

# Existence of Solutions for Boundary Value Problems of Conformable Fractional Differential Equations

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## Abstract

In this paper, we study a class of boundary value problems for conformable fractional differential equations under a new definition. Firstly, by using the monotone iterative technique and the method of coupled upper and lower solution, the sufficient condition for the existence of the boundary value problem is obtained, and the range of the solution is determined. Then the existence and uniqueness of the solution are proved by the proof by contradiction. Finally, a concrete example is given to illustrate the wide applicability of our main results.

## Keywords

Boundary Value Problems, Conformable Fractional Differential Equations, The Method of Coupled Upper and Lower Solution, Coupled Solution, Monotone Iteration

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## 1. Introduction

In recent years, there are few studies on boundary value problems of conformable fractional differential equations under new definitions [1] [2] [3]. And conformable fractional derivatives not only have good operational properties (Four Operational Rules of Derivatives, Chain Rule and Leibniz Rule), this definition can also construct fractional Newton equation and Euler-Lagrange equation from fractional variational method, this is of great significance to the study of uniform or uniformly accelerated motion of particles and to the solution of Newton's fractional-order mechanical problems [4] [5] (fractional-order harmonic oscillator, fractional-order damped oscillator and forced oscillator). And the method of upper and lower solution for monotone iteration can not only

gives the existence theorem, but also determines the value range of the solution. Therefore, this method has gradually become an important method for studying nonlinear differential equations [6] [7] [8] [9]. In addition, with the application of anti-periodic boundary value problems in various mathematical models and physical processes has been widely applied, the integral boundaries are also widely used in heat conduction, chemical engineering, groundwater flow, thermoelasticity, plasma physics and other fields. As a result, more and more studies have been made on this kind of problems [10] [11] [12] (anti-periodic boundary value problems, anti-periodic boundary value problems with integral boundaries). However, the indefinite sign of solutions of nonlinear differential equations determines that some problems (anti-periodic boundary value problems and their generalizations) cannot be studied directly by the method of upper and lower solutions for monotone iteration. But the development of nonlinear analysis theory provides a powerful tool for the study of these problems. In the generalized monotone iteration process, the method of coupled upper and lower solution becomes an important method to study this kind of problem by the flexible construction of the comparison theorem [13] [14] [15] [16]. Motivated by the above work, in this paper, the existence of solutions for a class of boundary value problems of conformable fractional differential equations under a new definition is proved by using the method of coupled upper and lower solution, and the range of solutions is obtained. Throughout this paper, we consider the existence of solutions of boundary value problems for the following uniform fractional differential equations

$$\begin{cases} x^{(\delta)}(t) = f(t, x(t)), & t \in (0, 1), \\ x(0) = -rx(1) + \lambda \int_0^1 x(s) ds \end{cases} \quad (1)$$

where  $x^{(\delta)}(t)$  is the conformable fractional derivatives of order  $\delta$  for  $t \in (0, 1)$  which is defined in [1], and  $\delta \in (0, 1]$ ,  $r > 0$ ,  $\lambda > 0$ ,  $\mathbb{R} = (-\infty, +\infty)$ ,  $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

## 2. Preliminaries

In this section, we present some definitions and lemmas which will be used in the proof of our main results.

**Definition 2.1.** (See [1]) Given a function  $x: [0, +\infty) \rightarrow \mathbb{R}$ . Then the conformable fractional derivative of  $x$  of order  $\delta$  is defined by

$$x^{(\delta)}(t) := \lim_{\varepsilon \rightarrow 0} \frac{x(t + \varepsilon t^{1-\delta}) - x(t)}{\varepsilon},$$

for all  $t > 0$ ,  $\delta \in (0, 1)$ . If the conformable fractional derivative of  $x$  of order  $\delta$  exists, then we simply say that  $x$  is  $\delta$ -differentiable. If  $x$  is  $\delta$ -differentiable in some  $t \in (0, a)$ ,  $a > 0$ , and  $\lim_{t \rightarrow 0} x^{(\delta)}(t)$  exists, then we define

$$x^{(\delta)}(0) := \lim_{t \rightarrow 0} x^{(\delta)}(t).$$

**Definition 2.2.** Let  $y_0(t), z_0(t) \in C([0, 1], \mathbb{R})$ , then  $y_0 = y_0(t), z_0 = z_0(t)$

are said to be coupled lower and upper solutions of (1), respectively, if

$$\begin{cases} y_0^{(\delta)}(t) \leq f(t, y_0(t)), & t \in (0, 1), \\ y_0(0) + rz_0(1) \leq \lambda \int_0^1 y_0(s) ds. \\ z_0^{(\delta)}(t) \geq f(t, z_0(t)), & t \in (0, 1), \\ z_0(0) + ry_0(1) \geq \lambda \int_0^1 z_0(s) ds. \end{cases}$$

**Definition 2.3.** Let  $y, z \in C([0, 1], \mathbb{R})$ , then the function pair  $(y, z)$  is said to be coupled solutions of (1), if

$$\begin{cases} y^{(\delta)}(t) = f(t, y(t)), & t \in (0, 1), \\ y(0) + rz(1) = \lambda \int_0^1 y(s) ds. \\ z^{(\delta)}(t) = f(t, z(t)), & t \in (0, 1), \\ z(0) + ry(1) = \lambda \int_0^1 z(s) ds. \end{cases}$$

Let  $\gamma, \rho \in C([0, 1], \mathbb{R})$ , then  $(\rho, \gamma)$  is said to be minimum and maximum coupled solutions of (1), if  $(\rho, \gamma)$  are coupled solutions of (1), and  $\rho(t) \leq y(t), z(t) \leq \gamma(t)$  for any coupled solution  $(y, z)$ .

**Lemma 2.1.** (See [1]) Let  $\delta \in (0, 1]$ , and assume  $x_1, x_2$  to be  $\delta$ -differentiable, then

- 1)  $(ax_1 + bx_2)^{(\delta)}(t) = ax_1^{(\delta)}(t) + bx_2^{(\delta)}(t)$ ;
- 2)  $(x_1x_2)^{(\delta)}(t) = x_1(t)x_2^{(\delta)}(t) + x_2(t)x_1^{(\delta)}(t)$ ;
- 3)  $\left(\frac{x_1}{x_2}\right)^{(\delta)}(t) = \frac{x_2(t)x_1^{(\delta)}(t) - x_1(t)x_2^{(\delta)}(t)}{x_2^2(t)}$ .

for  $t \in (0, 1)$ ,  $a, b \in \mathbb{R}$ .

**Lemma 2.2.** (See [1]) If  $x$  is differentiable,  $t > 0$ , then  $x^{(\delta)}(t) = t^{1-\delta} \frac{dx}{dt}(t)$ .

**Lemma 2.3** (See [3]) If  $x^{(\delta)}(t)$  exists, then for  $t \neq 0$ , we have  $x^{(\delta)}(t) = t^{1-\delta} x'(t)$ .

**Lemma 2.4.** Assume that  $g \in C([0, 1], \mathbb{R})$ , and  $\delta \in (0, 1]$ ,  $m \in \mathbb{R}$ ,  $M > 0$ , Define function  $p : [0, 1] \rightarrow \mathbb{R}$  as follows:

$$p(t) = me^{-\frac{M}{\delta}t^\delta} + \int_0^t s^{\delta-1} g(s) e^{\frac{M}{\delta}(s^\delta - t^\delta)} ds. \tag{2}$$

Then  $p(t)$  is the solution of the initial value problem as follows

$$\begin{cases} p^{(\delta)}(t) + Mp(t) = g(t), & t \in (0, 1], \\ p(0) = m \end{cases}$$

**Proof** Assume that  $p(t)$  is given by (2), then  $p$  is differentiable for  $t > 0$ , therefore we have

$$\begin{aligned} p^{(\delta)}(t) &= t^{1-\delta} t^{\delta-1} \left( -mMe^{-\frac{M}{\delta}t^\delta} - Me^{-\frac{M}{\delta}t^\delta} \int_0^t e^{\frac{M}{\delta}(s^\delta - t^\delta)} s^{\delta-1} g(s) + g(t) \right) \\ &= -M \left( me^{-\frac{M}{\delta}t^\delta} + \int_0^t s^{\delta-1} g(s) e^{\frac{M}{\delta}(s^\delta - t^\delta)} ds \right) + g(t) \\ &= -Mp(t) + g(t). \end{aligned}$$

from Lemma 2.2, and  $p(t)$  subject to the condition

$$p(0) = m.$$

□

**Lemma 2.5.** (Comparison Theorem) Let  $p \in C([0,1], \mathbb{R}), M > 0$ , and the following inequalities hold true

$$\begin{cases} p^{(\delta)}(t) + Mp(t) \leq 0, & t \in [0,1], \\ p(0) \leq 0. \end{cases}$$

then  $p(t) \leq 0$ , for  $t \in [0,1]$ .

**Proof** Let  $p^{(\delta)}(t) + Mp(t) = g(t), p(0) = m$ , then we have  $g(t) \geq 0, m \geq 0$  for  $t \in [0,1]$ , and we can draw a conclusion from (2.1) and Lemma 2.3. □

### 3. Conclusions

**Theorem 3.1.** Assume that  $y_0(t), z_0(t)$  are coupled lower and upper solutions of (1.1) with  $y_0(t) \leq z_0(t)$  for  $t \in [0,1]$ , let

$D = \{x \in C([0,1], \mathbb{R}) \mid y_0(t) \leq x \leq z_0(t)\}$ . And if  $y_0(t) \leq x_2 \leq x_1 \leq z_0(t)$ , then the following inequalities hold true

$$f(t, x_1) - f(t, x_2) \geq -M(x_1 - x_2). \tag{3}$$

for  $t \in [0,1]$  and  $M > 0$ . If we take  $y_0(t), z_0(t)$  as initial elements, the iterative sequences defined by

$$\begin{cases} y_n(t) = \left( \lambda \int_0^1 y_{n-1}(s) ds - rz_{n-1}(1) \right) e^{-\frac{M}{\delta}t^\delta} + \int_0^t s^{\delta-1} f_{y_{n-1}}(s) e^{\frac{M}{\delta}(s^\delta - t^\delta)} ds, & t \in [0,1] \\ z_n(t) = \left( \lambda \int_0^1 z_{n-1}(s) ds - ry_{n-1}(1) \right) e^{-\frac{M}{\delta}t^\delta} + \int_0^t s^{\delta-1} f_{z_{n-1}}(s) e^{\frac{M}{\delta}(s^\delta - t^\delta)} ds, & t \in [0,1] \end{cases} \tag{4}$$

are  $\{y_n(t)\}$  and  $\{z_n(t)\}$ , then

- 1)  $y_n(t) \rightarrow y^*(t)$  and  $z_n(t) \rightarrow z^*(t)$  uniformly and  $y^*, z^* \in D$ ;
- 2)  $(y^*, z^*)$  are coupled minimal and maximal solutions of (1.1) respectively in  $D$ ;
- 3) If  $x(t)$  is the solution of (1.1) in  $D$ , then we have  $y^* \leq x \leq z^*$ ; i.e., we have

$$y^*(t) \leq x(t) \leq z^*(t),$$

for  $t \in [0,1]$ .

**Proof 1).** There is a unique solution to the boundary value problem as follows

$$\begin{cases} y^{(\delta)}(t) = f(t, u(t)) - M(y(t) - u(t)), & t \in (0,1), \\ y(0) + rv(1) = \lambda \int_0^1 u(s) ds, \\ z^{(\delta)}(t) = f(t, v(t)) - M(z(t) - v(t)), & t \in (0,1), \\ z(0) + ru(1) = \lambda \int_0^1 v(s) ds, \end{cases}$$

which is given by

$$\begin{cases} y(t) = \left( \lambda \int_0^1 u(s) ds - rv(1) \right) e^{-\frac{M}{\delta} t^\delta} + \int_0^t s^{\delta-1} f_u(s) e^{\frac{M}{\delta}(s^\delta - t^\delta)} ds, & t \in [0,1], \\ z(t) = \left( \lambda \int_0^1 v(s) ds - ru(1) \right) e^{-\frac{M}{\delta} t^\delta} + \int_0^t s^{\delta-1} f_v(s) e^{\frac{M}{\delta}(s^\delta - t^\delta)} ds, & t \in [0,1]. \end{cases}$$

for  $u, v \in D$  and  $u \leq v$  from Lemma 2.2 and Lemma 2.3. Where

$f_v(t) = f(t, v(t)) + Mv(t)$ ,  $f_u(t) = f(t, u(t)) + Mu(t)$ . Define operator  $T : D \times D \rightarrow C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$

$$T(u, v)(t) = (T_1(u, v), T_2(u, v)),$$

where operators  $T_1, T_2$  are given by

$$\begin{cases} T_1(u, v) = \left( \lambda \int_0^1 u(s) ds - rv(1) \right) e^{-\frac{M}{\delta} t^\delta} + \int_0^t s^{\delta-1} f_u(s) e^{\frac{M}{\delta}(s^\delta - t^\delta)} ds, & t \in [0,1], \\ T_2(u, v) = \left( \lambda \int_0^1 v(s) ds - ru(1) \right) e^{-\frac{M}{\delta} t^\delta} + \int_0^t s^{\delta-1} f_v(s) e^{\frac{M}{\delta}(s^\delta - t^\delta)} ds, & t \in [0,1]. \end{cases}$$

respectively. Then the fixed point of operator  $T$  in  $D \times D$  means the coupled solutions of (1).

Let  $y_1 = T_1(y_0, z_0)$ ,  $z_1 = T_2(y_0, z_0)$ .

Here we prove that  $y_0 \leq y_1$ ,  $z_1 \leq z_0$ ,  $y_1 \leq z_1$ , and  $y_1, z_1$  are coupled lower and upper solutions of (1).

Whereas

$$\begin{cases} y_1^{(\delta)}(t) = f(t, y_0(t)) - M(y_1(t) - y_0(t)), & t \in (0,1), \\ y_1(0) + rz_0(1) = \lambda \int_0^1 y_0(s) ds, \\ z_1^{(\delta)}(t) = f(t, z_0(t)) - M(z_1(t) - z_0(t)), & t \in (0,1), \\ z_1(0) + ry_0(1) = \lambda \int_0^1 z_0(s) ds. \end{cases} \quad (5)$$

And  $y_0, z_0$  are coupled lower and upper solutions of (1), then we have

$$\begin{cases} (y_0^{(\delta)}(t) - y_1^{(\delta)}(t)) + M(y_0(t) - y_1(t)) \leq 0, \\ y_0(0) - y_1(0) \leq 0, \\ (z_1^{(\delta)}(t) - z_0^{(\delta)}(t)) + M(z_1(t) - z_0(t)) \leq 0, \\ z_1(0) - z_0(0) \leq 0. \end{cases}$$

for  $t \in [0,1]$ . And by Lemma 2.5, we have

$$y_0(t) \leq y_1(t), z_1(t) \leq z_0(t), t \in [0,1].$$

So we can easily get that

$$\begin{cases} y_1^{(\delta)}(t) = f(t, y_0(t)) - M(y_1(t) - y_0(t)) \leq f(t, y_1(t)), \\ y_1(0) + rz_1(1) \leq \lambda \int_0^1 y_1(s) ds, \\ z_1^{(\delta)}(t) = f(t, z_0(t)) - M(z_1(t) - z_0(t)) \geq f(t, z_1(t)), \\ z_1(0) + ry_1(1) \geq \lambda \int_0^1 z_1(s) ds. \end{cases}$$

from formula (3) and (5). *i.e.*,  $y_1, z_1$  are coupled lower and upper solutions of

(1).

We also get that

$$\begin{cases} y_1^{(\delta)}(t) - z_1^{(\delta)}(t) \leq -M(y_1(t) - z_1(t)), \\ y_1(0) - z_1(0) = \lambda \int_0^1 (y_0(s) - z_0(s)) ds + r(y_0(1) - z_0(1)) \leq 0. \end{cases}$$

from formula (5) and  $y_0 \leq z_0$ . Similarly, we have  $y_1 \leq z_1$ . by Lemma 2.5.

Let  $y_n = T_1(y_{n-1}, z_{n-1}), z_n = T_2(y_{n-1}, z_{n-1})$ , then from formula (4), we have that  $y_n, z_n$  are coupled lower and upper solutions of (1) for any  $n \geq 2$ , which is similar to the proof above. And

$$y_{n-1} \leq y_n \leq z_n \leq z_{n-1}.$$

In summary, we have

$$y_0(t) \leq y_1(t) \leq \dots \leq y_n(t) \leq \dots \leq z_n(t) \leq \dots \leq z_1(t) \leq z_0(t);$$

for  $t \in [0, 1]$ . Therefore, function sequences  $\{y_n(t)\}, \{z_n(t)\}$  are uniformly bounded, *i.e.*,

$$\|y_n\| \leq M_0, \|z_n\| \leq M_0.$$

for  $n = 0, 1, 2, \dots$  and  $M_0 > 0$ . Because  $f$  is continuous, we have

$$|f_{y_{n-1}}(t)| \leq M_1.$$

for  $n = 1, 2, 3, \dots, t \in [0, 1]$  and  $M_1 > 0$ . In addition, because that functions

$e^{\frac{M}{\delta}(s^\delta - t^\delta)}$  and  $e^{-\frac{M}{\delta}t^\delta}$  are continuous, we have

$$\begin{aligned} & |y_n(t_2) - y_n(t_1)| \\ &= \left| \left( \lambda \int_0^1 y_{n-1}(s) ds - rz_{n-1}(1) \right) \left( e^{-\frac{M}{\delta}t_2^\delta} - e^{-\frac{M}{\delta}t_1^\delta} \right) \right. \\ & \quad \left. + \int_0^{t_2} s^{\delta-1} f_{y_{n-1}}(s) e^{\frac{M}{\delta}(s^\delta - t_2^\delta)} ds - \int_0^{t_1} s^{\delta-1} f_{y_{n-1}}(s) e^{\frac{M}{\delta}(s^\delta - t_1^\delta)} ds \right| \\ &\leq \left| \left( \lambda \int_0^1 y_{n-1}(s) ds - rz_{n-1}(1) \right) \right| \left| e^{-\frac{M}{\delta}t_2^\delta} - e^{-\frac{M}{\delta}t_1^\delta} \right| \\ & \quad + \left| \int_0^{t_1} s^{\delta-1} f_{y_{n-1}}(s) \left( e^{\frac{M}{\delta}(s^\delta - t_2^\delta)} - e^{\frac{M}{\delta}(s^\delta - t_1^\delta)} \right) ds + \int_{t_1}^{t_2} s^{\delta-1} f_{y_{n-1}}(s) e^{\frac{M}{\delta}(s^\delta - t_2^\delta)} ds \right| \\ &\leq \left| \left( \lambda \int_0^1 y_{n-1}(s) ds - rz_{n-1}(1) \right) \right| \left| e^{-\frac{M}{\delta}t_2^\delta} - e^{-\frac{M}{\delta}t_1^\delta} \right| \\ & \quad + \left| \int_0^{t_1} s^{\delta-1} f_{y_{n-1}}(s) \left( e^{\frac{M}{\delta}(s^\delta - t_2^\delta)} - e^{\frac{M}{\delta}(s^\delta - t_1^\delta)} \right) ds \right| + \left| \int_{t_1}^{t_2} s^{\delta-1} f_{y_{n-1}}(s) e^{\frac{M}{\delta}(s^\delta - t_2^\delta)} ds \right| \\ &\rightarrow 0 \end{aligned}$$

if  $0 \leq t_1 < t_2 \leq 1$  and  $t_2 \rightarrow t_1$ . Hence,  $\{y_n(t)\}$  is equicontinuous, we can also get that  $\{z_n(t)\}$  is equicontinuous similarly.

In summary, by Ascoli-Arzela theorem [17], we can prove that  $\{y_n\}, \{z_n\}$  are convergent because of the monotonicity of Sequences, *i.e.*, there are two

functions  $y^*, z^*$ , such that

$$\lim_{n \rightarrow \infty} \|y_n - y^*\| = 0, \lim_{n \rightarrow \infty} \|z_n - z^*\| = 0.$$

and  $y^*, z^* \in D$ . Next we take limits on both sides of (4), then from Lebesgue Dominated Convergence Theorem, we have

$$T(y^*, z^*) = (T_1(y^*, z^*), T_2(y^*, z^*)) = (y^*, z^*).$$

if  $n \rightarrow \infty$ . i.e.,  $(y^*, z^*)$  are coupled solutions of (1).

2) Here we prove that  $(y^*, z^*)$  are coupled minimal and maximal solutions of (1) respectively in  $D$ .

Assume that  $(x_1, x_2)$  are a set of coupled solutions of (1), then the above problem is equivalent to prove that

$$y^* \leq x_1, x_2 \leq z^*.$$

Whereas  $x_1, x_2 \in D$ , therefore  $y_0 \leq x_1, x_2 \leq z_0$ . Assume that  $y_k \leq x_1, x_2 \leq z_k$  for  $k > 1$ , here we prove that  $y_{k+1} \leq x_1, x_2 \leq z_{k+1}$ .

Consider that

$$\begin{cases} y_{k+1}^{(\delta)}(t) = f(t, y_k(t)) - M(y_{k+1}(t) - y_k(t)), & t \in (0, 1), \\ y_{k+1}(0) + rz_k(1) = \lambda \int_0^1 y_k(s) ds. \\ z_{k+1}^{(\delta)}(t) = f(t, z_k(t)) - M(z_{k+1}(t) - z_k(t)), & t \in (0, 1), \\ z_{k+1}(0) + ry_k(1) = \lambda \int_0^1 z_k(s) ds. \end{cases}$$

And from Definition 2.3, we have that

$$\begin{cases} x_1^{(\delta)}(t) = f(t, x_1(t)), & t \in (0, 1), \\ x_1(0) + rx_2(1) = \lambda \int_0^1 x_1(s) ds, \\ x_2^{(\delta)}(t) = f(t, x_2(t)), & t \in (0, 1), \\ x_2(0) + rx_1(1) = \lambda \int_0^1 x_2(s) ds. \end{cases}$$

Then from (3), we get that

$$\begin{cases} x_1^{(\delta)}(t) - y_{k+1}^{(\delta)}(t) + M(x_1(t) - y_{k+1}(t)) \geq 0, & t \in (0, 1) \\ x_1(0) - y_{k+1}(0) \geq 0, \\ z_{k+1}^{(\delta)}(t) - x_2^{(\delta)}(t) + M(z_{k+1}(t) - x_2(t)) \geq 0, & t \in (0, 1) \\ z_{k+1}(0) - x_2(0) \geq 0. \end{cases}$$

In that way, we have

$$y_{k+1} \leq x_1, x_2 \leq z_{k+1},$$

according to Lemma 2.5. By Mathematical Induction, we can get

$$y_n \leq x_1, x_2 \leq z_n.$$

for  $n = 1, 2, 3, \dots$ . In addition, because of the convergence of iterative sequences, we have

$$y^* \leq x_1, x_2 \leq z^*,$$

if  $n \rightarrow \infty$ . *i.e.*,

$$y^*(t) \leq x_1(t), x_2(t) \leq z^*(t),$$

for  $t \in [0,1]$ . Therefore,  $(y^*, z^*)$  are coupled minimal and maximal solutions of (1) respectively in  $D$  from Definition 2.3.

3) Here we prove that if  $x$  is the solution of (1) in  $D$ , then  $y^* \leq x \leq z^*$ . In conclusion (2) above, let  $x_1(t) = x_2(t) = x(t)$ , because that  $x$  is the solution of (1) in  $D$ , therefore,  $(x_1, x_2)$  are a set of coupled solutions of (1). Obviously,  $x$  subject to

$$y^* \leq x \leq z^*.$$

In summary, Theorem 3.1 is proved.  $\square$

**Theorem 3.2.** Assume that  $f(t, x)$  is increasing in  $x$  on

$D = \{x \in C([0,1], \mathbb{R}) \mid y_0(t) \leq x \leq z_0(t)\}$ , and  $1 > r > 0, \lambda > 0$ , then there exists a unique solution of (1) in  $D = \{x \in C([0,1], \mathbb{R}) \mid y_0(t) \leq x \leq z_0(t)\}$ .

**Proof** By Theorem 3.1, we get that  $y_n(t) \rightarrow y^*(t)$  and  $z_n(t) \rightarrow z^*(t)$ . And we have  $y_0(t) \leq y^*(t) \leq z^*(t) \leq z_0(t)$  for  $t \in [0,1]$ . Then we have that  $y^*(t) - z^*(t) \leq 0$  for  $t \in [0,1]$ . Here we prove that  $y^*(t) = z^*(t)$ .

If  $f(t, x)$  is increasing in  $x$  on  $D = \{x \in C([0,1], \mathbb{R}) \mid y_0(t) \leq x \leq z_0(t)\}$ , assume that  $y^*(t) - z^*(t) < 0$ , then we have

$$\begin{cases} y^{*(\delta)}(t) - z^{*(\delta)}(t) = f(t, y^*(t)) - f(t, z^*(t)) \leq 0, \\ y^*(0) - z^*(0) = r(y^*(1) - z^*(1)) + \lambda \int_0^1 (y^*(s) - z^*(s)) ds. \end{cases}$$

considering the convergence of iterative sequences. Let  $h(t) = y^*(t) - z^*(t)$ , then we have  $h^{(\delta)}(t) = y^{*(\delta)}(t) - z^{*(\delta)}(t) \leq 0$  by Lemma 2.2, *i.e.*, the function  $h(t) = y^*(t) - z^*(t)$  is monotonically decreasing. Hence,  $h(0) \geq h(1)$ , therefore, we draw a contradiction from the conclusion that  $h(0) < h(1)$ , which can be obtained from the condition  $1 > r > 0, \lambda > 0$  and the boundary value conditions above. Therefore, we have  $h(t) = y^*(t) - z^*(t) = 0$ , *i.e.*,  $y^*(t) = z^*(t)$  is the solution of (1).  $\square$

On the basis of (1), we can also consider the existence of solutions of boundary value problems for the following uniform fractional differential equations:

$$\begin{cases} x^{(\delta)}(t) = f(t, x(t)), & t \in (0,1), \\ x^{(k)}(0) = -rx^{(k)}(1) + \lambda \int_0^1 x(s) ds, & k = 1, 2, \dots, n. \end{cases} \quad (6)$$

where  $x^{(\delta)}(t)$  is the conformable fractional derivatives of order  $\delta$  for  $t \in (0,1)$  which is defined in [1], and  $\delta \in (n, n+1]$ ,  $n \geq 1, r > 0, \lambda > 0$ ,  $\mathbb{R} = (-\infty, +\infty)$ ,  $f: [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous. Similarly, the existence of the solution can be proved by the method of coupled upper and lower solution, and the range of the solution can be obtained. Due to  $\delta \in (n, n+1]$ , so the original problem needs to be solved until the solution of the equation of order  $n$  before we construct the comparison theorem, which is the difficulty of (6).



## 4. Examples

To illustrate our main results, we present the following example.

**Example 4.1.** Consider the boundary value problem of conformable fractional differential equations under the following new definitions

$$\begin{cases} x^{(\frac{1}{2})}(t) = t(1-x^2(t)) - 3x(t), & t \in (0,1), \\ x(0) = -\frac{1}{3}x(1) + \frac{1}{2}\int_0^1 x(s) ds. \end{cases} \quad (7)$$

It is obvious that  $y_0(t) = -1, z_0(t) = 1$  are coupled lower and upper solutions of (7), and from the condition  $f(t, x) = t(1-x^2) - 3x$ , we can get that there exists a constant  $M \geq t(x_1 + x_2) + 3 > 0$  for  $-1 \leq x_2 \leq x_1 \leq 1$ , such that the formula (4) of Theorem 3.1 holds. Hence, problem (4) has at least one solution  $x \in [-1, 1]$  for  $t \in [0, 1]$  by Theorem 3.2.  $\square$

**Example 4.2.** Consider the boundary value problem of conformable fractional differential equations under the following new definitions

$$\begin{cases} x^{(\frac{1}{3})}(t) = \arctan x(t), & t \in (0,1), \\ x(0) + ax(1) = b\int_0^1 x(s) ds. \end{cases} \quad (8)$$

where  $1 > a > 0, b > 0$ , it is easy to get that

$$\begin{cases} y_0(t) = -\frac{3}{2}\pi t^{\frac{1}{3}} + \frac{3}{2}\frac{\pi}{a+b-1}\left(a + \frac{3}{4}b\right), & t \in [0,1], \\ z_0(t) = \frac{3}{2}\pi t^{\frac{1}{3}} - \frac{3}{2}\frac{\pi}{a+b-1}\left(a + \frac{3}{4}b\right), & t \in [0,1]. \end{cases}$$

which yield to

$$\begin{cases} -\frac{\pi}{2} = y_0^{(\frac{1}{3})}(t) \leq \arctan y_0(t), & t \in (0,1), \\ y_0(0) + az_0(1) \leq b\int_0^1 y_0(s) ds, \\ \frac{\pi}{2} = z_0^{(\frac{1}{3})}(t) \geq \arctan z_0(t), & t \in (0,1), \\ z_0(0) + ay_0(1) \geq b\int_0^1 z_0(s) ds. \end{cases}$$

Therefore,  $y_0(t), z_0(t)$  are coupled lower and upper solutions of (8), it is obvious that the formula (4) of Theorem 3.1 holds. Hence, problem (8) has at least one solution  $x \in [y_0(t), z_0(t)]$  for  $t \in [0, 1]$  by Theorem 3.2.  $\square$

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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