and the inequality $0 \leq N \leq 8\nu$ holds for the number of three-magnon BSs N .

Theorem 14. Let $J_1 < J < 2J_1$ and $\Lambda_1 \in]0, \pi[$, $\Lambda_2 \in]0, \pi[$, $\Lambda_3 \in]0, \pi[$ or $\Lambda_1 \in]\pi, 2\pi[$, $\Lambda_2 \in]\pi, 2\pi[$, $\Lambda_3 \in]\pi, 2\pi[$.

a) If $\cos \frac{\Lambda_1}{2} \leq \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} \leq \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_3}{2} \leq \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda_1}{2} \geq -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} \geq -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_3}{2} \geq -\frac{J-J_1}{2J_1}$, then the essential

spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the union of five intervals:

$\sigma_{ess}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + c, z_2 + d] \cup [z_3 + z_4 + e, z_3 + z_4 + f] \cup [z_5 + z_6 + e, z_5 + z_6 + f]$, and the inequality $4 \leq N \leq 8\nu + 4$ holds for the number of three-magnon BSs N .

b) If $\cos \frac{\Lambda_1}{2} > \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} > \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_3}{2} > \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda_1}{2} < -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} < -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_3}{2} < -\frac{J-J_1}{2J_1}$, then the essential

spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the union of seven intervals:

$\sigma_{ess}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + c, z_2 + d] \cup [z_3 + c, z_3 + d] \cup [z_4 + z_5 + e, z_4 + z_5 + f] \cup [z_6 + z_7 + e, z_6 + z_7 + f] \cup [z_8 + z_9 + e, z_8 + z_9 + f]$, and the inequality

$9 \leq N \leq 8\nu + 9$ holds for the number of three-magnon BSs N .

c) If $\cos \frac{\Lambda_1}{2} \leq \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} > \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_3}{2} > \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda_1}{2} \geq -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} < -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_3}{2} < -\frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda_1}{2} \geq -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} > \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_3}{2} > \frac{J-J_1}{2J_1}$, then the essential spectrum of the operator

$\tilde{H}_{3\Lambda}$ consists of the union of six intervals:

$\sigma_{\text{ess}}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + c, z_2 + d] \cup [z_3 + c, z_3 + d]$, and the $\cup [z_4 + z_5 + e, z_4 + z_5 + f] \cup [z_6 + z_7 + e, z_6 + z_7 + f]$

inequality $6 \leq N \leq 8\nu + 6$ holds for the number of three-magnon BSs N .

Theorem 15. Let $2J_1 < J < 3J_1$ and $\Lambda \in]0, \pi[$ or $\Lambda \in]\pi, 2\pi[$

a) If $\cos \frac{\Lambda_1}{2} > \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} > \frac{J-J_1}{2J_1}$, or $\cos \frac{\Lambda_1}{2} < -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} < -\frac{J-J_1}{2J_1}$, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of

the union of five intervals: $\sigma_{\text{ess}}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + c, z_2 + d]$, $\cup [z_3 + z_4 + e, z_3 + z_4 + f] \cup [z_5 + z_6 + e, z_5 + z_6 + f]$,

and the inequality $4 \leq N \leq 8\nu + 4$ holds for the number of three-magnon BSs N .

b) If $\cos \frac{\Lambda_1}{2} \leq \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} \leq \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda_1}{2} \geq -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} \geq -\frac{J-J_1}{2J_1}$, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of

the union of three intervals:

$\sigma_{\text{ess}}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + z_3 + e, z_2 + z_3 + f]$, and the inequality

$1 \leq N \leq 8\nu + 1$ holds for the number of three-magnon BSs N .

c) If $\cos \frac{\Lambda_1}{2} > \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} \leq \frac{J-J_1}{2J_1}$, or $\cos \frac{\Lambda_1}{2} < -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} \geq -\frac{J-J_1}{2J_1}$, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of

the union of four intervals:

$\sigma_{\text{ess}}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + c, z_2 + d] \cup [z_3 + z_4 + e, z_3 + z_4 + f]$, and

the inequality $2 \leq N \leq 8\nu + 2$ holds for the number of three-magnon BSs N .

d) If $\cos \frac{\Lambda_1}{2} \leq \frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} > \frac{J-J_1}{2J_1}$ or $\cos \frac{\Lambda_1}{2} \geq -\frac{J-J_1}{2J_1}$, $\cos \frac{\Lambda_2}{2} < -\frac{J-J_1}{2J_1}$, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of

the union of four intervals:

$\sigma_{\text{ess}}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + z_3 + e, z_2 + z_3 + f]$, and the inequality $\cup [z_4 + z_5 + e, z_4 + z_5 + f]$

$2 \leq N \leq 8\nu + 2$ holds for the number of three-magnon BSs N .

Theorem 16. Let $J = 3J_1$ and $\Lambda_1 \neq 0, \Lambda_2 \neq 0$. Then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the union of three intervals:

$\sigma_{ess}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + z_3 + e, z_2 + z_3 + f]$, and the inequality $1 \leq N \leq 8\nu + 1$ holds for the number of three-magnon BSs N .

Theorem 17. Let $J > 3J_1$ and $\Lambda_1 \neq 0, \Lambda_2 \neq 0$. Then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the union of five intervals:

$\sigma_{ess}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + c, z_2 + d] \cup [z_3 + z_4 + e, z_3 + z_4 + f] \cup [z_5 + z_6 + e, z_5 + z_6 + f]$, and the inequality

$4 \leq N \leq 8\nu + 4$ holds for the number of three-magnon BSs N .

If Λ_1 and Λ_2 and Λ_3 has the form

$$\Lambda_1 = (\Lambda_{1,1}, \Lambda_{1,2}, \dots, \Lambda_{1,\nu}) = (\Lambda_{1,0}, \Lambda_{1,0}, \dots, \Lambda_{1,0}),$$

$$\Lambda_2 = (\Lambda_{2,1}, \Lambda_{2,2}, \dots, \Lambda_{2,\nu}) = (\Lambda_{2,0}, \Lambda_{2,0}, \dots, \Lambda_{2,0}), \text{ and}$$

$\Lambda_3 = (\Lambda_{3,1}, \Lambda_{3,2}, \dots, \Lambda_{3,\nu}) = (\Lambda_{3,0}, \Lambda_{3,0}, \dots, \Lambda_{3,0})$, then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ is investigated to analogously to one-dimensional case.

In the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ is appear only two additional intervals and corresponding estimation for the number of three-magnon BSs, in the case of, when the operator $\tilde{H}_{2\Lambda_1}$ and $\tilde{H}_{2\Lambda_2}$ and $\tilde{H}_{2\Lambda_3}$ has a correspondingly, correspondingly to equal to number a and b and c, BSs, that the estimation $a(b+c) \leq N \leq 8\nu + a(b+c)$ changed to the estimation $(a+1)(b+c+1) \leq N \leq 8\nu + (a+1)(b+c+1)$.

For arbitrary values Λ_1 and Λ_2 and Λ_3 and $\nu \geq 2$, the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ is consists of the union of no more that $(2\nu+1)^3 + 1$ intervals, and the relation $N_1 \leq N \leq 8\nu + N_1$, holds for the number of three-magnon BSs N , where $0 \leq N_1 \leq (2\nu+1)^3$.

Theorem 18. If $J = J_1$ and the number ν be arbitrary. Then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the union of no more then three intervals: $\sigma_{ess}(\tilde{H}_{3\Lambda}) = [a, b] \cup [z_1 + c, z_1 + d] \cup [z_2 + z_3 + e, z_2 + z_3 + f]$, and the inequality $0 \leq N \leq 8\nu + 1$ holds for the number of three-magnon BSs N .

The cases, when $\Lambda_1 = 0$ or $\Lambda_2 = 0$ or $\Lambda_3 = 0$ investigated the similarly. Analogously is investigated essential spectrum and discrete spectrum of the operator $\tilde{H}_{3\Lambda}$ for the other cases.

Obviously, that the case, when the of spin component S^z take the value -1 coincide, with cases when spin component S^z take the value 1 .

5. Case, When $S_m^z \varphi_0 = 0$

We already say, what the spin component S_m^z can by take on a value $-1, 0, 1$, i.e. can by $S_m^z \varphi_0 = -\varphi_0$ or $S_m^z \varphi_0 = 0$ or $S_m^z \varphi_0 = \varphi_0$.

Now we consider the case, when the of spin component S^z take value 0 .

Hamiltonian of the system also has the form (3) and acts in the symmetrical Fo'ck space \mathcal{H} . In this case the vacuum vector φ_0 uniquely determined by the conditions: $S_m^+ \varphi_0 = 0, S_m^z \varphi_0 = 0, \|\varphi_0\| = 1$.

Theorem 19. The space \mathcal{H}_3 is invariant with respect of the operator H . The operator H_3 is a bounded self-adjoint operator. It generates the bounded

self-adjoint operator \bar{H}_3 , acting in the space $l_2\left(\left(Z^v\right)^3\right)$ according to the formula

$$\begin{aligned}
& (\bar{H}_3 f)(p, q, r) \\
&= -J \sum_{p, q, r, \tau} \left\{ \left[\delta_{p, q+\tau} + \delta_{p+\tau, q} + \delta_{p, r+\tau} + \delta_{p+\tau, r} + \delta_{q+\tau, r} + \delta_{q, r+\tau} \right] f(p, q, r) \right. \\
&\quad - \left(\frac{1}{2} \delta_{p-\tau, q} + \frac{1}{2} \delta_{p-\tau, r} \right) f(p-\tau, q, r) - \left(\frac{1}{2} \delta_{q-\tau, r} + \frac{1}{2} \delta_{p, q-\tau} \right) f(p, q-\tau, r) \\
&\quad - \left(\frac{1}{2} \delta_{q, r-\tau} + \frac{1}{2} \delta_{p, r-\tau} \right) f(p, q, r-\tau) - \left(\frac{1}{2} \delta_{p+\tau, q} + \frac{1}{2} \delta_{p+\tau, r} \right) f(p+\tau, q, r) \\
&\quad \left. - \left(\frac{1}{2} \delta_{q+\tau, r} + \delta_{p, q+\tau} \right) f(p, q+\tau, r) - \left(\frac{1}{2} \delta_{p, r+\tau} + \frac{1}{2} \delta_{q, r+\tau} \right) f(p, q, r+\tau) \right\} \\
&- J_1 \sum_{p, q, r, \tau} \left\{ \left(\delta_{p, q+\tau} + \delta_{p+\tau, q} + \delta_{p, r+\tau} + \delta_{p+\tau, r} + \delta_{q+\tau, r} + \delta_{q, r+\tau} + \frac{5}{2} \delta_{p, q+\tau} \delta_{q, r} \right. \right. \\
&\quad \left. \left. + \frac{5}{2} \delta_{q+\tau, r} \delta_{p, r} + \frac{5}{2} \delta_{q, r+\tau} \delta_{p, q} + \frac{5}{2} \delta_{p+\tau, r} \delta_{q, r} + \frac{5}{2} \delta_{q, r+\tau} \delta_{p, r} + \frac{5}{2} \delta_{q+\tau, r} \delta_{p, q} \right) f(p, q, r) \right. \\
&\quad - \left(\frac{1}{2} \delta_{p-\tau, q} + \frac{1}{2} \delta_{p-\tau, r} + \frac{3}{2} \delta_{q, r+\tau} \delta_{p, q} + \frac{3}{2} \delta_{q+\tau, r} \delta_{p, r} + \delta_{p, q+\tau} \delta_{q, r} \right) f(p-\tau, q, r) \\
&\quad - \left(\frac{1}{2} \delta_{p+\tau, q} + \frac{1}{2} \delta_{q, r+\tau} + \frac{3}{2} \delta_{q, r+\tau} \delta_{p, q} + \frac{3}{2} \delta_{p+\tau, r} \delta_{q, r} + \delta_{q, r+\tau} \delta_{p, r} \right) f(p, q-\tau, r) \\
&\quad - \left(\frac{1}{2} \delta_{p+\tau, r} + \frac{1}{2} \delta_{q+\tau, r} + \frac{3}{2} \delta_{q+\tau, r} \delta_{p, r} + \frac{3}{2} \delta_{p+\tau, r} \delta_{q, r} + \delta_{q+\tau, r} \delta_{p, q} \right) f(p, q, r-\tau) \\
&\quad - \left(\frac{1}{2} \delta_{p+\tau, q} + \frac{1}{2} \delta_{p+\tau, r} + \frac{3}{2} \delta_{p+\tau, r} \delta_{q, r} + \frac{3}{2} \delta_{q+\tau, r} \delta_{p, q} + \delta_{q, r+\tau} \delta_{p, r} \right) f(p+\tau, q, r) \\
&\quad - \left(\frac{1}{2} \delta_{p, q+\tau} + \frac{1}{2} \delta_{q+\tau, r} + \frac{3}{2} \delta_{p, r+\tau} \delta_{q, r} + \frac{3}{2} \delta_{q+\tau, r} \delta_{p, q} + \delta_{q+\tau, r} \delta_{p, r} \right) f(p, q+\tau, r) \\
&\quad - \left(\frac{1}{2} \delta_{p, r+\tau} + \frac{1}{2} \delta_{q, r+\tau} + \frac{3}{2} \delta_{p, r+\tau} \delta_{q, r} + \frac{3}{2} \delta_{q, r+\tau} \delta_{p, r} + \delta_{q, r+\tau} \delta_{p, q} \right) f(p, q, r+\tau) \\
&\quad + \delta_{q, r+\tau} \delta_{p, q} f(p-\tau, q-\tau, r) + \delta_{p, r} \delta_{q+\tau, r} f(p-\tau, q, r-\tau) \\
&\quad + \delta_{p+\tau, r} \delta_{q, r} f(p, q-\tau, r-\tau) + \delta_{q+\tau, r} \delta_{p, q} f(p+\tau, q+\tau, r) \\
&\quad + \delta_{q, r+\tau} \delta_{p, r} f(p+\tau, q, r+\tau) + \delta_{p, r+\tau} \delta_{q, r} f(p, q+\tau, r+\tau) \\
&\quad + \left(\frac{1}{4} \delta_{p, r} \delta_{q+\tau, r} + \frac{1}{4} \delta_{p, r+\tau} \delta_{q, r} \right) f(p-\tau, q+\tau, r) \\
&\quad + \left(\frac{1}{4} \delta_{p, q} \delta_{q, r+\tau} + \frac{1}{4} \delta_{p, r+\tau} \delta_{q, r} \right) f(p-\tau, q, r+\tau) \\
&\quad + \left(\frac{1}{4} \delta_{p+\tau, r} \delta_{q, r} + \frac{1}{4} \delta_{p, r} \delta_{q, r+\tau} \right) f(p+\tau, q-\tau, r) \\
&\quad + \left(\frac{1}{4} \delta_{p, q} \delta_{q+\tau, r} + \frac{1}{4} \delta_{q, r+\tau} \delta_{p, r} \right) f(p, q-\tau, r+\tau) \\
&\quad + \left(\frac{1}{4} \delta_{p+\tau, r} \delta_{q, r} + \frac{1}{4} \delta_{p, q} \delta_{q+\tau, r} \right) f(p+\tau, q, r-\tau) \\
&\quad \left. + \left(\frac{1}{4} \delta_{p, q} \delta_{q+\tau, r} + \frac{1}{4} \delta_{p, r} \delta_{q, r+\tau} \right) f(p, q+\tau, r) \right\}. \tag{25}
\end{aligned}$$

where $\delta_{k,j}$ is the Kronecker symbol. The operator H_3 acts on the vector $\psi \in \mathcal{H}_3$ according to the formula

$$H_3\psi = \sum_{p,q,r} (\bar{H}_3 f)(p, q, r) S_p^- S_q^- S_r^- \varphi_0. \tag{26}$$

Theorem 20. The Fourier transformation transforms the operator \bar{H}_3 into the bounded self-adjoint operator $\tilde{H}_3 = \mathcal{F}\bar{H}_3\mathcal{F}^{-1}$, acting in the space $\tilde{\mathcal{H}}_3$ according to the formula

$$\begin{aligned} & (\tilde{H}_3 f)(\lambda, \mu, \gamma) \\ &= -4(J + J_1) \left[\int_{T^v} h_1(\lambda, \mu, s) f(s, \lambda + \mu - s, \gamma) ds \right. \\ & \quad \left. + \int_{T^v} h_1(\lambda, \gamma, s) f(s, \mu, \lambda + \gamma - s) ds + \int_{T^v} h_1(\mu, \gamma, s) f(\lambda, s, \mu + \gamma - s) ds \right] \\ & \quad + J \int_{T^v} \int_{T^v} h_2(\lambda, \mu, \gamma, s, t) f(s, t, \lambda + \mu + \gamma - s - t) ds dt, \end{aligned} \tag{27}$$

where

$$\begin{aligned} h_1(\lambda, \mu, s) &= \sum_{i=1}^v [\cos(\lambda_i - s_i) + \cos(\mu_i - s_i) - \cos s_i - \cos(\lambda_i + \mu_i - s_i)], \\ h_2(\lambda, \mu, \gamma, s, t) &= \sum_{i=1}^v [4\cos(s_i + t_i) + 4\cos(\lambda_i + \mu_i + \gamma_i - t_i) + 4\cos(\lambda_i + \mu_i + \gamma_i - s_i) \\ & \quad + 6\cos(\mu_i + \gamma_i - s_i - t_i) + 6\cos(\lambda_i + \gamma_i - s_i - t_i) + 6\cos(\lambda_i + \mu_i - s_i - t_i) \\ & \quad + 3\cos(\lambda_i - s_i) + 3\cos(\lambda_i - t_i) + 3\cos(\mu_i - t_i) + 3\cos(\mu_i - s_i) \\ & \quad + 3\cos(\gamma_i - s_i) + 3\cos(\gamma_i - t_i) - 4\cos s_i - 4\cos t_i - 4\cos(\lambda_i + \mu_i - s_i) \\ & \quad - 4\cos(\lambda_i + \mu_i - t_i) - 5\cos(\mu_i - s_i - t_i) - 5\cos(\lambda_i - s_i - t_i) \\ & \quad - 5\cos(\gamma_i - s_i - t_i) - 6\cos(\mu_i + \gamma_i - s_i) - 6\cos(\lambda_i + \gamma_i - s_i) \\ & \quad - 5\cos(\lambda_i + \mu_i + \gamma_i - s_i - t_i)]. \end{aligned}$$

The spectral properties of the considered energy operator of three-magnon systems in the isotropic ferromagnetic non-Heisenberg model are closely related to those of its two-magnon subsystems. We first study the spectrum and bound states of two-magnon subsystems.

Theorem 21. The space \mathcal{H}_2 is invariant with respect of the operator H . The operator H_2 is a bounded self-adjoint operator. It generates the bounded self-adjoint operator \bar{H}_2 , acting in the space $l_2((Z^v)^2)$ according to the formula

$$\begin{aligned} & (\bar{H}_2 f)(p, q) \\ &= -(J + J_1) \sum_{p,q,\tau} \left\{ [\delta_{p,q+\tau} + \delta_{p+\tau,q}] f(p, q) - \frac{1}{2} \delta_{p-\tau,q} f(p - \tau, q) \right. \\ & \quad \left. - \frac{1}{2} \delta_{p,q-\tau} f(p, q - \tau) + \frac{1}{2} \delta_{p+\tau,q} f(p + \tau, q) + \frac{1}{2} \delta_{p,q+\tau} f(p, q + \tau) \right\}. \end{aligned} \tag{28}$$

The operator H_2 acts on the vector $\psi \in \mathcal{H}_2$ according to the formula

$$H_2\psi = \sum_{p,q} (\bar{H}_2 f)(p, q) S_p^- S_q^- \varphi_0. \tag{29}$$

Theorem 22. The Fourier transformation transforms the operator \bar{H}_2 into

the bounded self-adjoint operator $\tilde{H}_2 = \mathcal{F}\tilde{H}_2\mathcal{F}^{-1}$, acting in the space $\tilde{\mathcal{H}}_2$ according to the formula

$$(\tilde{H}_2 f)(x, y) = \int_{T^\nu} h_1(x, y, s) f(s, x + y - s) ds, \quad (30)$$

where

$$h_1(x, y, s) = -4(J + J_1) \sum_{i=1}^{\nu} \left[\cos \frac{x_i - y_i}{2} - \cos \frac{x_i + y_i}{2} \right] \cos \left(\frac{x_i + y_i}{2} - s_i \right).$$

Let the full quasi-momentum of the system $x + y = \Lambda$ be fixed. Let $L_2(\Gamma_\Lambda)$ be the space of functions that are quadratically integrable over the manifold $\Gamma_\Lambda = \{(x, y) : x + y = \Lambda\}$. It is known [21] that the operator \tilde{H}_2 and space $\tilde{\mathcal{H}}_2$ can be expanded into the direct integrals $\tilde{H}_2 = \oplus_{T^\nu} \tilde{H}_{2\Lambda} d\Lambda$, $\tilde{\mathcal{H}}_2 = \oplus_{T^\nu} \tilde{\mathcal{H}}_{2\Lambda} d\Lambda$ of the operators $\tilde{H}_{2\Lambda}$ and the spaces $\tilde{\mathcal{H}}_{2\Lambda}$ such that the spaces $\tilde{\mathcal{H}}_{2\Lambda}$ are invariant with respect to the operators $\tilde{H}_{2\Lambda}$ and the operators $\tilde{H}_{2\Lambda}$ act in the space $\tilde{\mathcal{H}}_{2\Lambda}$ according to the formula

$$(\tilde{H}_{2\Lambda} f_\Lambda)(x) = \int_{T^\nu} h_{1\Lambda}(x, t) f_\Lambda(t) dt, \quad (31)$$

where $h_{1\Lambda}(x, t) = h_1(x, \Lambda - x, t)$ and $f_\Lambda(x) = f(x, \Lambda - x)$.

Theorem 23. Let full quasi-momentum of the system Λ be arbitrary. Then the operator \tilde{H}_2 has a unique BS φ with the energy value $z = -2(J + J_1)$ and it is ν -fold degenerated.

Let the full quasi-momentum of the system $x + y + z = \Lambda$ be fixed. Then the operator \tilde{H}_3 and space $\tilde{\mathcal{H}}_3$ can be expanded into the direct integrals $\tilde{H}_3 = \oplus_{T^\nu} \tilde{H}_{3\Lambda} d\Lambda$, $\tilde{\mathcal{H}}_3 = \oplus_{T^\nu} \tilde{\mathcal{H}}_{3\Lambda} d\Lambda$.

We now determine the structure of the essential spectrum of the three-magnon system and then estimate the number of three-magnon BSs in this system. Comparing Formulas (27) and (30) and using tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces [22], we can verify that the operator $\tilde{H}_{3\Lambda}$ can be represented in the form

$$\tilde{H}_{3\Lambda} = \tilde{H}_{2\Lambda_1} \otimes I + I \otimes (\tilde{H}_{2\Lambda_2} + \tilde{H}_{2\Lambda_3}) + \tilde{K}_\Lambda, \quad (32)$$

where I is the unit operator in the space $\tilde{\mathcal{H}}_1$, and \tilde{K}_Λ are finite-dimensional operator (see (27)).

Theorem 24. Let full quasi-momentum of the system Λ be arbitrary. Then the essential spectrum of the operator $\tilde{H}_{3\Lambda}$ consists of the three points: 0, z_1 and $z_2 + z_3$, where z_1 and z_2 and z_3 is a eigenvalue of the operators $\tilde{H}_{2\Lambda_1}$ and $\tilde{H}_{2\Lambda_2}$ and $\tilde{H}_{2\Lambda_3}$, correspondingly, and the inequality $1 \leq N \leq 8\nu + 1$ holds for the number of three-magnon BSs N .

The finding results shown the structure of essential spectra and discrete spectrum of three-magnon system, in the cases, when the component S^z of spin is receive the value 1 and 0, is strongly different one another.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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