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Entropy Number of Diagonal Operator

Jin Chen*, **Wenjing Lu, Hanyue Xiao, Yanan Wang, Xin Tan**

School of Science, Xihua University, Chengdu, China
Email: *751237185@qq.com

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Abstract

In this paper, the entropy number of diagonal operator is discussed. On the one hand, the order of entropy number of the finite dimensional diagonal operator $D_m : l_p^m \to l_q^m (1 \le q < p \le \infty)$ is estimated. On the other hand, the order of entropy number of a class of infinite dimensional diagonal operator $D : l_p \to l_q (1 \le q < p \le \infty)$ is estimated.

Keywords

Entropy Number, The Finite Dimensional Diagonal Operator, The Infinite Dimensional Diagonal Operator

1. Research Background

Entropy number, as an important content of approximation theory, has been paid attention by many researchers. Its estimation and many optimal problems, such as information-based complexity, optimal number estimation and so on are closely related. The main research object of entropy number is based on the basic function class which is widely used in modern analysis and calculation mathematics, including Sobolev, Hölder-Nikolskii. The main purpose of Besov and some classes of analytic functions is to find the best approximation set and the best approximation method for the class of functions in a certain sense, and to estimate the order of the best approximation.

2. Introduction

The entropy number of function space has been extensively and deeply studied during the last decade, and a series of splendid results have also been achieved [1–11]. However, estimating the entropy numbers of the function space is not an easy task. The common method is to convert the entropy number of estimating function space into the entropy number of estimating sequence space. Hence, it is particularly important to estimate the entropy number of the sequence space. This paper discusses the entropy number of diagonal operator. First, we introduce the notion of the entropy number.

Definition 1.1 Let X be a bounded linear space, $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, and B be a subset of X. Then the quantity

$$\begin{aligned} &\varepsilon_n(B, X) \\ &:= \inf\{\varepsilon > 0 : B \textit{ can be covered by } 2^{n-1} \textit{ balls with radius } \varepsilon \textit{ in } X\} \\ &= \inf_{M_n} \sup_{x \in B_X} \inf_{y \in M_n} \|x - y\| \end{aligned} \tag{1}$$

is called entropy number of B. Where M_n runs over all the subset in Y with $| M_n | \leq 2^{n-1}$.

Definition 1.2 Let $(X, \| \cdot \|)_X$ and $(Y, \| \cdot \|)_Y$ be two normed linear spaces, $T : X \to Y$ be a bounded linear operator, and $n \in \mathbb{N}$. Then the quantity

$$\varepsilon_n(T) := \varepsilon_n(T : X \to Y) = \varepsilon_n(T(B_X), Y) \tag{2}$$

is called the entropy number of operator T. Where B_X is the unit ball of X.

The detailed inforation of entropy number can be found in reference [1]. In this paper, we estimate the entropy number of diagonal operator. First, we introduce the sequence space .

$l_p (1 \leq p \leq \infty)$ denotes the classical real sequence space with norm

$$\|x\|_{l_p} = \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{n \in \mathbb{N}} |x_n|, & p = \infty. \end{cases} \tag{3}$$

where, $x = (x_n)$ is a real sequence. The unit ball of l_p is denoted by B_p.

We also denote the m-dimensional sequence with norm $\| \cdot \|_{l_p^m}$ as usual by l_p^m , and $B_p^m(r)$ denotes the ball of radius r in l_p^m , especially, $B_p^m := B_p^m(1)$.

Now, we present the diagonal operator.

$D_m : l_p^m \to l_q^m (1 \leq p, q \leq \infty)$ denotes the m-dimensional diagonal operator. Where $D_m x = (\sigma_1 x_1, \sigma_2 x_2, \cdots, \sigma_m x_m)$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m > 0$, $x = \{x_n\}_{n=1}^m \in l_p^m$.

$D : l_p \to l_q (1 \leq p, q \leq \infty)$ presents the infinite dimensional diagonal operator. Where $Dx = (\sigma_1 x_1, \sigma_2 x_2, \cdots \sigma_n x_n, \cdots)$, $x = \{x_n\} \in l_p$.

This paper mainly discusses the entropy number of operator $D_m : l_p^m \to l_q^m (1 \leq p < q \leq \infty)$ and $D : l_p \to l_q (1 \leq p < q \leq \infty)$, while, estimates their sharp orders. The structure of the paper is as follows: In the second section, we deal with the entropy number of finite dimensional diagonal operator. The entropy number of infinite dimensional diagonal operator is considered in the third section.

For convenience, it is assumed all the time, that $c_i (i = 1, 2, \ldots)$ is a positive constant which is associated with the parameters p, q. For two positive functions $a(y)$ and $b(y), y \in D$, if $a(y) \leq c_1 b(y)$, then $a(y) \ll b(y)$. If $c_2 a(y) \geq b(y)$, then $a(y) \gg b(y)$. If $a(y) \gg b(y)$ and

$a(y) \ll b(y)$, then $a(y) \asymp b(y)$.

3. The Entropy Number of Finite Dimensional Diagonal Operator

Shütt discussed the entropy numbers of the diagonal operator in finite dimension case in [2]:

Theorem 1.1 [2] Assuming that $1 \le p < q \le \infty$, $n \in \mathbb{N}$, and I is the identity operator from l_p^m to l_q^m. Then

$$\varepsilon_n(I) \asymp \begin{cases} 1, & 1 \le n \le logm, \\ (\frac{log(1+\frac{m}{n})}{n})^{\lambda}, & logm \le n \le m, \\ 2^{-\frac{n}{m}} m^{-\lambda} & n \ge m. \end{cases} \tag{4}$$

In this section, the entropy number of finite dimensional diagonal operator is discussed. The following theorem is the main result of this section. To prove this, first,we give t lemma.

Lemma 1.2 Let $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$ (let us say p, q is a conjugate), if $f \in L_p$, the hölder inequality implies

$$\left| \int_{\Omega} f(t) g(t) d\mu \right| \le \left(\int_{\Omega} \mid f(t) \mid^p d\mu \right)^{\frac{1}{p}} \left(\int_{\Omega} \mid g(t) \mid^q d\mu \right)^{\frac{1}{q}} \tag{5}$$

Theorem 1.2 Let $1 \le q < p \le \infty$, $n \in \mathbb{N}$. Then

$$\begin{aligned} & \sup_{1 \le k \le m} 2^{-\frac{n}{k}} (\sigma_1 \ldots \sigma_k)^{\frac{1}{k}} k^{\frac{1}{q} - \frac{1}{p}} \ll \varepsilon_n(D_m) \\ \ll & \sup_{1 \le k \le m} 2^{-\frac{n}{k}} (\sigma_1 \ldots \sigma_k)^{\frac{1}{k}} m^{\frac{1}{q} - \frac{1}{p}} \end{aligned} \tag{6}$$

Proof. Estimating the upper bound of the theorem 1.2.

Let $D' : l_q \to l_q$ be the diagonal operator from l_p to l_q . Where $x = (x_n) \in l_p, D'x = (\sigma_k x_k)$, $\sigma_n = 0$, if $n > m$. Then D' is a bounded linear operator from l_q to l_q. According to ([1], P16. Proposition 1.3.2), and noting to $\sigma_k = 0$, if $k > m$, one has

$$\varepsilon_n(D') \ll \sup_{1 \le k \le \infty} 2^{-\frac{n}{k}} (\sigma_1 \ldots \sigma_k)^{\frac{1}{k}} = \sup_{1 \le k \le m} 2^{-\frac{n}{k}} (\sigma_1 \ldots \sigma_k)^{\frac{1}{k}} \tag{7}$$

Following hölder inequality that $B_p^m \subseteq B_q^m(m^{\frac{1}{q} - \frac{1}{p}})$.
So

$$\begin{aligned} \varepsilon_n(D_m) &= \varepsilon_n(D_m(B_p^m), l_q^m) \le \varepsilon_n(D_m) = \varepsilon_n(D_m(B_q^m(m^{\frac{1}{q}-\frac{1}{p}})), l_q^m) \\ &= m^{\frac{1}{q}-\frac{1}{p}} \varepsilon_n(D_m(B_q^m), l_q^m) \\ &= m^{\frac{1}{q}-\frac{1}{p}} \varepsilon_n(D_m : l_q^m \to l_q^m) \end{aligned} \tag{8}$$

It is easy to see that $D_m = P_m D'$, where P_m is the projection operator from l_q to l_q^m. Therefore

$$\begin{aligned} \varepsilon_n(D_m : l_q^m \to l_q^m) &= \varepsilon_n(P_m D' : l_q \to l_q^m) \\ &\le \parallel P_m \parallel \varepsilon_n(D' : l_q \to l_q) \\ &\ll \varepsilon_n(D' : l_q \to l_q). \end{aligned} \tag{9}$$

From formulas (8), (9) and (7), we get

$$\varepsilon_n(D_m) \ll m^{\frac{1}{q}-\frac{1}{p}} \sup_{1\le k\le n} 2^{-\frac{n}{k}}(\sigma_1\ldots\sigma_k)^{\frac{1}{k}} \tag{10}$$

The proof of the lower bound of the estimation Theorem 1.2.

For $\forall k, 1\le k\le m$, let D_k be the diagonal operator from l_p^k to l_q^k, I_k be the embedding operator from l_p^k to l_p^m, and P_k be the projection operator from l_q^m to l_q^k.

Obviously, $D_k = P_k D_m I_k$. So

$$\varepsilon_n(D_k : l_p^k \to l_q^k) \quad = \quad \varepsilon_n(P_k D_m I_k : l_p^k \to l_q^k)$$

$$\le \quad \| P_k \| \| I_k \| \, \varepsilon_n(D_m : l_p^m \to l_q^m) \le \varepsilon_n(D_m : l_p^m \to l_q^m) \tag{11}$$

For $\forall \varepsilon > \varepsilon_n(D_k)$, let $y_1,\ldots,y_l \in l_q^k$ $(l\le 2^{n-1})$, so that

$$D_k(B_p^k) \subset \sum_{j=1}^{l}\{y_j + \varepsilon B_q^k\}.$$

Thus

$$Vol_k(D_k(B_p^k)) \le 2^n \varepsilon^k Vol_k(D_k(B_q^k)). \tag{12}$$

Since

$$Vol_k(B_p^k) \asymp k^{-\frac{k}{p}},\ Vol_k(B_q^k)) \asymp k^{-\frac{k}{q}},$$

$$Vol_k(D_k(B_p^k)) \gg \sigma_1\ldots\sigma_k Vol_k(B_p^k),$$

Therefore

$$\varepsilon \gg 2^{-\frac{n}{k}}(\sigma_1\ldots\sigma_k)^{\frac{1}{k}} k^{\frac{1}{q}-\frac{1}{p}}. \tag{13}$$

Let $\varepsilon \to \varepsilon_n(D_k)$. Formulas (11) implies

$$\varepsilon_n(D_m) \ge \varepsilon_n(D_k) \gg 2^{-\frac{n}{k}}(\sigma_1\ldots\sigma_k)^{\frac{1}{k}} \cdot k^{\frac{1}{q}-\frac{1}{p}}. \tag{14}$$

By the arbitrariness of k, we got

$$\varepsilon_n(D_m) \gg \sup_{1\le k\le m} 2^{-\frac{n}{k}}(\sigma_1\cdots\sigma_k)^{\frac{1}{k}} \cdot k^{\frac{1}{q}-\frac{1}{p}}. \tag{15}$$

The proof of Theorem 1.2 is completed.

4. The Entropy Numbers of a Class of Infinite Dimensional Diagonal Operator

K*ü*hn discussed the entropy number of infinite dimensional diagonal operator that satisfies certain decay conditions in [3]. Logarithms are always taken in base 2, log=$\log_2$.

Theorem 3.1 ([3]) Let $1\le p,\, q\le\infty$, $n\in\mathbb{N}$, $\alpha > \max\{\frac{1}{q}-\frac{1}{p}, 0\}$.

(1) If $\sigma_k \asymp k^{-\alpha}$, then $\varepsilon_n(D) \asymp k^{\frac{1}{q}-\frac{1}{p}-\alpha}$.

(2) If $\sigma_k \asymp \sigma_{2k}$, and $\sup_{n\ge k} \frac{\sigma_n}{\sigma_k}(\frac{n}{k})^{\alpha} \le \infty$, then $\varepsilon_n(D) \asymp k^{\frac{1}{q}-\frac{1}{p}}\sigma_n$.

Theorem 3.2 ([3]) Let $1\le p,\, q\le\infty$, $n\in\mathbb{N}$, $\lambda = \frac{1}{p}-\frac{1}{q}$.

(1) If $\sigma_k \asymp (\log k)^{-\lambda}$, then $\varepsilon_n(D) \asymp k^{-\lambda}$.

(2) If $\sigma_k \asymp \sigma_{k^2}$, $\sup\limits_{n\geq k} \frac{\sigma_n}{\sigma_k}(\frac{1+\log n}{1+\log k})^{\lambda} < \infty$, then $\varepsilon_n(D) \asymp (\frac{\log k}{k})^{\lambda}\sigma_n$.

(3) If $\inf\limits_{n\geq k} \frac{\sigma_n}{\sigma_k}(\frac{1+\log n}{1+\log k})^{\lambda} > 0$, then $\varepsilon_n(D) \asymp \sigma_{2^k}$.

(4) If $\alpha > 0, \beta \in \mathbb{R}$,if$\sigma_k \asymp (\log k)^{-\alpha}(\log\log k)^{\beta}$, then

$$\varepsilon_n(D) = \begin{cases} n^{-\lambda}(\log n)^{\lambda-\alpha}(\log\log n)^{\beta}, & \alpha > \lambda \, or \, \alpha = \lambda, \beta \leq 0 \\ n^{-\lambda}(\log n)^{\beta}, & \alpha > \lambda \, or \, \alpha = \lambda, \, \beta \geq 0. \end{cases}$$

(5) If $\sigma_k \asymp \sigma_{2k}, \sup\limits_{n\geq k} \frac{\sigma_n}{\sigma_k}(\frac{1+\log n}{1+\log k})^{\lambda} < \infty$,then $n^{-\lambda}\sigma_n \ll \varepsilon_n(D)(\frac{\log n}{n})^{\lambda}\sigma_n$.

(6) If $\alpha > 0, 0 < \delta < 1, \sigma_k \asymp \exp(-\alpha(\log h)^{\delta})$, then

$$(\frac{(\log n)^{1-\delta}}{n})^{\lambda}\sigma_n \ll \varepsilon_n(D) \ll (\frac{(\log n)^{1-\delta}\log\log n}{n})^{\lambda}\sigma_n.$$

Obviously, the entropy number of the diagonal operator D for $\sigma_k \asymp 2^{-k}$ is not discussed in theorem 3.1 and theorem 3.2. We continues to above work, extending to get the entropy number of a class of infinite dimensional diagonal operator , which satisfies the special decay conditions and contains the case of $\sigma_k \asymp 2^{-k}$. See details in Theorem 3.3.

Theorem 3.3 Assuming $1 \leq q < p \leq \infty$, $n \in \mathbb{N}$. Let D be a diagonal operator from l_p to l_q, and satisfy the following conditions:

(1) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \cdots > 0$.

(2) There exists $M > 0$ such that for all k, implies

$$(\sum_{j=k}^{\infty}(\frac{\sigma_j}{\sigma_k})^{\lambda})^{1/\lambda} \leq M, \quad for \; all \; k.$$

where $\lambda = \frac{1}{q} - \frac{1}{p}$. Then

$$\varepsilon_n(D) := \varepsilon_n(D : l_p \to l_q) \asymp \sup_{1\leq k<\infty} 2^{-\frac{n}{k}}(\sigma_1 \cdots \sigma_k)^{1/k} k^{\frac{1}{q}-\frac{1}{p}}.$$

Proof. Estimating the upper bound of Theorem 3.3.
Let

$$\delta(n) = 8 \cdot \sup_{1\leq k<\infty} 2^{-\frac{n}{k}}(\sigma_2 \cdots \sigma_k)^{1/k}(\frac{Vol_k(B_p^k)}{Vol_k(B_q^k)})^{1/k}.$$

Then

$$\sigma_n \leq (\sigma_1 \cdots \sigma_n)^{\frac{1}{n}} \leq 2 \cdot 2^{-\frac{n}{n}}(\sigma_1 \cdots \sigma_n)^{\frac{1}{n}}.$$

From $B_q^n \subseteq B_p^n$, we get $Vol_k(B_p^n) \geq Vol_k(B_q^n)$.
Thus, there exists $r \in \mathbb{N}$, such that $\sigma_r \leq \frac{\delta(n)}{4}$.

(1) If $\sigma_1 \leq \frac{\delta(n)}{4}$, then

$$\varepsilon_n(D) \leq \|D\| = (\sum_{K=1}^{\infty}\sigma_k^{\lambda})^{1/\lambda} = \sigma_1(\sum_{K=1}^{\infty}(\frac{\sigma_k}{\sigma_1})^{\lambda})^{1/\lambda} \ll \sigma_1 \leq \frac{\delta(n)}{4}$$
$$\ll \sup_{1\leq k<\infty} 2^{-\frac{n}{k}}(\sigma_1 \cdots \sigma_k)^{1/k} k^{\frac{1}{q}-\frac{1}{p}}.$$

(2) If there exists $k \in \mathbb{N}$, such that $\sigma_{k+1} \leq \frac{\delta(n)}{4} < \sigma_k$, let

$$\begin{aligned} D^k : l_p &\rightarrow l_q \\ x = (x_j) &\mapsto D^k x = (\sigma_1 x_1, \cdots, \sigma_{2^k} x_{2^k}, 0, \cdots), \end{aligned}$$

and

$$\begin{aligned} D_{2^k} : l_p^{2^k} &\rightarrow l_q^{2^k} \\ x = (x_1 \cdots x_{2^k}) &\mapsto D_{2^k} x = (\sigma_1 x_1, \cdots, \sigma_{2^k} x_{2^k}) \end{aligned}$$

Then

$$\varepsilon_n(D^k) = \varepsilon_n(D_{2^k}).$$

So, there is a maximum natural number N and $y_1 \cdots y_{2^{N-1}} \in D^k(B_p)$, such that

$$\|y_i - y_j\|_{l_q} > \frac{M\delta(n)}{2}(i \neq j)$$

and

$$D^k(B_p) \subseteq \bigcup_{j=1}^{2^{N-1}} \{y_j + \frac{M\delta(n)}{2} B_q\}. \tag{16}$$

That is

$$\varepsilon_N(D^k) \leq \frac{M\delta(n)}{2}.$$

Obviously, $D = (D - D^k) + D^k$, This gives

$$\begin{aligned} \varepsilon_N(D) &\leq \|D - D^k\| + \varepsilon_N(D^k) \\ &\leq (\sum_{j=k+1}^{\infty} \sigma_j^{\lambda})^{1/\lambda} + \frac{M\delta(n)}{2} \\ &\leq \sigma_{k+1}(\sum_{j=k+1}^{\infty} (\frac{\sigma_j}{\sigma_{k+1}})^{\lambda})^{1/\lambda} + \frac{M\delta(n)}{2} \\ &\leq \frac{M\delta(n)}{4} + \frac{M\delta(n)}{2} = \frac{3M\delta(n)}{4} \end{aligned} \tag{17}$$

For $\forall j(1 \leq j \leq 2^{N-1}, y_j = (y_1, \cdots y_{2^k}, 0, \cdots)$. Let $y_j' = (y_1, \cdots y_{2^k})$ Thereby, from the definition of D^k and D_{2^k}. Noting $\frac{\delta(n)}{4} < \sigma_k \leq \sigma_{k-1} \leq \cdots \leq \sigma_1$, we can get

$$\bigcup_{j=1}^{2^{N-1}} \{y_j' + \frac{\delta(n)}{4} B_q^{2^k}\} \subset D^{(k)}(B_P^{2^k}) + \frac{\delta(n)}{4} B_q^{2^k} \subset 2D^{(k)}(B_P^{2^k}).$$

Thus

$$2^{N-1}(\frac{\delta(n)}{4})^2 \cdot Vol_k(B_P^k) \quad \leq \quad 2^k \cdot \sigma_1 \cdots \sigma_k \cdot Vol_k(B_q^k)$$

That is

$$2^{N-1} \leq (\frac{8}{\delta(n)})^k \cdot \sigma_1 \cdots \sigma_k \frac{Vol_k(B_p^k)}{Vol_k(B_q^k)} \tag{18}$$

By the definition of σ_n, one has

$$\frac{8}{\delta(n)} \leq 2^{\frac{n-1}{k}}(\sigma_1 \cdots \sigma_k)^{-1/k}(\frac{Vol_k(B_p^k)}{Vol_k(B_q^k)})^{-1/k} \tag{19}$$

Formulas (18) and (19) imply now that $2^{N-1} \leq 2^{n-1}$, so $N \leq n$. Thus, according to (18), we finally arrive that

$$\begin{aligned}\varepsilon_n(D) &\leq \frac{3}{4}M\delta(n)\\ &\leq M \sup_{1\leq k<\infty} 2^{-\frac{n}{k}}(\sigma_1 \cdots \sigma_k)^{1/k}k^{\frac{1}{q}-\frac{1}{p}}\\ &\ll \sup_{1\leq k<\infty} 2^{-\frac{n}{k}}(\sigma_1, \ldots \sigma_k)^{\frac{1}{k}}k^{\frac{1}{q}-\frac{1}{p}}\end{aligned}$$

Estimating the lower bound of Theorem 3.3.

In order to estimating the lower bound of theorem 3.3, we use the method of Theorem 2.1, meanwhile keeping $m = \infty$, in this case, the lower bound of Theorem 3.3 can be estimated, and the proof is finished.

5. The Conclusion

This paper gives the entropy number of diagonal operator in finite and infinite dimensions, and discusses the entropy numbers of diagonal operator satisfying special diagonal elements.

Supporting Information

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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