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The Existence of Solution of a Critical Fractional Equation

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Abstract

In this paper, we study the existence of solution of a critical fractional equation; we will use a variational approach to find the solution. Firstly, we will find a suitable functional to our problem; next, by using the classical concept and properties of the genus, we construct a mini-max class of critical points.

Keywords

Variational Approach, Fractional Laplacian, Minimax Principle, Genus

1. Introduction

In this paper, we focus our attention on the following problem:

$$\begin{cases} (-\Delta)^s u = \lambda V(x)|u|^{p-1} + \beta K(x)|u|^{2_s^*-1} & \text{in } \Omega \\ u = 0 & \text{in } R^n \setminus \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n, $\lambda > 0$, $0 < s < 1$ and $n > 2s$, $1 < p < 2_s^*$, $K(x) \in C(R^n) \cap L^\infty(R^n)$, $V(x) \geq 0$ and $V(x) \in C(R^n) \cap L^q(R^n)$ with $q = \frac{2_s^*}{2_s^* - p}$ here $(-\Delta)^s$ denotes the fractional Laplace operator defined, up to a normalization factor, by

$$(-\Delta)^s u(x) = \int_{R^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \mathrm{d}y, \quad x \in R^n. \tag{1.2}$$

The aim of this paper is to study the existence of solutions, we will see that if $1 < p < 2$, then by concentration-compactness principle, together with mini-max arguments, we can prove the existence of solutions for (1.1). We now summarize the main result of the paper.

Theorem 1.1. Let $1<p<2$, $K(x)\in C(R^n)\cap L^\infty(R^n)$ and $0\le V(x)\in C(R^n)\cap L^q(R^n)$ with $q=\frac{2_s^*}{2_s^*-p}$. Moreover, $V(x)>0$ is bounded on Ω. Then

1) For any $\lambda>0$, there exists $\tilde{\beta}>0$, then for any $0<\beta<\tilde{\beta}$, (1.1) has a consequence of weak solutions $\{u_n\}$.

2) For any $\beta>0$, there exist $\tilde{\lambda}>0$, then for any $0<\lambda<\tilde{\lambda}$, (1.1) has a consequence of weak solutions $\{u_n\}$.

We denote by $H^s(R^n)$ the usual fractional Sobolev space endowed with the so-called Gagliardo norm

$$\|u\|_{H^s(R^n)}=\|u\|_{L^2(R^n)}+\left(\int_{R^n\times R^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}, \tag{1.3}$$

Then we defined

$$X_0^s(\Omega)=\left\{u\in H^s(R^n): u=0 \text{ } a.e. \text{ in } R^n\setminus\Omega\right\} \tag{1.4}$$

endowed with the norm

$$\|u\|_{X_0^s(\Omega)}=\left(\int_{R^n\times R^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}, \tag{1.5}$$

we refer to [1] for a general definition of $X_0^s(\Omega)$ and its properties.

Observe that by [[2], Proposition 3.6] we have the following identity

$$\|u\|_{X_0^s(\Omega)}=\left\|(-\Delta)^{\frac{s}{2}}u\right\|_{L^2(R^n)}. \tag{1.6}$$

In this work, the Sobolev constant is given by (can be seen in [[3], theorem 7.58])

$$S(n,s):=\inf_{u\in H^s(R^n)\setminus\{0\}}Q_{n,s}(u)>0, \tag{1.7}$$

where

$$Q_{n,s}(u):=\frac{\int_{R^n\times R^n}\frac{\left|u(x)-u(y)^2\right|}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y}{\left(\int_{R^n\times R^n}|u(x)|^{2_s^*}\mathrm{d}x\right)^{\frac{2}{2_s^*}}},\quad u\in H^s(R^n) \tag{1.8}$$

2. Statements of the Result

We will use a variational approach to find a solution of (1.1). Firstly, we will associate a suitable functional to our problem, the Euler-Lagrange functional related to problem (1) is given by $J:X_0^s(\Omega)\to R$ defined as follow

$$J(u_n)=\frac{1}{2}\|u_n\|_{X_0^s(\Omega)}^2-\frac{\lambda}{p}\int_\Omega V(x)|u_n|^p\,\mathrm{d}x-\frac{\beta}{2_s^*}\int_\Omega K(x)|u_n|^{2_s^*}\,\mathrm{d}x. \tag{2.1}$$

To proof that J satisfy the Palais Smale condition at level c, we need the following lemma.

Lemma 2.1 [4] Letting ϕ be a regular function that satisfies that for some $\tilde{c} > 0$

$$|\phi(x)| \le \frac{\tilde{c}}{1+|x|^{n+s}}, \quad x \in R^n \tag{2.2}$$

and

$$|\nabla\phi(x)| \le \frac{\tilde{c}}{1+|x|^{n+s}}, \quad x \in R^n \tag{2.3}$$

Let $B : X_0^{\frac{s}{2}}(\Omega) \times X_0^{\frac{s}{2}}(\Omega) \to R$ be a bilinear form defined by

$$B(f,g)(x) := 2\int_R \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+s}} \mathrm{d}y. \tag{2.4}$$

then, for every $s \in (0,1)$, there exist positive constant c_1 and c_2, such that for $x \in R^n$, one has

$$\left|(-\Delta)^{\frac{s}{2}}\phi(x)\right| \le \frac{c}{1+|x|^{n+s}} \text{ and } |B(\phi,\phi)(x)| \le \frac{c}{1+|x|^{n+s}}. \tag{2.5}$$

To establish the next auxiliary result we consider a radial, nonincreasing cut-off function

$$\phi \in C_0^\infty(R^n) \text{ and } \phi_\varepsilon(x) := \phi\left(\frac{x}{\varepsilon}\right) \tag{2.6}$$

Lemma 2.2. [4] Letting $\{u_m\}$ be a uniformly bounded in $X_0^s(\Omega)$ and $\phi_\varepsilon \in C_0^\infty(R^n)$ the function defined in (2.6). Then,

$$\lim_{\varepsilon\to 0}\lim_{m\to 0}\left|\int_{R^n} u_m(x)(-\Delta)^{\frac{s}{2}}\phi_\varepsilon(x)(-\Delta)^{\frac{s}{2}}u_m(x)\mathrm{d}x\right| = 0. \tag{2.7}$$

Lemma 2.3. [4] With the same assumptions of Lemma 2.8 we have that

$$\lim_{\varepsilon\to 0}\lim_{m\to 0}\left|\int_{R^n}(-\Delta)^{\frac{s}{2}}u_m(x)\mathrm{d}xB(u_m,\phi_\varepsilon)(x)\right| = 0. \tag{2.8}$$

where B is defined in (2.4).

Lemma 2.4. [5] (Minimax principle) Assume that $E \in C(X,\mathbb{R})$, and $\mathcal{A}$ is a family of nonempty subset of X, denote

$$c = \inf_{A\in\mathcal{A}}\sup_{x\in A} E(x) \tag{2.9}$$

If the following conditions holds:

1) c is a finite real number;

2) there exists an $\bar{\varepsilon} > 0$, such that $\mathcal{A}$ is invariant with respect to the family of mappings;

$$\mathcal{T} = \{T \in (X,X) \mid T(x) = x, \text{ if } E(x) < c - \bar{\varepsilon}\}, \tag{2.10}$$

that is, for any $T \in \mathcal{T}$, there holds

$$A \in \mathcal{A} \Rightarrow \mathcal{T}(A) \in \mathcal{A}$$

Then, E possesses a $(PS)_c$ sequence at level c define as (6.1.1); Furthermore, if E satisfies the $(PS)_c$ condition (or the $(PS)_c$ condition at level c), then c is a critical value of E.

3. Proof of Theorem 1.1

Firstly, recalling that J is said to satisfy the Palais Smale condition at level c if any sequence $\{u_n\} \in X_0^s(\Omega)$ such that $J(u_n) \to c$ and $J'(u) \to 0$ has a convergent subsequence.

Lemma 3.1. The $(PS)_c$ sequence $\{u_n\}$ for J is bounded.

Proof. Note that $\{u_n\} \subset X_0^s(\Omega)$ satisfies

$$\begin{aligned} J(u_n) &= \frac{1}{2}\|u_n\|^2_{X_0^s(\Omega)} - \frac{\lambda}{p}\int_\Omega V(x)|u_n|^p \,\mathrm{d}x - \frac{\beta}{2_s^*}\int_\Omega K(x)|u_n|^{2_s^*} \,\mathrm{d}x \\ &= c + o_n(1) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \langle J'(u_n), \phi \rangle &= \int_\Omega (-\Delta)^s u_n \mathrm{d}x - \lambda \int_\Omega V(x)|u_n|^{p-2} u\phi \mathrm{d}x - \beta \int_\Omega K(x)|u_n|^{2_s^*-2} u\phi \mathrm{d}x \\ &= o_n(1)\|\phi\|_{X_0^s(\Omega)}, \qquad \forall \phi \in X_0^s(\Omega) \end{aligned} \tag{3.2}$$

where $o_n(1) \to 0$ as $n \to \infty$. Choose $\phi = u_n \in X_0^s(\Omega)$ as test function in (3.2), we get that

$$\begin{aligned} o_n(1)\|u_n\|_{X_0^s(\Omega)} &= \langle J'(u_n), u_n \rangle \\ &= \|u_n\|^2_{X_0^s(\Omega)} - \lambda \int_\Omega V(x)|u_n|^p \,\mathrm{d}x - \beta \int_\Omega K(x)|u_n|^{2_s^*} \,\mathrm{d}x \\ &= c + o_n(1). \end{aligned} \tag{3.3}$$

therefore, by (3.1) and (3.2), we have

$$\begin{aligned} & c + o_n(1) - \frac{1}{2_s^*} o_n(1)\|u_n\|_{X_0^s(\Omega)} \\ &= \frac{1}{2}\|u_n\|^2_{X_0^s(\Omega)} - \frac{\lambda}{p}\int_\Omega V(x)|u_n|^p \,\mathrm{d}x - \frac{\beta}{2_s^*}\int_\Omega K(x)|u_n|^{2_s^*} \,\mathrm{d}x \\ &\quad - \frac{1}{2_s^*}\|u_n\|^2_{X_0^s(\Omega)} - \frac{\lambda}{2_s^*}\int_\Omega V(x)|u_n|^p \,\mathrm{d}x - \frac{\beta}{2_s^*}\int_\Omega K(x)|u_n|^{2_s^*} \,\mathrm{d}x \\ &\geq \frac{s}{n}\|u_n\|^2_{X_0^s(\Omega)} - \left(\frac{\lambda}{p} - \frac{1}{2_s^*}\right)\|V(x)\|_{L^q(\Omega)}\|u_n\|^p_{L^{2_s^*}} \\ &\geq \frac{s}{n}\|u_n\|^2_{X_0^s(\Omega)} - \left(\frac{\lambda}{p} - \frac{1}{2_s^*}\right) S(n,s)^{-\frac{p}{2}}\|V(x)\|_{L^q(\Omega)}\|u_n\|^p_{X_0^s(\Omega)}. \end{aligned} \tag{3.4}$$

which yields the boundeness of $\{u_n\}$ in $X_0^s(\Omega)$,since $1 < p < 2$.

If $K(x) \in L^\infty(\mathbb{R}^n)$, then for $2 < p < 2_s^*$, similar to the proof of $1 < p < 2$, we get

$$c+o_n(1)+o_n(1)\|u_n\|_{X_0^s(\Omega)} \geq \left(\frac{p-2}{2p}\right)\|u_n\|_{X_0^s(\Omega)}^2 - \frac{(p-2_s^*)\beta}{2_s^*} S^{-\frac{2_s^*}{2}} \|u_n\|_{X_0^s(\Omega)}^{2_s^*}$$

Which also yields the boundedness of $(PS)_c$ sequence $\{u_n\}$.

Lemma 3.2. Assume that $c<0$. Then

1) For any $\lambda>0$, there exists $\beta_0>0$, such that for any $0<\beta<\beta_0$, then J satisfies $(PS)_c$.

2) For any $\beta>0$ there exists $\lambda_0>0$ such that for any $0<\lambda<\lambda_0$, then J satisfies $(PS)_c$.

Proof. By Lemma3.1 $\{u_n\}$ is bounded in $X_0^s(\Omega)$, up to a subsequence, we get that

$$\begin{aligned} &u_n \to u && x\in X_0^s(\Omega). \\ &u_n \to u && x\in L^r(\Omega),\ 1\leq r<2_s^*. \\ &u_n \to u && \text{a.e. } x\in\Omega. \end{aligned} \tag{3.5}$$

Following [6] it is easy to prove that $X_0^s(\Omega)$ could also be the $X_0^s(\Omega)$-norm. Applying [[7], Theorem1.5], we have that the exist an index. Set $I\subseteq N$ a sequence of point $\{x_k\}_{x\in I\subset\Omega}$ and two sequences of nonnegative real numbers $\{\mu_k\}_{k\in I}, \{v_k\}_{k\in I}$, such that

$$\left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 \to \mu\left|(-\Delta)^{\frac{s}{2}} u\right|^2 + \sum_{k\in I}\mu_k\delta_{x_k}. \tag{3.6}$$

moreover

$$|u_n|^{2_s^*} \to \mu|u|^{2_s^*} + \sum_{k\in I} v_k\delta_{x_k}. \tag{3.7}$$

in the sense of measures, with

$$v_k \leq S(s,n)^{-\frac{2_s^*}{2}} \mu_k^{\frac{2_s^*}{2}} \quad \text{for every } k\in I \tag{3.8}$$

here δ_{x_k} denotes the Dirac Delta at x_k, while $S(n,s)$ is the constant given in (1.7), we consider $\phi\in C_0^\infty(R^n)$ a nonincreasing cut-off function satisfying

$$\phi=1 \text{ in } B_1(x_{k_0}) \text{ and } \phi=0 \text{ in } B_2(x_{k_0})^c \tag{3.9}$$

Set $\phi_\varepsilon(x)=\phi\left(\frac{x}{\varepsilon}\right), x\in R^n$ taking the derivative of (1.6), for any $u,\phi\in X_0^s(\Omega)$. We obtain that

$$\int_{R^n\times R^n} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y = \int_{R^n}\phi(x)(-\Delta)^s u(x)\mathrm{d}x \tag{3.10}$$

Then, taking $\phi_\varepsilon u_n$ as a test function in $J'(u_n)\to 0$

$$\lim_{n\to 0}\int_{\mathbb{R}^n}\phi_\varepsilon u_n(-\Delta)u_n\mathrm{d}x - \left(\lambda\int_{B_{2\varepsilon}(x_{k_0})} V(x)u_n^p\phi_\varepsilon\mathrm{d}x + \beta\int_{B_{2\varepsilon}(x_{k_0})} K(x)u_n^{2_s^*}\phi_\varepsilon\mathrm{d}x\right)=0 \tag{3.11}$$

by (3.10), we have

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^n} u_n(x)(-\Delta)^{\frac{s}{2}} u_n(x)(-\Delta)^{\frac{s}{2}}\phi_\varepsilon(x)\mathrm{d}x\\&-2\int_{\mathbb{R}^n}(-\Delta)^{\frac{s}{2}} u_n(x)\int_{\mathbb{R}^n}\frac{(\phi_\varepsilon(x)-\phi_\varepsilon(y))(u_n(x)-u_n(y))}{|x-y|^{n+s}}\mathrm{d}x\mathrm{d}y\\&=\lim_{n\to\infty}\lambda\int_{B_{2\varepsilon}(x_{k_0})}V(x)|u_n|^p(x)\phi_\varepsilon(x)\mathrm{d}x+\beta\int_{B_{2\varepsilon}(x_{k_0})}K(x)|u_n|^{2_s^*}(x)\phi_\varepsilon(x)\mathrm{d}x\\&-\int_{B_{2\varepsilon}(x_{k_0})}\left((-\Delta)^{\frac{s}{2}}u_n\right)^2\phi_\varepsilon(x)\mathrm{d}x.\end{aligned}\tag{3.12}$$

therefore, by (3.5) (3.6) and (3.7) we get

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^n} u_n(x)(-\Delta)^{\frac{s}{2}} u_n(x)(-\Delta)^{\frac{s}{2}}\phi_\varepsilon(x)\mathrm{d}x\\&-2\int_{\mathbb{R}^n}(-\Delta)^{\frac{s}{2}} u_n(x)\int_{\mathbb{R}^n}\frac{(\phi_\varepsilon(x)-\phi_\varepsilon(y))(u_n(x)-u_n(y))}{|x-y|^{n+s}}\mathrm{d}x\mathrm{d}y\\&=\lim_{\varepsilon\to0}\lambda\int_{B_{2\varepsilon}(x_{k_0})}V(x)|u_n|^p(x)\phi_\varepsilon(x)\mathrm{d}x+\int_{B_{2\varepsilon}(x_{k_0})}\phi_\varepsilon(x)\mathrm{d}v\\&-\beta\int_{B_{2\varepsilon}(x_{k_0})}K(x)\phi_\varepsilon(x)\mathrm{d}\mu.\end{aligned}\tag{3.13}$$

Since ϕ is regular function with compact support, it is easy to see that it satisfies the hypothesis of Lemma 2.1, by Lemma 2.2 and Lemma 2.3 applied to the sequence $\{u_n\}$, it follows that the left hand side of (3.13) goes to zero. We obtain that

$$\begin{aligned}&\lim_{\varepsilon\to0}\left(\lambda\int_{B_{2\varepsilon}(x_{k_0})}V(x)|u_n|^p(x)\phi_\varepsilon(x)\mathrm{d}x+\int_{B_{2\varepsilon}(x_{k_0})}\phi_\varepsilon(x)\mathrm{d}v-\beta\int_{B_{2\varepsilon}(x_{k_0})}K(x)\phi_\varepsilon(x)\right)\mathrm{d}\mu\\&=\beta K(x_{k_0})v_{k_0}-\mu_{k_0}=0.\end{aligned}\tag{3.14}$$

Clearly, if $K(x)\le 0$, we get $\mu_{k_0}=v_{k_0}=0$; if $K(x_{k_0})>0$, by (3.8), we get

$$v_{k_0}=0 \quad\text{or}\quad v_{k_0}\ge\left[\frac{S(n,s)}{\beta K(x_{k_0})}\right]^{\frac{n}{2s}}.$$

suppose that $v_{k_0}\ne 0$, we know that

$$0>c=\lim_{n\to\infty}\left[J(u_n)-\frac{1}{2_s^*}\langle J'(u_n),u_n\rangle\right]\tag{3.15}$$

according to the embedded theorem, we have

$$\begin{aligned}0>c&\ge\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|u_n\|^2_{X_0^s(\Omega)}-\left(\frac{\lambda}{p}-\frac{\lambda}{2_s^*}\right)\int_\Omega V(x)|u_n|^p\mathrm{d}x\\&=\frac{s}{n}\|u_n\|^2_{X_0^s(\Omega)}-\left(\frac{\lambda}{p}-\frac{\lambda}{2_s^*}\right)\int_\Omega V(x)|u_n|^p\mathrm{d}x\\&\ge\frac{s}{n}S^{-1}(n,s)\|u_n\|^2_{L^{2_s^*}(\Omega)}-\left(\frac{\lambda}{p}-\frac{\lambda}{2_s^*}\right)S^{-\frac{p}{2}}(n,s)\|V(x)\|_{L^q(\Omega)}\|u_n\|^p_{L^{2_s^*}}.\end{aligned}\tag{3.16}$$

This yields that

$$\|u\|_{L^{2_s^*}(\Omega)} \le C\lambda^{\frac{1}{2-p}}. \tag{3.17}$$

Thus, if $v_{k_0} \ge \left[\frac{S(n,s)}{\beta K(x_{k_0})}\right]^{\frac{n}{2s}}$, we get that

$$\begin{aligned}
0 > c &= \lim_{n\to\infty}\left[J(u_n) - \frac{1}{2_s^*}\langle J'(u_n), u_n\rangle\right] \\
&\ge \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|u\|^2_{X_0^s(\Omega)} + \frac{s}{n}\mu_{k_0} - \left(\frac{\lambda}{p} - \frac{\lambda}{2_s^*}\right)\int_\Omega V(x)|u|^p \mathrm{d}x \\
&\ge \frac{s}{n}S^{-1}(n,s)\|u\|^2_{L^{2_s^*}(\Omega)} + \frac{s}{n}\mu_{k_0} - \left(\frac{\lambda}{p} - \frac{\lambda}{2_s^*}\right)S^{-\frac{p}{2}}(n,s)\|V(x)\|_{L^q(\Omega)}\|u\|^p_{L^{2_s^*}} \\
&\ge \frac{s}{n}S^{-1}(n,s)\|u\|^2_{L^{2_s^*}(\Omega)} + \frac{s}{n}\mu_{k_0} - \left(\frac{\lambda}{p} - \frac{\lambda}{2_s^*}\right)S^{-\frac{p}{2}}(n,s)\|V(x)\|_{L^q(\Omega)}\|u\|^p_{L^{2_s^*}} \\
&\ge \frac{s}{n}S(n,s)v_{k_0}^{-\frac{2_s^*}{2}} - \left(\frac{\lambda}{p} - \frac{\lambda}{2_s^*}\right)S^{-\frac{p}{2}}(n,s)\|V(x)\|_{L^p(\Omega)}\|u\|^p_{L^{2_s^*}} \\
&\ge \frac{s}{n}S^{\frac{n}{2s}}(n,s)\left[\beta K(x_{k_0})\right]^{\frac{2s-n}{2s}} - C\lambda^{\frac{2}{2-p}}.
\end{aligned} \tag{3.18}$$

However, if $\beta > 0$ is given, we can choose $\lambda_0 > 0$ so small for every $0 < \lambda < \lambda_0$ that last term on the right-hand side above is greater than 0 which is contradiction when $2 < p < 2_s^*$

$$\begin{aligned}
0 > c &= \lim_{n\to\infty}\left[J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle\right] \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2_{X_0^s(\Omega)} - \left(\frac{\beta}{2_s^*} - \frac{\beta}{p}\right)\int_\Omega K(x)|u|^{2_s^*}\mathrm{d}x \\
&\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2_{X_0^s(\Omega)} - \left(\frac{\beta}{2_s^*} - \frac{\beta}{p}\right)\int_{\Omega\cap\{K(x)<0\}} K(x)|u|^{2_s^*}\mathrm{d}x \\
&\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2_{X_0^s(\Omega)} - \left(\frac{\beta}{2_s^*} - \frac{\beta}{p}\right)\|K(x)\|_{L^\infty}\|u\|^{2_s^*}_{L^{2_s^*}}
\end{aligned}$$

β is the same as λ greater than 0. We see that $v_{k_0} \ge \left[\frac{S(n,s)}{\beta K(x_{k_0})}\right]^{\frac{n}{2s}}$ cannot occur if λ_0 or β_0 are choose properly. Thus $\mu_k = v_k = 0$. As consequence, we obtain that $(u_n)_+ - u \to 0$ in $L^{2_s^*}(\Omega)$, that is $\lim_{n\to\infty}\int_{R^n}|(u_n)_+|^{2_s^*}\mathrm{d}x = \int_{R^n}|u|^{2_s^*}\mathrm{d}x$. This implies convergence of $\lambda V(x)|u_n|^{p-1} + \beta K(x)|u_n|^{2_s^*-1}$ in $L^{2_s^*}(\Omega)$. Finally using the continuity of the inverse operator $(-\Delta)^s$. We obtain strong convergence of u_n in $X_0^s(\Omega)$. #

Next, by using the classical concept and properties of the genus, we construct a min-max class of the critical point.

For a Banach space X, We define the set

$\mathcal{A}=\{A\subset X\setminus\{0\}: A \text{ is closed in } X \text{ and symmetric with respect to the orign}\}$

For $\mathcal{A}\in A$, define

$$\gamma(A):=\inf\{m\in N,\exists\phi\in C(A,R^m\setminus\{0\}),\phi(x)=-\phi(-x)\} \tag{3.19}$$

If there is no mapping ϕ as above for any $m\in N$, there $\gamma(A)=+\infty$. we refer to [8]for the properties of the genus.

Proposition 3.3. [8] Let $A,B\subset \mathrm{A}$,

1) If there exists an odd map $f\in C(A,B)$, then $\gamma(A)\le\gamma(B)$;

2) If $A\subset B$, then $\gamma(A)\le\gamma(B)$;

3) $\gamma(A\cup B)\le\gamma(A)+\gamma(B)$;

4) If S is a sphere centered at the origin in R^m, then $\gamma(s)=m$;

5) If A is compact, there exists a symmetric Neighborhood N of A, such that $\gamma(\bar{N})=\gamma(A)$.

According Holder inequality, we get that

$$\begin{aligned}J(u)&=\frac{1}{2}\|u\|_{X_0^s}^2-\frac{\lambda}{p}\int_\Omega V(x)|u|^p\mathrm{d}x-\frac{\beta}{2_s^*}\int_\Omega K(x)|u|^{2_s^*}\mathrm{d}x\\&\ge\frac{1}{2}\|u\|_{X_0^s}^2-C_1\lambda\|u\|_{X_0^s}^p-C_2\beta\|u\|_{X_0^s}^{2_s^*}\end{aligned} \tag{3.20}$$

We define the function

$$Q(t):=\frac{1}{2}t^2-C_1\lambda t^p-C_2\beta t^{2_s^*} \tag{3.21}$$

Then it is easy to see that given $\beta>0$, there exists $\lambda_1>0$ so small that for every $0<\lambda<\lambda_1$, there exists $0<T_0<T_1$ such that $Q(t)<0$ for $0\le t\le T_0$, $Q(t)>0$ for $T_0<t<T_1$.and $Q(t)<0$ $t>T_1$. Analogously, for given $\lambda>0$, we can choose $\beta_1>0$ with the property that T_0,T_1 as above for each $0<\beta<\beta_1$. Clearly, $Q(T_0)=Q(T_1)=0$.

As in [9], Let $\tau:\mathbb{R}^+\to[0,1]$ be a nonincreasing C^∞ function such that $\tau(t)=1$ if $0\le\tau\le T_0$ and $\tau(t)=0$. if $\tau\ge T_0$. Set $\Psi(u)=\tau(\|u\|_{X_0^s(\Omega)})$, we make the following truncation of the function J:

$$\tilde{J}(u)=\frac{1}{2}\|u\|_{X_0^s}^2-\frac{\lambda}{p}\int_\Omega V(x)|u|^p\mathrm{d}x-\frac{\beta}{2_s^*}\psi(u)\int_\Omega K(x)|u|^{2_s^*}\mathrm{d}x \tag{3.22}$$

then

$$\tilde{J}(u)\ge\tilde{Q}\|u\|_{X_0^s(\Omega)}. \tag{3.23}$$

where $\tilde{Q}(t):=\frac{1}{2}t^2-C_1\lambda t^p-C_2\beta t^{2_s^*}\psi(t)$.

It is clear that $\tilde{J}(u)\in C^1$ and is bounded from below.

Lemma 3.4. [10] 1) For any $\lambda>0$ and $0<\beta<\beta_1$ or any $\beta>0$ and $0<\lambda<\lambda_1$, if $\tilde{J}(u)<0$, then $\|u\|_{X_0^s(\Omega)}<T_0$ and $\tilde{J}(u)=J(u)$.

2) For any $\lambda>0$, there exists such that if $0<\beta<\bar{\beta}$ and $c<0$ then $\tilde{J}$ satisfies $(PS)_c$.

3) For any $\beta > 0$,there exists $\tilde{\lambda} > 0\left(\tilde{\lambda} \le \lambda_1\right)$ such that if $0 < \lambda < \tilde{\lambda}$ and $c < 0$ then $\tilde{J}$ satisfies $(PS)_c$.

Lemma 3.5. Denote $\tilde{J}^\alpha := \left\{u \in X_0^s(\Omega), \tilde{J}(u) \le \alpha\right\}$. Then for any $m \in N$, there is $\varepsilon_m < 0$ such that $\gamma\left(\tilde{J}^{\varepsilon_m}\right) \ge m$.

Proof. Denote by $X_0^s(\Omega)$ the closure of $C_0^\infty(\Omega)$ with the respect to norm $\|u\|_{X_0^s(\Omega)} = \left(\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}$, $V(x) > 0$ in Ω. Extending functions in $X_0^s(\Omega)$ by 0 outside Ω. Let X_m be a m-dimensional subspace of $X_0^s(\Omega)$. For any $u \in X_m, u \ne 0$. We write $u = r_m w$ with $w \in X_m$ and $\|w\|_{X_0^s(\Omega)} = 1$. From the assumptions of $V(x)$, it is easy to see for every $w \in X_m$ with $\|w\|_{X_0^s(\Omega)} = 1$ that there exists $d_m > 0$ such that

$$\int_\Omega V(x)|w|^p \mathrm{d}x \ge d_m \tag{3.24}$$

For $0 < r_m < T_0$. Since all the norms are equivalent, we get

$$\begin{aligned}\tilde{J}(u) = J(u) &= \frac{1}{2}\|u\|_{X_0^s(\Omega)}^2 - \frac{\lambda}{p}\int_\Omega V(x)|u|^p \mathrm{d}x - \frac{\beta}{2_s^*}\int_\Omega K(x)|u|^{2_s^*}\mathrm{d}x \\ &\le \frac{1}{2}\|u\|_{X_0^s(\Omega)}^2 - \frac{\lambda}{p}\int_\Omega V(x)|u|^p \mathrm{d}x + \frac{\beta}{2_s^*}\left|\int_\Omega K(x)|u|^{2_s^*}\mathrm{d}x\right| \\ &\le \frac{1}{2}r_m^2 - \lambda c d_m + c\beta r_m^{2_s^*} := \varepsilon_m.\end{aligned}$$

Therefore for given λ and β. we can choose $r_m \in (0, T_0)$ sufficiently small so that $\tilde{J}(u) \le \varepsilon_m < 0$. #

Let $S_{r_m} = \left\{u \in X_0^s(\Omega) : \|u\|_{X_0^s(\Omega)} = r_m\right\}$. Then $S_{r_m} \cap X_m \subset \tilde{J}^{\varepsilon_m}$, Hence by proposition 3.3 (2) and (4) $r\left(\tilde{J}^{\varepsilon_m}\right) \ge r\left(S_{r_m} \cap X_m\right) \ge m$.

We denote $\Gamma_m = \{A \in \mathrm{A} : \gamma(A) \ge m\}$ and let

$$C_m := \inf_{A \in \Gamma_m} \sup_{u \in A} J(u) \tag{3.25}$$

then

$$-\infty < C_m \le \varepsilon_m < 0, \quad m \in N \tag{3.26}$$

because $\tilde{J}^{\varepsilon_m} \in \Gamma_m$ and $\tilde{J}$ is bounded from below.

Proposition 3.6. Let λ, β be as in Lemma 3.5 (2) and (3). Then all c_m given by (3.25) are critical values of $\tilde{J}$ and $c_m \to 0$ as $m \to 0$.

Proof. Denote $K_\varepsilon = \left\{u \in X_0^s(\Omega) : \tilde{J}(u) = c, \tilde{J}'(u) = 0\right\}$. Then by Lemma 3.4 (2) and (3), if $c < 0$, K_c is compact. It is clear that $C_m \le C_{m+1}$. By (3.26) $C_m < 0$. Hence $C_m \to \bar{C} \le 0$. Moreover, since $(PS)_c$ satisfied, it follows from a standard argument (see [11]) that all C_m are critical values of $\tilde{J}$. Now, we claim that $\bar{c} = 0$. If $\bar{c} < 0$ because $K_{\bar{c}}$ is compact and $K_{\bar{c}} \in A$, it follows from Proposition 3.3 (5) that $\gamma(K_{\bar{c}}) = m_0 < +\infty$ and there exists $\delta > 0$ such that $\gamma(K_{\bar{c}}) = \gamma(N_\delta(K_{\bar{c}})) = m_0$. By the deformation Lemma [9], there exists $\varepsilon > 0(\bar{c} + \varepsilon < 0)$ and an odd homeomorphism $\varsigma(\cdot) : X_0^s(\Omega) \to X_0^s(\Omega)$ such that

$$\varsigma\left(\tilde{J}^{\bar{c}+\varepsilon} \setminus N_\delta\left(K_{\bar{c}}\right)\right) \subset \tilde{J}^{\bar{c}-\varepsilon} \tag{3.27}$$

Since c_m is increasing anad converges to $\bar{c}$. there exists $m \in N$ such that

$$c_m > \bar{c} - \varepsilon . \tag{3.28}$$

And exists a $A \in \Gamma_{m+m_0}$ such that

$$\sup_{u \in A} \tilde{J}(u) < \bar{c} + \varepsilon \tag{3.29}$$

By Proposition 3.3 (3), we obtain

$$\gamma\left(\overline{A \setminus N_\delta\left(K_{\bar{c}}\right)}\right) \geq \gamma(A) - \gamma\left(N_\delta\left(K_{\bar{c}}\right)\right) \geq m \tag{3.30}$$

By Proposition 3.3 (1), we obtain

$$\gamma\left(\overline{\varsigma\left(A \setminus N_\delta\left(K_{\bar{c}}\right)\right)}\right) \geq m \tag{3.31}$$

therefore

$$\varsigma\left(A \setminus N_\delta\left(K_{\bar{c}}\right)\right) \in \Gamma_m$$

consequently, from (3.28), we get

$$\sup_{u \in \varsigma\left(A \setminus N_\delta\left(K_{\bar{c}}\right)\right)} \tilde{J}(u) \geq c_m > \bar{c} - \varepsilon \tag{3.32}$$

on the other hand, by (3.27) and (3.29)

$$\varsigma\left(A \setminus N_\delta\left(K_{\bar{c}}\right)\right) \subset \varsigma\left(\tilde{J}^{\bar{c}+\varepsilon} \setminus N_\delta\left(K_{\bar{c}}\right)\right) \subset \tilde{J}^{\bar{c}-\varepsilon} \tag{3.33}$$

which implies that

$$\sup_{u \in \varsigma\left(A \setminus N_\delta\left(K_{\bar{c}}\right)\right)} \tilde{J}(u) \leq \bar{c} - \varepsilon \tag{3.34}$$

this contradicts to (3.32).Hence $c_m \to 0$. #

By (1) of Lemma 3.4 $\tilde{J}(u) = J(u)$ if $\tilde{J}(u) < 0$. This and Proposition 3.6 give Theorem1.1.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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